FORM C
APPROVED FOR USE IN
PURDUE UNIVERSITY

Friday March 12 Lecture 22

No Live Class Monday....

Until now Θ was a scalar rv.

$\Lambda = \mathbb{R}^m$

Cost function is a mapping $\mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^+$

And the separable case .... ie

$$C(a, \theta) = \sum_{i=1}^{m} C_i(a_i, \theta_i)$$

Then if separable the posterior cost

$$E\{C(\hat{\theta}(y), \Theta) \mid Y = y\} = \sum_{i=1}^{m} E\{C_i(\hat{\theta}_i(y), \Theta_i) \mid Y = y\}$$

MMSE $\quad \|a - \theta\|^2 = \sum_{i=1}^{m} (a_i - \theta_i)^2$

MMAE $\quad \sum_{i=1}^{m} |a_i - \theta_i|$

MAP The generalization of scalar is

$$C(a, \theta) = \begin{cases} 1 & \text{if } \max_i |a_i - \theta_i| > \Delta \\ 0 & \text{if } \max_i |a_i - \theta_i| < \Delta \end{cases}$$

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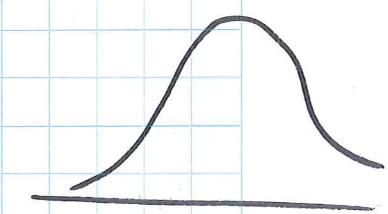
Then the posterior cost is

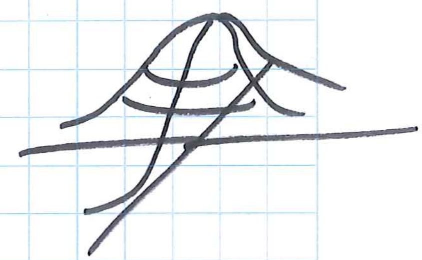
$$E\{C(\hat{\theta}(y), \Theta) \mid Y = y\}$$

$$= 1 - P\{|\hat{\theta}_1(y) - \Theta_1| \le \Delta, |\hat{\theta}_2(y) - \Theta_2| \le \Delta, \ldots, |\hat{\theta}_m(y) - \Theta_m| \le \Delta\}$$

$$= 1 - P\left\{\bigcap_{i=1}^{m} \{|\hat{\theta}_i(y) - \Theta_i| \le \Delta\}\right\}.$$

$\Rightarrow$ get for the estimator the conditional mode.





$\Rightarrow$ Note: In general, this is not the same as the vector whose ith component is cond. mode of Θ_i given $Y = y$.

Further Vector Case Considerations ... Generalize the Squared error

$$\mathcal{A} = \mathbb{R}^m$$

$$C(a, \theta) = (a-\theta)^T A (a-\theta)$$

A ⟶ sym, pos. def.

* allows weighting errors in components
* joint weighting of components.

Then to get Bayes estimate

$$E\{(\hat{\theta}(y) - \Theta)^T A (\hat{\theta}(y) - \Theta) \mid Y=y\}$$

$$= \hat{\theta}(y)^T A \hat{\theta}(y) - 2\hat{\theta}^T(y) A E\{\Theta \mid Y=y\} + E\{\Theta^T A \Theta \mid Y=y\}$$

To mimize: Take gradient wrt $\hat{\theta}(y)$, set equal zero, solve

$$\nabla_{\hat{\theta}(y)} \{\text{above}\} = 2A\hat{\theta}(y) - 2AE\{\Theta \mid Y=y\}$$

$$\Longrightarrow \hat{\theta}_B(y) = E\{\Theta \mid Y=y\}$$

* Cond. mean estimator is <u>still</u> MMSE.
* The min. Bayes however.

$$r(\hat{\Theta}_B) = \text{Trace}\left\{ A\, E\{\mathrm{Cov}(\Theta | Y)\} \right\}.$$

$\boxed{\text{Special Case}}$ $\Gamma = \mathbb{R}^n$ $\Delta = \mathbb{R}^m$ Y, Θ jointly Gaussian

$\mu_Y \quad \mu_\Theta$

$\Sigma_{YY} \quad \Sigma_{\Theta\Theta}$

$\Sigma_{Y\Theta}$

Know that cond. dist. of Θ given $Y=y$ is Gaussian

$$\hat{\mu}_\Theta(y) = \mu_\Theta + \Sigma_{\Theta Y} \Sigma_{YY}^{-1} (y - \mu_Y)$$

Cond. Cov. of Θ given $Y=y$ $= \hat{\Sigma}_\Theta = \Sigma_{\Theta\Theta} - \Sigma_{\Theta Y} \Sigma_{YY}^{-1} \Sigma_{Y\Theta}$

To Do

$Y = H\Theta + N$

$\Theta \sim N(\mu_\Theta, \Sigma_\Theta)$

$N \sim N(0, \Sigma)$

$\Theta \perp\!\!\!\perp N$

H known fixed $n \times m$

Example $\quad Z = B + V$ $\quad$ B is binary, ± 1 with equal prob.
$V \sim N(0, \sigma^2)$ $\quad B \perp V$.

Want 1) MAP est of B.
2) MMSE est of B.
3) which to pick?

We need post. mass function of B given $Z = z$. We use Bayes formula ...

$$w(b|z) = \frac{f(z|b)\,w(b)}{\int f(z|b)\,w(b)\,db}$$

... but of course B is discrete so properly interprete integral.

Z given $B = b$ is $\sim N(b, \sigma^2)$

$$f_{Z|B}(z|b) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(z-b)^2/2\sigma^2}$$

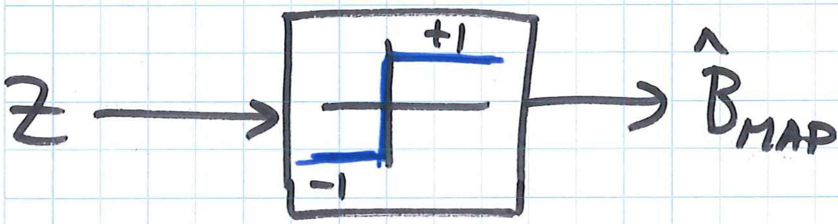
$$\Rightarrow \text{Integral in den} = \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma}\left[e^{-(z-1)^2/2\sigma^2} + e^{-(z+1)^2/2\sigma^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(z^2+1)/2\sigma^2} \cosh\left(z/\sigma^2\right)$$

Now turning to MAP ...

$$\hat{B}_{MAP}(z) = \arg\left\{ \max\left\{ \underbrace{\frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(z+1)^2/2\sigma^2}}_{b=-1},\ \underbrace{\frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(z-1)^2/2\sigma^2}}_{b=+1} \right\}\right\}$$

$$= \operatorname{argmax}\{bz\} = \begin{cases} +1 & \text{if } z > 0 \\ -1 & \text{if } z < 0. \end{cases}$$



If instead we do MMSE

$$E\{B \mid Z = z\} = \tanh\left(z/\sigma^2\right)$$