FRI 3/5 LECT 19

Bayesian Param Est

$Y \in \Gamma \quad \Theta \in \Lambda \quad \hat{\theta} : \Gamma \to \Lambda$ an estimator

$$R_\theta(\hat{\theta}) = E\{C(\hat{\theta}(Y), \theta) \mid \Theta = \theta\}$$ cond. risk

$$r(\hat{\theta}) = E\{C(\hat{\theta}(Y), \Theta)\}$$
$$= E\{E\{C(\hat{\theta}(Y), \Theta) \mid \Theta\}\}$$
$$= E\{\underbrace{E\{C(\hat{\theta}(Y), \Theta) \mid Y\}}_{\text{Posterior Cost given } Y.}\}$$

Last Time $C(\hat{\theta}, \theta) = |\hat{\theta} - \theta|^2 \Longrightarrow$ Bayes estimator turned out to be conditional mean estimator

$$\hat{\theta}(y) = E\{\Theta \mid Y = y\}$$

Example Min. Mean Absolute Error (MMAE)

$$C(a, \theta) = |a - \theta|$$

Goal – Derive the form of the MMAE Estimator.

Fact : X is a r.v. s.t. $P(X \geq 0) = 1$ then

$$E\{X\} = \int_0^\infty P(X > x)\, dx$$

Note $C(\hat{\Theta}(y), \Theta) = |\hat{\Theta}(y) - \Theta| \geq 0.$

$$E\{|\hat{\Theta}(y) - \Theta| \mid Y=y\} = \int_0^\infty P\{|\hat{\Theta}(y) - \Theta| > x \mid Y=y\}\, dx$$

Then break integral up into two parts

$$\text{above} = \int_0^\infty P\{\Theta > x + \hat{\Theta}(y) \mid Y=y\}\, dx + \int_0^\infty P\{\Theta < -x + \hat{\Theta}(y) \mid Y=y\}\, dx$$

C.o.V. $t = x + \hat{\Theta}(y)$ in 1st integ. ; $t = -x + \hat{\Theta}(y)$ in second.

Thus the Bayes Est $\hat{\Theta}_{MABS}(y)$ is any pt. st.

$$P(\Theta < t \mid Y=y) \leq P(\Theta > t \mid Y=y) \quad \text{for } t < \hat{\Theta}_{MABS}(y)$$

and

$$P(\Theta < t \mid Y=y) \geq P(\Theta > t \mid Y=y) \quad \text{for } t > \hat{\Theta}_{MABS}(y)$$

This is called the <u>median</u> of the Posterior Dist of Θ given $Y=y$.

Sometimes called the cond. median. est.

Apply the C.O.V. and manipulate ....

$$E\{|\hat{\Theta}(y)-\Theta|\,|Y=y\} = \int_{\hat{\Theta}(y)}^{\infty} P(\Theta>t\,|Y=y)\,dt + \int_{-\infty}^{\hat{\Theta}(y)} P(\Theta<t\,|Y=y)\,dt$$

$$\frac{\partial}{\partial\hat{\Theta}(y)} E\{|\hat{\Theta}(y)-\Theta|\,/Y=y\}$$

$$= P(\Theta<\hat{\Theta}(y)\,|Y=y) - P(\Theta>\hat{\Theta}(y)\,|Y=y).$$

$\lim_{\hat{\Theta}(y)\to+\infty}$ above $=+1$ $\quad$ $\lim_{\hat{\Theta}(y)\to-\infty}$ above $=-1$ $\quad$ and it is increasing.

$\Longrightarrow$ Conclude : The posterior cost achieves a min. cost over $\hat{\Theta}(y)$ at any point or set of points where deriv. changes sgn.

This means that the Bayes estimate denoted $\hat{\Theta}_{ABS}(y)$ is any point such that

$$P\{\Theta < t \mid Y=y\} \le P\{\Theta > t \mid Y=y\} \qquad t < \hat{\Theta}_{ABS}(y)$$

and

$$P\{\Theta < t \mid Y=y\} \ge P\{\Theta > t \mid Y=y\} \qquad t > \hat{\Theta}_{ABS}(y)$$

A point $\hat{\Theta}_{ABS}(y)$ satisfying the condition above is a median of the conditional distribution of Θ given $Y=y$. So the MMAE estimator is a conditional median estimate.

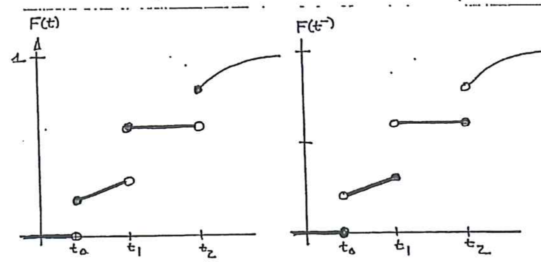
Let $F(t)$ be the conditional distribution of Θ given $Y=y$. Then the derivative of the previous page involves the function

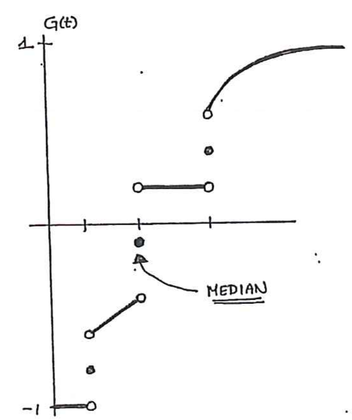
$$F(t^-) - [1 - F(t)] = F(t) + F(t^-) - 1 \triangleq G(t)$$

defined for $t \in \mathbb{R}$. Properties of distribution functions tell us:

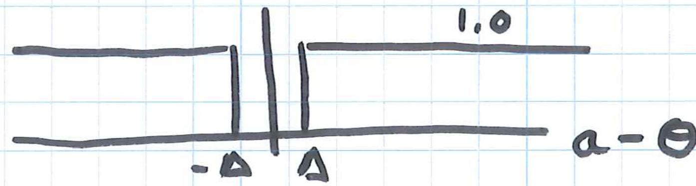
$$\lim_{t\to\infty} G(t) = 1 \qquad \lim_{t\to-\infty} G(t) = -1$$

$G(t)$ is nondecreasing for increasing t.





Example **Maximum A Posterior (MAP)**

$C(a, \theta) =$ 

uniform cost.

$$E\{C(\hat{\theta}(y), \Theta) \mid Y=y\} = P\{|\hat{\theta}(y) - \Theta| > \Delta \mid Y=y\}$$
$$= 1 - P\{|\hat{\theta}(y) - \Theta| \leq \Delta \mid Y=y\}.$$

As a connection back to det. theory imagine situation where Θ takes only $\theta_1, \theta_2 \ldots \theta_M$ values where $|\theta_i - \theta_j| > \Delta$ for $i \neq j$

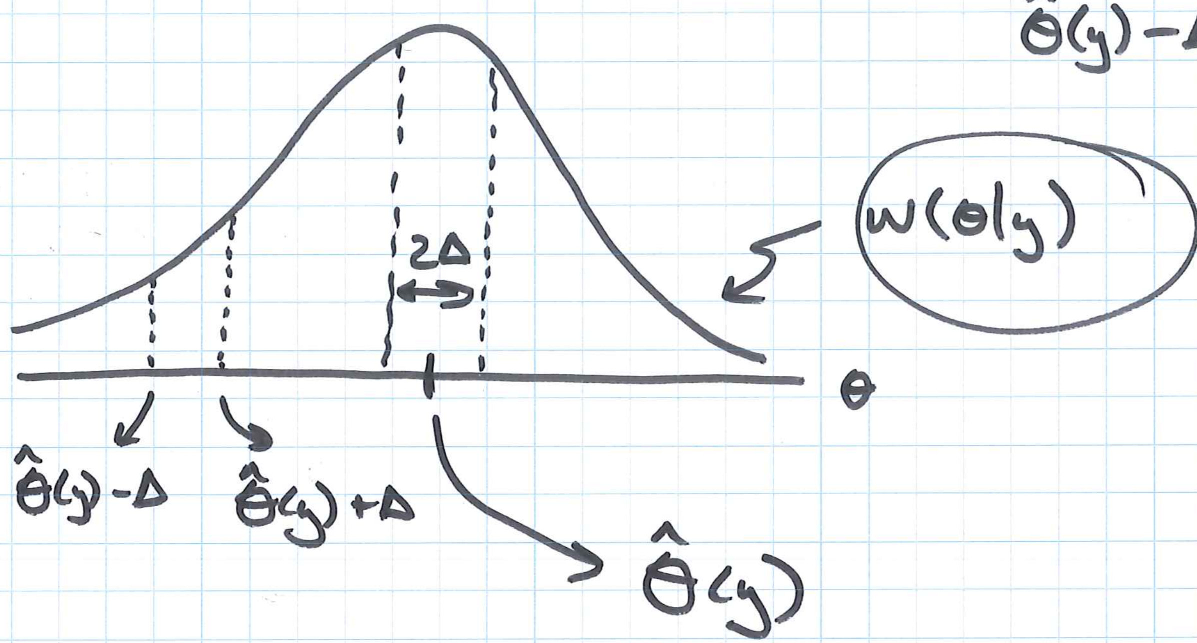
$\Rightarrow$ This becomes the Bayes Detector (M-ary) for unif. cost.

$\Rightarrow$ Estimator maximizes $w(\theta|y)$.

In case of cont. r.v. Θ

$$E\{C(\hat{\Theta}(y), \Theta) \mid Y = y\} = 1 - \int_{\hat{\Theta}(y)-\Delta}^{\hat{\Theta}(y)+\Delta} w(\theta|y)\, d\theta$$

we want to max this integ. over $\hat{\Theta}(y)$



But in the limit as $\Delta \to 0$ The choice is to maximize the posterior pdf of Θ given $Y = y$. (MAP).

Also called cond. mode estimate