Wed. March 31 Lect. $\boxed{30}$

CRLB Thm

θ scalar param.
Y observations
$\hat{\theta}(Y)$ unbiased estimator of θ

$$\Rightarrow \mathrm{Var}_\theta\left\{\hat{\theta}(Y)\right\} \geq \frac{1}{I_\theta}$$

where

$$I_\theta = E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f_\theta(Y)\right)^2\right\}$$

$$= -E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log f_\theta(Y)\right\}$$

Achieving the CRLB?

Subject to any needed conditions on derivatives
...

There exists an unbiased estimator whose variance attains the CRLB if and only if

$$\frac{\partial}{\partial\theta} \log f_\theta(y) = a(\theta)\left[\tilde{\theta}(y) - \theta\right] \quad (*)$$

and in this case $a(\theta) = I_\theta$.

Say $(*)$

$$E_\theta\left\{\frac{\partial}{\partial\theta} \log f_\theta(Y)\right\} = a(\theta)\left[E_\theta\left\{\tilde{\theta}(Y)\right\} - \theta\right]$$

$$= \int_\Gamma \frac{\partial}{\partial\theta} \log f_\theta(y)\, f_\theta(y)\, dy = \int_\Gamma \frac{\partial}{\partial\theta} f_\theta(y)\, dy = \frac{\partial}{\partial\theta} \underbrace{\int_\Gamma f_\theta(y)\, dy}_{=0}$$

To see why condition works ... look again at Schwarts:

U, V are rvs. with finite m.s. values

$$\left(E\{UV\}\right)^2 \leq E\{U^2\}E\{V^2\}$$

with equality iff

$$\lambda U + \mu V = 0$$

for some real # λ, μ not both zero.

⟹ Back to CRLB --- ie back to proof. Spec. to unbiased case where we had

$$1 = E_\theta\left\{\left[\hat{\theta}(Y) - \theta\right]\left[\frac{\partial}{\partial\theta}\log f_\theta(Y)\right]\right\}$$

$$= \mathrm{Cov}_\theta\left\{\hat{\theta}(Y), \frac{\partial}{\partial\theta}\log f_\theta(Y)\right\}$$

$$1 = 1^2 = E_\theta\left\{\left[\hat{\theta}(Y)-\theta\right]\left[\frac{\partial}{\partial\theta}\log f_\theta(Y)\right]\right\}^2$$

$$\leq \underbrace{E_\theta\left\{\left[\hat{\theta}(Y)-\theta\right]^2\right\}}_{\mathrm{Var}_\theta\{\hat{\theta}(Y)\}} \underbrace{E_\theta\left\{\left[\frac{\partial}{\partial\theta}\log f_\theta(Y)\right]^2\right\}}_{I_\theta}$$

and now equality iff. $\exists$ real number $\lambda_\theta, \mu_\theta$ not both zero st.

$$\lambda_\theta\left[\hat{\theta}(Y)-\theta\right] + \mu_\theta\left[\frac{\partial}{\partial\theta}\log f_\theta(Y)\right] = 0$$

with prob 1 under P_θ

$$\iff \frac{\partial}{\partial\theta}\log f_\theta(y) = \underbrace{\left(-\frac{\lambda_\theta}{\mu_\theta}\right)}_{a(\theta)}\left[\hat{\theta}(y)-\theta\right]$$

Plug into top equation shows $a(\theta) = I_\theta$.

Example $Y = [Y_1 \dots Y_n]$ a 0-1 vector that results from Bernoulli trials with suc. prob. θ.

$$P_\theta(y) = \theta^{m(y)} (1-\theta)^{n-m(y)}$$

$m(y) = \#$ succ. in n trials $= \sum_{k=1}^{n} y_k$

$$\frac{\partial}{\partial\theta} \log P_\theta(y) = \frac{\partial}{\partial\theta}\left\{ m(y)\log\theta + (n-m(y))\log(1-\theta)\right\}$$

$$= \frac{m(y)}{\theta} - \frac{n-m(y)}{1-\theta}$$

$$= \frac{n}{\theta(1-\theta)}\Big[\underbrace{\frac{m(y)}{n}}_{\hat{\theta}(y)} - \theta\Big]$$

$\therefore$ achieve CRLB is unbiased and

$$\mathrm{Var}_\theta\left\{\tilde{\theta}(Y)\right\} = \frac{\theta(1-\theta)}{n}$$

Example Λ an open interval

$$f_\theta(y) = c(\theta)\, e^{g(\theta)T(y)}\, h(y)$$ ie a one param exp. fam.

$$\log f_\theta(y) = \log c(\theta) + g(\theta)T(y) + \log h(y)$$

$$\int_\Pi f_\theta(y)\,dy = 1 = c(\theta)\int_\Pi e^{g(\theta)T(y)}\, h(y)\,dy.$$

$$\Rightarrow c(\theta) = \left[\int_\Pi e^{g(\theta)T(y)}\, h(y)\,dy\right]^{-1}$$

$$\Rightarrow \log c(\theta) = -\log \int_\Pi e^{g(\theta)T(y)}\, h(y)\,dy$$

Now take $\frac{\partial}{\partial\theta} \log f_\theta(y)$ ....

$$\frac{\partial}{\partial\theta}\log f_\theta(y) = g'(\theta)\,T(y) - \frac{\partial}{\partial\theta}\log\int_\Pi e^{g(\theta)T(y)}\,h(y)\,dy$$

$$= g'(\theta)\,T(y) - \frac{g'(\theta)\int_\Pi T(y)\,e^{g(\theta)T(y)}\,h(y)\,dy}{\int_\Pi e^{g(\theta)T(y)}\,h(y)\,dy}$$

$$= g'(\theta)\left[T(y) - E_\theta\{T(Y)\}\right]$$

ie we have the right form.

$\Rightarrow$ If $T(Y)$ is unbiased then is MVUE.

Generalize CRLB to Vector Case

$\theta \in \mathbb{R}^m$ The analogy of Fisher info is the $m \times m$ Fisher info matrix

$$[I_\theta]_{i,j} = E_\theta\left\{\left(\frac{\partial}{\partial\theta_i}\log f_\theta(Y)\right)\left(\frac{\partial}{\partial\theta_j}\log_\theta f_\theta(Y)\right)\right\}$$

$$= -E_\theta\left\{\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f_\theta(Y)\right\}$$

Then if $\hat{\theta}(Y)$ is an unbiased est. of θ

$$\text{Var}_\theta\{\hat{\theta}(Y)\} = E_\theta\left\{(\hat{\theta}(Y)-\theta)(\hat{\theta}(Y)-\theta)^T\right\}$$

$$\geq I_\theta^{-1} \overset{ie}{\Longleftrightarrow} \text{Var}_\theta\{\hat{\theta}(Y)\} - I_\theta^{-1} \geq 0$$

positive semi-definite

For interpretations ....

$$A \geqslant 0 \iff v^T A v \geqslant 0 \quad \forall \text{ vectors } v.$$

To get indiv. bounds ....

$$v = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow \text{ith spot.}$$