Monday 3/22 Lecture 26

Rao-Blackwell

$E_\theta\{\hat{g}(Y)\} = g(\theta).$

Suppose $\hat{g}(Y)$ which is an unbiased estimate of $g(\theta)$ ... a real-valued funct. of $\theta \in \Lambda$ (allows θ to be a vector...).

Suppose also that T is suff. for θ.

Define $\tilde{g}(T(y)) \triangleq E_\theta\{\hat{g}(Y) \mid T(Y) = T(y)\}.$

Then:

* $\tilde{g}(T(Y))$ is also an unbiased estimator of $g(\theta)$.
* $\text{Var}_\theta\{\tilde{g}(T(Y))\} \leq \text{Var}_\theta\{\hat{g}(Y)\}$ with equality if and only if

$$P_\theta\{\hat{g}(Y) = \tilde{g}(T(Y))\} = 1.$$

R-B Proof

1) Show $\tilde{g}(T(Y))$ is a valid estimator for $g(\theta)$ ie cannot depend on θ. A consequence of T suff.

$$\tilde{g}(t) = E_\theta\{\hat{g}(Y) \mid T = t\}$$

$$= \int_\Gamma \hat{g}(y)\, f_{Y|T;\theta}(y|t)\, dy$$

$\rightarrow = f_{Y|T}(y|t)$ since T suff for θ.

2) $\tilde{g}(T(Y))$ is an unbiased est. of $g(\theta)$.

$$E_\theta\{\tilde{g}(T)\} = E_\theta\{E_\theta\{\hat{g}(Y)|T\}\} = E_\theta\{\hat{g}(Y)\} = g(\theta)$$

def of $\tilde{g}$

Law of Total Exp: $E[X] = E[E[X|Y]]$

3) $$\mathrm{Var}_\theta\{\tilde{g}(T)\} = E_\theta\{(\tilde{g}(T))^2\} - (g(\theta))^2$$

$$\mathrm{Var}_\theta\{\hat{g}(Y)\} = E_\theta\{(\hat{g}(Y))^2\} - (g(\theta))^2$$

$$\mathrm{Var}_\theta\{\tilde{g}\} \le \mathrm{Var}_\theta\{\hat{g}\} \iff E_\theta\{\tilde{g}^2\} \le E_\theta\{\hat{g}^2\}.$$

Examine this mean sq value.

$$E_\theta\{(\tilde{g}(T))^2\} = E_\theta\{(E_\theta\{\hat{g}(Y)|T\})^2\}$$

Claim $$\left(E_\theta\{\hat{g}(Y)|T\}\right)^2 \le E_\theta\{(\hat{g}(Y))^2|T\}.$$

Jensen's Inequality

X, Z jointly dist. r.v.s. $\phi(\cdot)$ convex on $\mathbb{R}$
and $E|X| < \infty$, $E|\phi(X)| < \infty$

Then

$$\phi(EX) \le E\phi(X)$$

$$\phi(E[X|Z]) \le E[\phi(X)|Z] \quad \text{w.p. } 1$$

Back to R-B Proof Claim: $\phi(x) = x^2$ is convex and strictly so.

We will need: W, V, T are jointly dist.

$W \le V$ wp 1 and Expectations all exist.

$$E[W|T] \le E[V|T] \text{ wp } 1.$$

$$E_\theta\{(\tilde{g}(T))^2\} = E_\theta\left\{\left(E_\theta\{\hat{g}(Y)|T\}\right)^2\right\}$$

$$\leq \underbrace{E_\theta\left\{E_\theta\{(\hat{g}(Y))^2 \,|T\}\right\}}_{E_\theta\{(\hat{g}(Y))^2\}}$$

$\square$ most of R-B.

you think about the equality condition

Implications of R-B.

* With a suff. stat. we can improve the variance of any unbiased estimator of $g(\theta)$ which is not already a funct. of T.

* If T is suff. for θ and if there is only one funct. of T which is an unbiased est. of $g(\theta)$ then that funct. must be MVUE for $g(\theta)$.

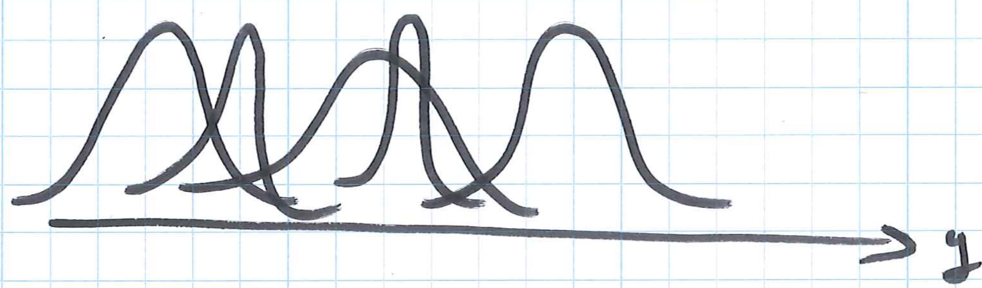
* Very interested in suff. stat. with this property.

Completeness

A family $\{P_\theta : \theta \in \Lambda\}$ of prob. dists is said to be <u>complete</u> if

$$E_\theta\{f(Y)\} = 0 \implies P_\theta\{f(Y) = 0\} = 1.$$
$$\forall \theta \in \Lambda$$

In order to understand consider a simple case ...



$\Gamma = \{ y_1 \; y_2 \; \cdots \; y_n \}$

f is a funct on Γ

$E_\theta\{ f(Y) \} = f^T p_\theta$

$f = [f(y_1) \; f(y_2) \cdots f(y_n)]^T$

$p_\theta = [p_\theta(y_1) \cdots \; p_\theta(y_n)]^T$

$p_\theta(y_i) > 0 \;\; \forall \theta, \forall i$

$$f^T p_\theta = 0 \quad \forall \theta \implies f = 0$$

$$\{p_\theta\} \text{ complete} \iff \text{span}\{p_\theta : \theta \in \Delta\} = \mathbb{R}^n.$$