Lecture 28

Friday March 26

R.B. Thm $\hat{g}(Y)$ unbiased est. of $g(\theta)$; T suff. for θ

Define $\tilde{g}(T(y)) = E_\theta\{\hat{g}(Y) \mid T(Y) = T(y)\}$

Then

* $\tilde{g}(T(Y))$ unbiased est. of $g(\theta)$
* $Var_\theta\{\tilde{g}(T(Y))\} \le Var_\theta\{\hat{g}(Y)\}$ with "=" if and only if

$$P_\theta\{\hat{g}(Y) = \tilde{g}(T(Y))\} = 1.$$

Implications

① With a suff. stat. can "improve" any unbiased est.

② If T is suff for θ and if there is only one funct. of T which is unbiased est. of $g(\theta)$ then it is MVUE.

⋮

Suppose T has prev. uniqueness prop. and say $g^*(T)$ is that est.

Say $\hat{g}(Y)$ is any other unbiased est. of $g(\theta)$.

$$E\{\hat{g}(Y) \mid T\}$$

$\Rightarrow$ RB says this is unbiased

$$\Rightarrow g^*(T) = E\{\hat{g}(Y) \mid T\}. \Longrightarrow g^*(T) \text{ is a MVUE.}$$

Need to search for such statistics T.

This Leads to Discussion of Completeness ...

A family $\{P_\theta : \theta \in \Lambda\}$ is said to be complete if

$$E_\theta\{f(Y)\} = 0 \quad \forall \theta \in \Lambda \Longrightarrow P_\theta\{f(Y) = 0\} = 1$$

Example $\Gamma = \{0, 1, \dots n\}$ $\Lambda = \{\theta : 0 \le \theta \le 1\}$

$$P_\theta(y) = \binom{n}{y} \theta^y (1-\theta)^{n-y} \qquad y = 0, 1, \dots n.$$

is a complete family.

Why? Let $f(\cdot)$ an arb. funct. on Γ.

$$E_\theta\{f(Y)\} = \sum_{y=0}^{n} f(y) \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

$$= (1-\theta)^n \sum_{y=0}^{n} a_y x^y$$

where $a_y = f(y) \dfrac{n!}{(n-y)!\,y!}$

$x = \dfrac{\theta}{1-\theta}$

Completeness ....

$= 0 \quad \forall x > 0 \implies a_y = 0 \quad y = 0, 1, \dots n.$

$\implies f(y) = 0$ " "

Completeness + Sufficiency

Suppose $T(Y)$ is suff. for a complete family $\{P_\theta : \theta \in \Delta\}$ and suppose $E_\theta\{|Y|\} < \infty$ $\forall \theta$

Define

$$f(y) = y - E_\theta\{Y \mid T(y)\}$$

and note that it does not dep. on θ...

Then for each $\theta \in \Delta$

$$E_\theta\{f(Y)\} = E_\theta\left\{Y - E_\theta\{Y \mid T(Y)\}\right\}$$

$$= 0 \quad \forall \theta.$$

$\Longrightarrow$ $Y = E_\theta\{Y \mid T(Y)\}$ with prob. 1

ie Y is a funct of T $\Longrightarrow$ T is one-to-one, T is a trivial suff. stat.

Where is this notion useful?

Suppose T is suff. for θ. Let Q_θ denote the dist of $T(Y)$ when $Y \sim P_\theta$.

If $\{Q_\theta : \theta \in \Lambda\}$ is complete say T is a complete suff. statistic.

Notes

① When this happens, no further data compression can be achieved without losing information.

② Recall: a minimal suff. stat. is one that is a function of every other suff. stat.

[minimal suff. stats. do not always exist]

But if $\{P_\theta\}$ has a min. suff. stat. and if T is complete suff. then it is minimal.

③ Any unbiased estimator that is a funct. of a comp. suff stat has !ness prop. and is MVUE.

Exponential Families

A class of dists. $\{P_\theta : \theta \in \Lambda\}$ is said to be an exp. family if $\exists$ real-valed functs $C, Q_1, Q_2 \ldots Q_m$ $T_1 \ldots T_m$ and h st. P_θ has a density

$$f_\theta(y) = C(\theta) \exp\left\{ \sum_{l=1}^{m} Q_l(\theta) T_l(y) \right\} h(y).$$

$\forall \theta$.

Following:

Gaussians

Poisson

Laplace

Binomial

Geometric

$$P_\theta(y) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

$$= \underbrace{(1-\theta)^n}_{C(\theta)} \exp\{\underbrace{y}_{T_1(y)} \underbrace{\log\left(\frac{\theta}{1-\theta}\right)}_{Q_1(\theta)}\} \underbrace{\binom{n}{y}}_{h(y).}$$

$Y_i \quad 1 \le i \le n \quad \sim N(\xi, \sigma^2) \qquad \theta = (\xi, \sigma^2)$

$= f_Y(y_1 \cdots y_n) =$ usual ~~2π~~

$\Rightarrow$ a 2 param. exp. fam.

$$C(\theta) = \exp\left\{-\frac{n}{\sigma^2}\xi^2\right\} \frac{1}{(\sqrt{2\pi}\,\sigma)^n}$$

$Q_1(\theta) = \xi/\sigma^2 \qquad T_1(y) = \sum_{i=1}^n y_i \qquad h(y) = 1$

$Q_2(\theta) = -1/2\sigma^2 \qquad T_2(y) = \sum_{i=1} y_i^2$

Completeness Thm

$\Gamma = \mathbb{R}^n \quad \Lambda \subseteq \mathbb{R}^m$ and each P_θ has a density

$$f_\theta(y) = c(\theta) \exp\left\{ \sum_{\ell=1}^{m} \theta_\ell T_\ell(y) \right\} h(y)$$

where c, T_ℓ, h are real-valued functs.

Then

$$T(y) = \begin{bmatrix} T_1(y) \\ T_2(y) \\ \vdots \\ T_m(y) \end{bmatrix}$$

is a complete suff. stat. for θ if Λ contains an m-dim. rectangle.

Cramer-Rao Lower Bound:

"the more a pdf depends on the unknown param, the easier it ought to be to estimate it"

$Y = \theta + N$ $\qquad N \sim N(0, \sigma^2)$

$\theta \in \mathbb{R}$ unknown param.

$$f_\theta(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}(y-\theta)^2\right\}.$$

$\Longrightarrow Y = y_1 \Longrightarrow f_\theta(y_1)$

when I think as "function of θ" its called the likelihood funct.