Wed. April 7 – Lect. 33

ML Properties ...

Recall CRLB: $\hat{\theta}(y)$ unbiased for θ

$$\mathrm{Var}_\theta\{\hat{\theta}(Y)\} \geq \left(E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f_\theta(Y)\right)^2\right\}\right)^{-1}$$

$$= \left(-E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log f_\theta(Y)\right\}\right)^{-1}$$

+ Equality $\iff \frac{\partial}{\partial\theta}\log f_\theta(y) = a(\theta)\left[\hat{\theta}(y) - \theta\right]$

Such are called <u>efficient</u>.

<u>Thm</u> Subject to usual smoothness conditions if an efficient est. exists it will be ML.

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$\hat{\theta}$ is efficient $\Longleftrightarrow$

$$\frac{\partial}{\partial\theta}\log f_\theta(y) = a(\theta)\left[\hat{\theta}(y) - \theta\right]$$

Recall that ML est. is found from ...

$$\left.\frac{\partial}{\partial\theta}\log f_\theta(y)\right|_{\theta=\hat{\theta}_{ML}(y)} = 0$$

$$0 = \left. a(\theta)\left[\hat{\theta}(y) - \theta\right]\right|_{\theta=\theta_{ML}}$$

So must have

either $a(\theta_{ML}) = 0$ or $\left[\hat{\theta}(y) - \theta_{ML}\right] = 0$

$a(\theta_{ML}) = 0$ is not a good soln since it results in a $\hat{\theta}_{ML}(y)$ that is indep. of obs.

$\Longrightarrow$ Then the CRLB $= \infty$.

$\hat{\theta}(y) = \hat{\theta}_{ML}(y) \longrightarrow$ This is a legitimate estimator

Thm An ML estimator is a function of any suff. stat. for θ.

$\Longrightarrow$ Comes from NF Fact. Thm

$$f_\theta(y) = g_\theta(T(y))\, h(y)$$

Example $\quad Y_i = s(\theta) + N_i \quad 1 \le i \le n$

$N_i \sim \text{iid} \sim N(0, \sigma^2)$.

$$f_\theta(y) = C(y) \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - s(\theta))^2 \right\}$$

$$\frac{\partial}{\partial \theta} \log f_\theta(y) = \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - s(\theta)) \frac{\partial s(\theta)}{\partial \theta}$$

$\hookrightarrow$ In general can't write in form needed for existence of eff. est.

Look a solving likelihood …

$$\left[\frac{\partial s(\theta)}{\partial \theta} \frac{1}{\sigma^2} \right] \left[\frac{1}{n} \sum_{i=1}^{n} y_i - s(\theta) \right] \Bigg|_{\theta = \hat{\theta}_{ML}(y)} = 0$$

If range of $s(\theta)$ as $\theta \in \Lambda$ includes $\frac{1}{n}\sum_{i=1}^{n} y_i$ then a soln to likelihood equation exists and it will satisfy ...

$$s\left(\hat{\theta}_{ML}(y)\right) = \frac{1}{n}\sum_{i=1}^{n} y_i$$

Thus if $s^{-1}(\cdot)$ exists then

$$\hat{\theta}_{ML}(y) = s^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)$$

<u>Note:</u>

- If $s^{-1}(\cdot)$ does not exist then the ML est. prob. for θ is not well posed.
- If range of $s(\theta)$ does not include the sample mean $\Longrightarrow$ ML est will corresp. to some boundary pt. of range.

FORM C
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$Y_k = \sqrt{\theta}\, s_k R_k + N_k \qquad k = 1, 2, \ldots n$

$\{s_k : 1 \le k \le n\}$ Known sig.

N_k and R_k are iid $\sim N(0,1)$ $\qquad N_k \perp\!\!\!\perp R_k$

$\theta \geq 0$ is an unknown param.

(a) Likelihood eqN for θ, Y_k

(b) CRLB

(c) If s_k is ± 1 seq. Find MLE explicitely.

(d) Compute bias, var. compare CRLB.

$Y_1\ Y_2 \cdots Y_n$ are indep. $\sim N(0, 1+\theta s_k^2)$

$$\log f_\theta(y) = \sum_{k=1}^{n} \log\left\{ \frac{1}{\sqrt{2\pi}\sqrt{1+\theta s_k^2}} e^{-y_k^2/2(1+\theta s_k^2)} \right\}$$

$$= \sum_{k=1}^{n} \left\{ -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(1+\theta s_k^2) - \frac{y_k^2}{2(1+\theta s_k^2)} \right\}$$

Likelihood eqN:

$$\frac{\partial}{\partial\theta}\log f_\theta(y) = -\frac{1}{2}\sum_{k=1}^{n}\left\{ \frac{s_k^2}{1+\theta s_k^2} - \frac{y_k^2 s_k^2}{(1+\theta s_k^2)^2} \right\}$$

$$= 0.$$

$$\therefore \sum_{k=1}^{n} \frac{s_k^2(y_k^2 - 1 - \theta s_k^2)}{(1+\theta s_k^2)^2} = 0$$

$\hat{\theta}_{ML}(y)$ is among the solns of this.

CRLB

$$\frac{\partial^2}{\partial\theta^2}\log f_\theta(y) = \sum_{k=1}^{n}\left\{\frac{-s_k^4 y_k^2}{(1+\theta s_k^2)^3} + \frac{s_k^4}{2(1+\theta s_k^2)^2}\right\}$$

$$I_\theta = -E_\theta\{\text{above}\}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\frac{s_k^4}{(1+\theta s_k^2)^2}$$

$$\text{CRLB} = \frac{1}{I_\theta}$$

(c) Say $s_k^2 = 1$.

$$\Longrightarrow \text{Find } \hat{\theta}_{ML}(y) = \left(\frac{1}{n}\sum_{k=1}^{n} y_k^2\right) - 1$$

(d) Also find

$$E_\theta\{\hat{\theta}_{ML}(Y)\} = \theta \text{ ie unbiased.}$$

$$\mathrm{Var}_\theta\left\{\hat{\theta}_{ML}(Y)\right\} = \frac{2(1+\theta)^2}{n}$$

$$= \text{CRLB for this case.} \quad S_k^2 = 1$$

Background for Next Lect.

Strong Law Large Numb.: $X_1, X_2, \ldots$ an iid seq. of rvs having a finite mean

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to E\{X_1\}$$

as $n \to \infty$ with prob. one.

Defs $X_1, X_2, \ldots$ a seq. of rvs.

$X_n \to X$ with prob. 1 if $P\{\lim_{n\to\infty} X_n = X\} = 1$.
(almost everywhere or almost surely)

$$\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$$

$X_n \to X$ in probability if

$$\lim_{n\to\infty} P\{|X_n - X| \geq \varepsilon\} = 0 \quad \forall \varepsilon > 0.$$