Wednesday March 24 Lecture 27

Suff. Stat. Examples

#1 $Y_k = A + W_k \quad k = 0, 1 \ldots N-1 \quad W_k \sim N(0, \sigma^2)$ iid

$A \in \mathbb{R}$ param.

$$f_{Y;A}(y) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{k=0}^{N-1} (y_k - A)^2 \right\}$$

expand...

$$= \underbrace{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \left(NA^2 - 2A \sum_{k=0}^{N-1} y_k \right) \right\}}_{g(T(y);A)} \underbrace{\exp\left\{ -\frac{1}{2\sigma^2} \sum_{k=0}^{N-1} y_k^2 \right\}}_{h(y)}$$

Then N.P. Fact. Thm $\Rightarrow T(y) = \sum_{k=0}^{N-1} y_k$ is suff for A.

$\Rightarrow$ Actually any one-to-one function of $T(y)$ would do.

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#2 Same model ... $A=0$, σ^2 unknown param.

$$f_{Y;\sigma^2}(y) = \underbrace{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=0}^{N-1} y_k^2\right\}}_{g(T(y);\sigma^2)} \cdot \underbrace{1}_{h(y)}$$

$$\Rightarrow T(y) = \sum_{k=0}^{N-1} y_k^2 \quad \text{is suff. for } \sigma^2 ..$$

Another approach : Go to Definition

$Y, T(Y)$ r.v.s. pdfs from model $f_{Y;\theta}(y)$

$$f_{Y,T;\theta}(y,t) = f_{Y;\theta}(y)\,\delta(T(y)=t)$$

Then define

$$f_{Y|T;\theta}(y|t) = \frac{f_{Y,T;\theta}(y,t)}{f_{T;\theta}(t)} = \frac{f_{Y;\theta}(y)\,\delta(T(y)=t)}{f_{T;\theta}(t)}$$

Suff. if this is indep of θ.

Now for #2 $\quad T(y) = \sum_{k=0}^{N-1} y_k^2 = \sigma^2 \left\{ \begin{array}{l} \text{a chi-sq. r.v.} \\ \text{with } N \text{ d.o.f.} \end{array} \right\}$

If S is χ^2 N d.o.f.

$$f_S(s) = \begin{cases} \frac{1}{2^{N/2}\Gamma(N/2)} e^{-s/2} s^{\frac{N}{2}-1} & s>0 \\ 0 & s<0 \end{cases}$$

$$\frac{T}{\sigma^2} = S \implies f_{T;\sigma^2}(t) = \begin{cases} \frac{1}{\sigma^2}\frac{1}{2^{N/2}\Gamma(N/2)} e^{-t/2\sigma^2}\left(\frac{t}{\sigma^2}\right)^{\frac{N}{2}-1} & t>0 \\ 0 & t<0. \end{cases}$$

$$ds = \frac{dt}{\sigma^2}$$

$$f_{Y|T;\theta}(y|t) = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2} t(y)\right\} \delta(T(y)=t)}{\frac{1}{\sigma^2} \frac{1}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})} e^{-t/2\sigma^2} \left(\frac{t}{\sigma^2}\right)^{\frac{N}{2}-1}}$$

$$= \frac{\Gamma(\frac{N}{2})}{\pi^{N/2}\, t^{\frac{N}{2}-1}} \quad \delta(T(y)=t).$$

This does not dep on σ^2

$\therefore$ T is suff. for σ^2.

#3 Phase of Sinusoid

$$Y_n = A\cos(2\pi f_0 n + \phi) + W_n \qquad n = 0, 1, \dots, N-1$$

$$W_n \sim N(0, \sigma^2) \text{ iid.}$$

A, f_0, σ^2 all known.

Parameter is ϕ.

$$f_{Y;\phi}(y) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{n=0}^{N-1} \left(y_n - A\cos(2\pi f_0 n + \phi)\right)^2 \right\}$$

$$\sum y_n^2 - 2A \overbrace{\left(\sum y_n \cos 2\pi f_0 n\right)}^{T_1(y)} \cos\phi + 2A \overbrace{\left(\sum y_n \sin 2\pi f_0 n\right)}^{T_2(y)} \sin\phi$$
$$+ \sum A^2 \cos^2(2\pi f_0 n + \phi).$$

$$\Rightarrow f_{Y;\phi}(y) = g\left(T_1(y), T_2(y); \phi\right) \cdot h(y)$$

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Example $Y_n = r^n + W_n$ $\quad W_n \sim N(0, \sigma^2)$ iid $\to$ known.

$n = 0, 1, \dots N-1$

Param. r.

$$f_{Y;r}(y) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \underbrace{\sum_{n=0}^{N-1} (y_n - r^n)^2}\right\}$$

$$\sum_{n=0}^{N-1} y_n^2 - 2\sum_{n=0}^{N-1} r^n y_n + \sum_{n=0}^{N-1} r^{2n}.$$

Cannot factor into a funct. of the y_n $n=0, \dots N-1$ alone (at least a funct indep of r).

Using Sufficiency to MVUE

#1 $\quad Y_k = A + W_k \qquad W_k \sim N(0, \sigma^2)$ iid $\quad A \in \mathbb{R}$ param.

↑ known

$$T(y) = \sum_{k=0}^{N-1} y_k \quad \text{is suff. for } A.$$

Two approaches

ⓐ Find any unbiased est. of A say $\tilde{A}(y) = y_0$

$$\Longrightarrow \hat{A}(y) = E\{\tilde{A} \mid T(y)\}$$

ⓑ Just find a funct. $g(\cdot)$ st.

$$\hat{A}(y) = g(T(y))$$

is unbiased.

Approach (a) $\hat{A} = E\{\tilde{A} \mid T\}$ $\tilde{A}$ and T are jointly Gaussian with means

$$E\{\tilde{A}\} = E\{Y_0\} = A$$
$$E\{T\} = NA$$

and some cov. matrix

$$\Sigma = \begin{bmatrix} Var\{\tilde{A}\} & Cov\{\tilde{A}, T\} \\ Cov\{T, \tilde{A}\} & Var\{T\} \end{bmatrix}$$

1st day ...

$$E\{\tilde{A} \mid T = t\} = \underbrace{E\{\tilde{A}\}}_{A} + \frac{\overset{\sigma^2}{Cov\{\tilde{A}, T\}}}{\underset{N\sigma^2}{Var\{T\}}}\left[t - \underbrace{E\{T\}}_{NA}\right]$$

$$Cov\{\tilde{A}, T\} = Cov\{Y_0, \sum_{k=0}^{N-1} Y_k\} = \sum_{k=0}^{N-1} Cov\{Y_0, Y_k\} = \sigma^2$$

$$E\{\tilde{A} \mid T=t\} = A + \frac{\sigma^2}{N\sigma^2}(t - NA)$$

$$= \frac{1}{N} t = \frac{1}{N} \sum_{k=0}^{N-1} y_k$$

Approach is easier.