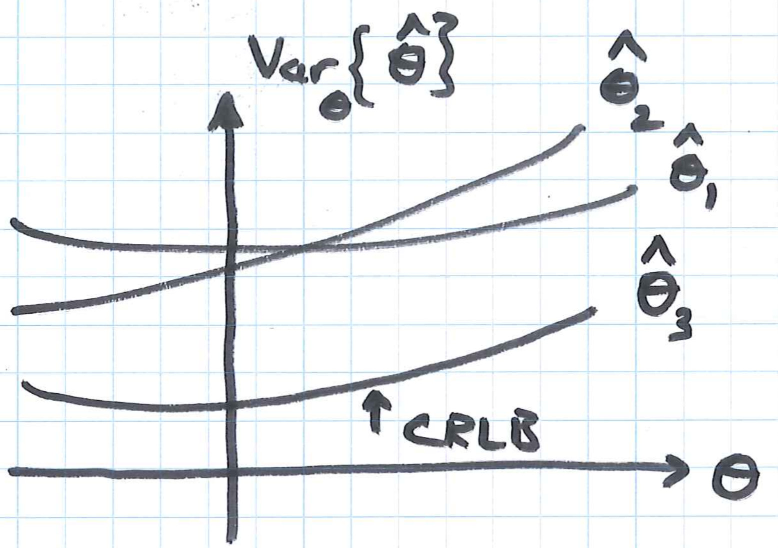
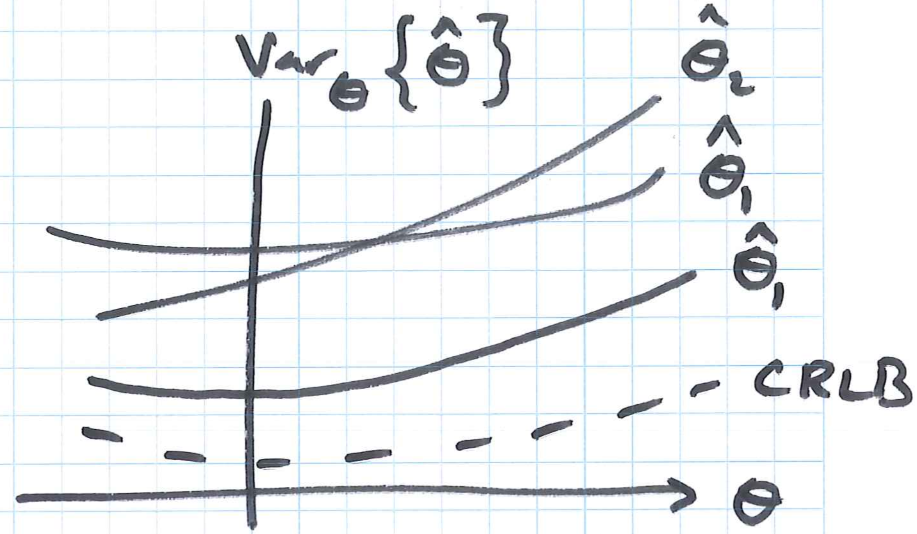
Friday April 2 Lecture 31

An estimate that is unbiased + attains the CRLB is called <u>efficient</u>.

Efficient estimators are MVUE



Efficient + MVUE



$\hat{\theta}_1$ could be MVUE, but it's not efficient.

Example : Phase Estimation

$$Y_k = A\cos(2\pi f_0 k + \phi) + N_k \qquad k = 0, 1 \dots N-1$$

A, f_0 assumed known

ϕ unknown param.

$N_k \sim N(0, \sigma^2)$ iid

$\Longrightarrow$ Easy to write down $f_\phi(y)$

$$\frac{\partial}{\partial\phi} \log f_\phi(y) = -\frac{A}{\sigma^2}\sum_{k=0}^{N-1}\left[y_k \sin(2\pi f_0 k + \phi) - \frac{A}{2}\sin(4\pi f_0 k + 2\phi)\right]$$

Condition for attainment of CRLB

$$\neq a(\phi)\left[\tilde{\Phi}(y) - \phi\right]$$

Take another deriv.

$$\frac{\partial^2}{\partial\phi^2} \log f_\phi(y) = -\frac{A}{\sigma^2}\sum_{k=0}^{N-1}\left[y_k \cos(2\pi f_0 k + \phi) - A\cos(4\pi f_0 k + 2\phi)\right]$$

Taking $-E_\phi\left\{\frac{\partial^2}{\partial\phi^2}\log f_\phi(Y)\right\}$

$$= \frac{NA^2}{2\sigma^2}\left[1 - \underbrace{\frac{1}{N}\sum_{k=0}^{N-1}\cos(4\pi f_0 k + 2\phi)}_{\approx 0 \text{ for } f_0 \text{ not near } 0 \text{ or } 1/2.}\right]$$

$$\approx \frac{NA^2}{2\sigma^2}$$

$$\mathrm{Var}_\phi\left\{\hat{\phi}(Y)\right\} \geq \frac{2\sigma^2}{NA^2}$$

CRLB for generic Signal in WGN

$$Y_k = s[k;\theta] + W_k \quad k = 0, 1 \dots N-1 \quad W_k \sim \text{iid } N(0, \sigma^2)$$

$\hookrightarrow$ know functional form but not the values of θ.

$$f_{Y;\theta}(y)$$

$$\frac{\partial^2}{\partial\theta^2} \log f_{Y;\theta}(y) = +\frac{1}{\sigma^2} \sum_{k=0}^{N-1} \left\{ \left(y_k - s[k;\theta]\right) \frac{\partial^2}{\partial\theta^2} s[k;\theta] - \left(\frac{\partial}{\partial\theta} s[k;\theta]\right)^2 \right\}$$

$$I_\theta = -E_\theta\left\{\text{above}\right\} = \sum_{k=0}^{N-1} \left(\frac{\partial}{\partial\theta} s[k;\theta]\right)^2$$

$$\mathrm{Var}_\theta\left\{\hat{\theta}(Y)\right\} \geq \frac{\sigma^2}{\sum_{k=0}^{N-1} \left(\frac{\partial}{\partial\theta} s[k;\theta]\right)^2}$$

Special Case: Sinusoidal Freq. Est.

$$s[k;\theta] = A\cos(2\pi f_0 k + \phi) \qquad k = 0, 1, \ldots N-1$$

unknown

known

$$\frac{\partial s[k; f_0]}{\partial f_0} = A \cdot 2\pi k \sin(2\pi f_0 k + \phi)$$

$$\sum_{k=0}^{N-1} \left(\quad \right)^2 = \sum_{k=0}^{N-1} A^2 4\pi^2 k^2 \sin^2(2\pi f_0 k + \phi).$$

$$\Rightarrow \mathrm{Var}_{f_0}\{\hat{f}_0\} \geq \frac{\sigma^2}{A^2 4\pi^2 \sum_{k=0}^{N-1} k^2 \sin^2(2\pi f_0 k + \phi)}.$$

Transformation of Parameters

CRLB Thm .... θ param. $Y \in \Gamma$ $\quad f_{Y;\theta}(y)$

$$\alpha = g(\theta)$$

unknown $\quad$ known $\quad$ unknown

Say have $\hat{\alpha} = \hat{\alpha}(Y)$ is an unbiased estimator of $\alpha = g(\theta)$ ie $E_\theta\{\hat{\alpha}(Y)\} = \alpha = g(\theta)$

$$\Longrightarrow \quad Var_\theta\{\hat{\alpha}(Y)\} \geq \frac{\left(\frac{\partial}{\partial\theta} g(\theta)\right)^2}{I_\theta}$$

Examples $s[k;\theta] = A \quad \forall k$ (ie $A = \theta$ is the param.).

$$\Rightarrow \mathrm{Var}_A\{\hat{A}(Y)\} \geq \frac{\sigma^2}{N}$$

Easy to see ... $\overline{Y} \triangleq \frac{1}{N}\sum_{k=0}^{N-1} Y_k$ is eff & MVUE

Say $\alpha = g(A) = A^2$

For this prob. $I_\theta = I_A = \frac{1}{\sigma^2}\sum_{k=0}^{N-1}(1)^2 = \frac{N}{\sigma^2}$

$$\left(\frac{\partial g(A)}{\partial A}\right)^2 = (2A)^2$$

$$\mathrm{Var}_A\{\hat{\alpha}\} \geq \frac{(2A)^2}{N/\sigma^2} = \frac{4A^2\sigma^2}{N}$$

Common Sense ... a good estimator ought to be

$$(\bar{Y})^2$$

$$E_A\{(\bar{Y})^2\} = (E_A\{\bar{Y}\})^2 + Var_A\{\bar{Y}\}$$

$$= A^2 + \frac{\sigma^2}{N}$$

$$\neq A^2$$

ie it is <u>not</u> unbiased, so should not compare to CRLB.

<u>By example</u>

Efficiency of estimators <u>not</u> <u>preserved</u> under $(\cdot)^2$ transformation.