[Created 3/29 to partially replace lecture of 3/8]

Form C
APPROVED FOR USE IN
PURDUE UNIVERSITY

Example : MVUE of Signal Amplitude

Model : $Y_k = N_k + \mu s_k \qquad k = 1, 2, \ldots n$

$N_k \sim N(0, \sigma^2)$

$s = [s_1\ s_2 \ldots s_n]^T$

μ is a amplitude term.

Unknown : $\mu \in \mathbb{R}$

Knowns : σ^2 and s

Find pdf of obs. vector $Y = [Y_1 \ldots Y_n]^T$

$$f_{Y;\theta}(y) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k - \mu s_k)^2\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{1/2}} \left(\exp\left\{-\frac{1}{2\sigma^2}\right\}\sum_{k=1}^{n} y_k^2\right)\left(\exp\left\{+\frac{\mu}{\sigma^2}\sum_{k=1}^{n} s_k y_k\right\}\right) \cdot \left(\exp\left\{-\frac{\mu^2}{2\sigma^2}\sum_{k=1}^{n} s_k^2\right\}\right)$$

Define $\theta_1 = \mu/\sigma^2 \quad T_1(y) = +\sum_{k=1}^{n} s_k y_k$

Define

$$C(\theta_1) = \exp\left\{-\frac{\sigma^2\theta_1^2}{2}\sum_{k=1}^{n} s_k^2\right\} \Big/ (2\pi\sigma^2)^{n/2}$$

$$h(y) = \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=1}^{n} y_k^2\right\}$$

Then see that pdf of Y is of form

$$f_{Y;\theta_1}(y) = C(\theta_1)\exp\left\{\theta_1 T_1(y)\right\} h(y)$$

Assume μ is an arb. real #. With this $\theta_1 = \mu/\sigma^2$ lives in

$$\Delta = \left\{\theta_1 : -\infty < \theta_1 < \infty\right\}$$

clearly this param. set contains a one-dim. rectangle.

$\Longrightarrow T_1(y)$ is a complete suff. stat. for θ_1.

Wish to estimate $\mu = +\sigma^2\theta_1 \triangleq g(\theta_1)$

Approach: Find any unbiased est. of $g(\theta_1)$ and cond. on $T_1(y)$ to get MVUE.

Note: $E_\theta\{Y_1\} = \mu s_1$ so assuming that $s_1 \neq 0$ then

$$\hat{g}(y) \triangleq y_1/s_1$$

is an unbiased estimator of $g(\theta)$.

Thus $\tilde{g}(T_1(y)) = E_\theta\{\hat{g}(Y) \mid T_1(Y) = T_1(y)\}$

will be an MVUE.

Therefore: How to compute?

$\hat{g}(Y)$ and $T_1(Y)$ are lin. functs of Y, a Gaussian rvec.
$\therefore \hat{g}(Y), T_1(Y)$ J.G.

$$E_\theta\{\hat{g}(Y)\} = \mu$$

$$E_\theta\{T_1(Y)\} = n\mu\overline{s^2} \quad \text{where} \quad \overline{s^2} \triangleq \frac{1}{n}\sum_{k=1}^{n} s_k^2$$

$$\mathrm{Var}_\theta\{\hat{g}(Y)\} = \sigma^2/s_1^2$$

$$\mathrm{Var}_\theta\{T_1(Y)\} = n\sigma^2\overline{s^2}$$

$$\mathrm{Cov}_\theta\{\hat{g}(Y), T_1(Y)\} = E_\theta\left\{(\hat{g}(Y)-\mu)(T_1(Y)-n\mu\overline{s^2})\right\}$$

$$= E_\theta\left\{\frac{Y_1}{s_1}\sum_{k=1}^{n} s_k Y_k\right\} - 2n\mu^2\overline{s^2} + n\mu^2\overline{s^2}$$

$$= E_\theta\{Y_1^2\} + \sum_{k=2}^{n}\frac{s_k}{s_1}E_\theta\{Y_1Y_k\} - 2n\mu^2\overline{s^2} + n\mu^2\overline{s^2}$$

$$= \sigma^2 + \mu^2 s_1^2 + \mu^2\sum_{k=2}^{n} s_k^2 - n\mu^2\overline{s^2}$$

$$\therefore \ \mathrm{Cov}_\theta\{\hat{g}(Y), T_1(Y)\} = \sigma^2$$

Now use formulas for cond. mean ...

$$\tilde{g}(T_1(y)) = E_\theta\{\hat{g}(Y)\} + \frac{\mathrm{Cov}_\theta\{\hat{g}(Y), T_1(Y)\}}{\mathrm{Var}_\theta\{T_1(Y)\}}\left[T_1(y) - E_\theta\{T_1(Y)\}\right]$$

$$= \frac{1}{n\,s^2}\sum_{k=1}^{n} s_k y_k \quad \longrightarrow \text{This is MVUE for ampl. param. } \mu.$$

The var. of this estimator is

$$\mathrm{Var}_\theta\{\tilde{g}(T_1(Y))\} = \sigma^2 / n s^2$$