LEC 17

Class Ex #3

Location Testing with Uniform Noise

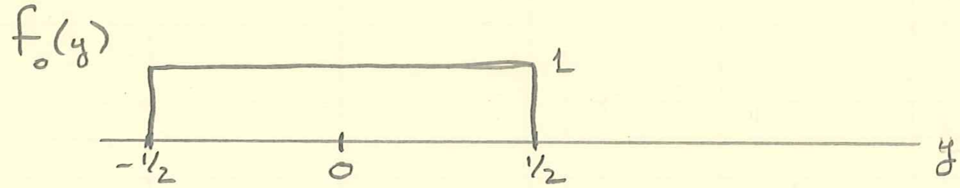
$H_0 : Y = U$

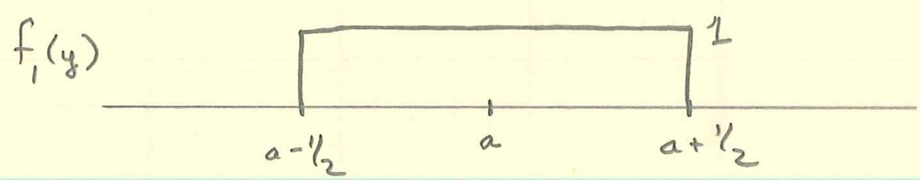
vs.

$H_1 : Y = a + U$

a = a deterministic parameter with $0 < a < 1$

$U \sim$ unif. on $[-1/2, 1/2]$.





For a fixed value of "a" the space in which $Y=y$ can fall is the interval

$$\left[-1/2, a+1/2\right]$$

and $L(y) = f_1(y)/f_0(y)$ can only take 3 values:

$$-\tfrac{1}{2} \le y < a - \tfrac{1}{2} \Rightarrow f_1(y) = 0,\ f_0(y) = 1 \Rightarrow L(y) = 0$$

$$a - \tfrac{1}{2} \le y < \tfrac{1}{2} \Rightarrow f_1(y) = f_0(y) = 1 \Rightarrow L(y) = 1$$

$$\tfrac{1}{2} \le y \le a + \tfrac{1}{2} \Rightarrow f_1(y) = 1,\ f_0(y) = 0 \Rightarrow L(y) = +\infty$$

$\therefore$ L is a discrete r.v. It's pmf under either hypothesis can be summarized by:

	$P(L=0)$	$P(L=1)$	$P(L=+\infty)$
Under H_0	a	$1-a$	0
Under H_1	0	$1-a$	a

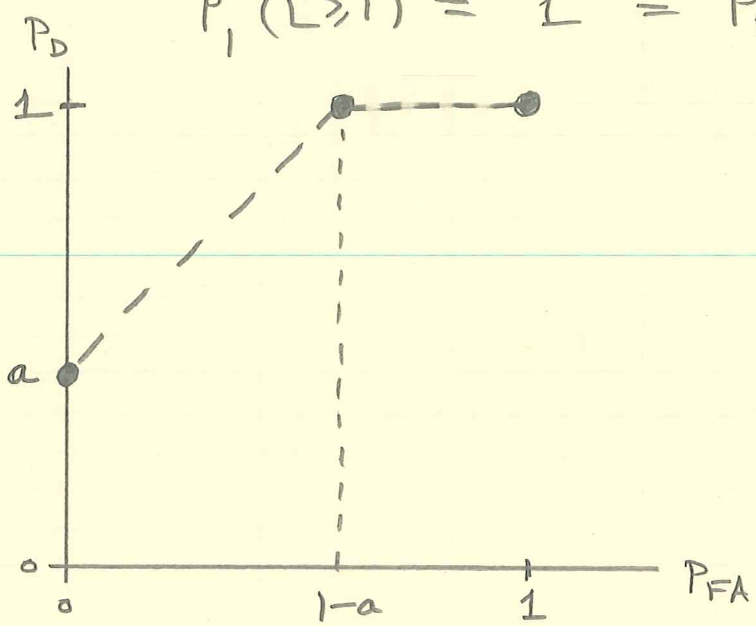
Without randomizing, what thresholds could we pick for a test of form

$$L \underset{<}{\geq} \tau \quad ?$$

$$\tau = 0 \implies \underset{P_{FA}}{P_0(L \geq 0)} = \underset{P_D}{P_1(L \geq 0)} = 1$$

$$\tau = 1 \implies P_0(L \geq 1) = 1-a = P_{FA}$$

$$P_1(L \geq 1) = 1 = P_D$$



If we picked a threshold larger than 1, say $\tau = 2$ ... then

$$P_0(L \geq 2) = 0 = P_{FA}$$

$$P_1(L \geq 2) = a = P_D$$

Therefore with a non-randomized test we can only achieve the three points

$$(P_{FA}, P_D) = (0, a),\ (1-a, 1),\ (1, 1)$$

on the ROC.

The three nonrandomized decision rules are

$$\delta_{\tau=0}(y) = \begin{cases} 1 & \geq \\ & L(y) \quad 0 \\ 0 & < \end{cases} \qquad P_{FA}(\delta_{\tau=0}) = 1, \quad P_D(\delta_{\tau=0}) = 1$$

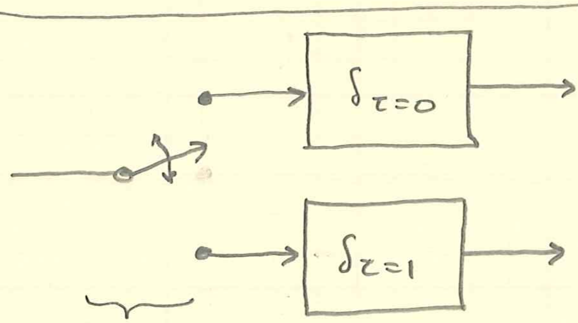
$$\delta_{\tau=1}(y) = \begin{cases} 1 & \geq \\ & L(y) \quad 1 \\ 0 & < \end{cases} \qquad P_{FA}(\delta_{\tau=1}) = 1-a, \quad P_D(\delta_{\tau=1}) = 1$$

$$\delta_{\tau=2}(y) = \begin{cases} 1 & \geq \\ & L(y) \quad 2 \\ 0 & < \end{cases} \qquad P_{FA}(\delta_{\tau=2}) = 0, \quad P_D(\delta_{\tau=2}) = a$$

Type 1

There are two types of randomized dr we might consider.

Select Between $\delta_{\tau=0}$ and $\delta_{\tau=1}$



pick top branch with prob γ

$$\begin{aligned} P_{FA} &= P_{FA}(\delta_{\tau=0})\,\gamma + P_{FA}(\delta_{\tau=1})(1-\gamma) \\ &= 1\cdot\gamma + (1-a)(1-\gamma) \\ &= \cancel{\gamma} + 1 - a - \cancel{\gamma} + a\gamma \\ &= 1 - a + a\gamma \longrightarrow \text{This can be chosen between } 1-a \text{ and } 1 \text{ by varying } \gamma. \end{aligned}$$

Therefore if $1-a < \alpha \leq 1$ then can solve

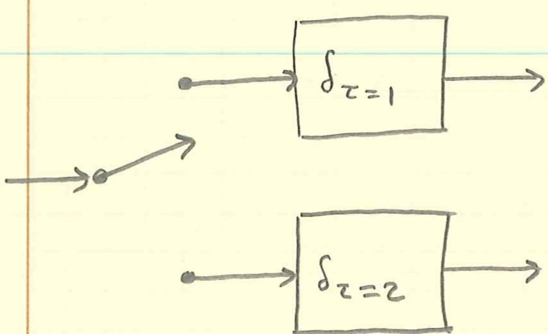
$$1 - a + a\gamma = \alpha$$

$$a\gamma = \alpha - 1 + a$$

$$\gamma = \frac{\alpha - (1-a)}{a}$$

This would hit the desired P_{FA}. But note that all of these would have $P_D = 1$. Therefore, there would be no reason not to pick $\gamma = 0$ and the non-randomized rule $\delta_{\tau=1}$.

Select Between $\delta_{\tau=1}$ and $\delta_{\tau=2}$



pick top branch with prob. γ.

$$\begin{aligned} P_{FA} &= P_{FA}(\delta_{\tau=1})\gamma + P_{FA}(\delta_{\tau=2})(1-\gamma) \\ &= (1-a)\gamma + 0\cdot(1-\gamma) \\ &= (1-a)\gamma \qquad \text{where } \gamma \text{ varies from 0 to 1.} \end{aligned}$$

$\Longrightarrow$ Can hit any P_{FA} between 0 and $1-a$. For such an α:

$$\gamma = \frac{\alpha}{1-a}$$

Then

$$P_D = \underbrace{P_D(\delta_{\tau=1})}_{1}\left(\frac{\alpha}{1-a}\right) + \underbrace{P_D(\delta_{\tau=2})}_{a}\left(\frac{1-a-\alpha}{1-a}\right)$$

$$P_D = \frac{\alpha}{1-a} + \frac{a(1-a-\alpha)}{1-a} = \frac{\alpha + a(1-a-\alpha)}{1-a}$$

$$= \frac{\alpha - \alpha a + a(1-a)}{1-a} = \frac{\alpha(1-a) + a(1-a)}{1-a}$$

$$= \alpha + a$$

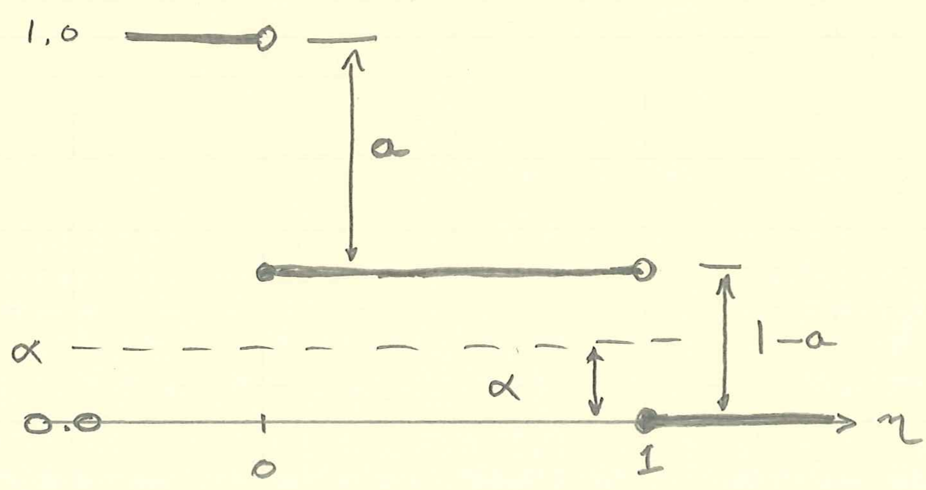
$\therefore$ The ROC points we can hit are

$$(\alpha, \alpha + a) \qquad 0 \le \alpha \le 1-a$$

Type 2 Randomized d.r. Picks one of the form given in the proof of the N.P. Lemma. There we started with the complementary cdf of the likelihood under H_0 ie

$$g(\eta) = P_0(L(Y) > \eta)$$

Of course, under H_0 $L(Y)$ only takes the values 0 and 1.



If $0 < \alpha < 1-a$, Then we are in Case 2 of the existence proof of NP lemma and the NP test would be

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if} \quad > \\ \frac{\alpha}{1-a} & L(y) = 1 \\ 0 & \quad < \end{cases}$$

This is of course the same as the prev. dr.

One more approach.

Easy to see that $L(y)$ is a monotone non-decreasing function of y. Therefore, we ought to be able to base a test on y.

Consider a test of this form:

$$y \overset{\text{say } H_1}{\underset{\text{say } H_0}{\gtrless}} \tau$$

Since Y is continuous under H_0 (H_1 also) there is no need to randomize. To set the false alarm at α st $0 < \alpha < 1$ solve

$$P_0(Y \geq \tau) = \frac{1}{2} - \tau \quad \text{for} \quad -\frac{1}{2} < \tau < \frac{1}{2}$$

$$\overset{\text{set}}{=} \alpha$$

$$\Longrightarrow \frac{1}{2} - \alpha = \tau$$

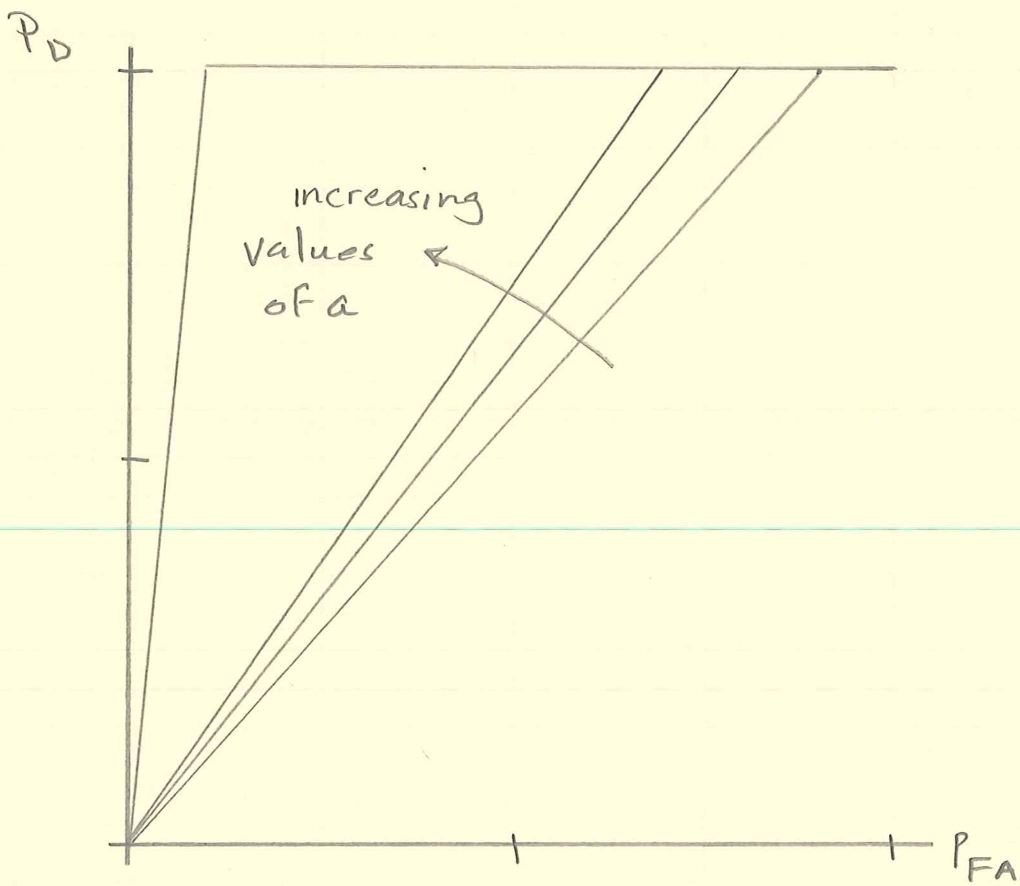
Then the prob. of detection can be computed from

$$P_D = P_1(Y \geq \tau) = a + \frac{1}{2} - \tau \quad \text{for } a - \frac{1}{2} < \tau < a + \frac{1}{2}$$

$$= a + \frac{1}{2} - \left(\frac{1}{2} - \alpha\right) = a + \alpha$$

The ROC would be the parametric curve

$$\left(P_F, P_D\right) = \left(\frac{1}{2} - \tau, \; a + \frac{1}{2} - \tau\right)$$



Class Example # 2

Possible UMP Test

$H_0 : Y_k \sim N(0,1)$ iid $\sigma^2 > 1.$

vs

$H_1 : Y_k \sim N(0,\sigma^2)$

$$f_{0,k}(y_k) = \frac{1}{\sqrt{2\pi}} e^{-y_k^2/2} \qquad f_{\sigma^2,k}(y_k) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y_k^2/2\sigma^2}$$

$$\Rightarrow L_k(y_k) = \frac{\exp\left\{-\frac{y_k^2}{2}\left(\frac{1}{\sigma^2}-1\right)\right\}}{\sigma}$$

$$\Rightarrow \log L_k(y_k) = -\frac{y_k^2}{2}\left(\frac{1-\sigma^2}{\sigma^2}\right) - \log\sigma$$

$$= y_k^2\left(\frac{\sigma^2-1}{2\sigma^2}\right) - \log\sigma$$

$$\Rightarrow \log L(y_1 \cdots y_n) = \left(\frac{\sigma^2-1}{2\sigma^2}\right)\sum_{k=1}^{n} y_k^2 - n\log\sigma$$

Therefore, so long as $\sigma^2 > 1$ (one sided), then

$$\log L(y_1 \cdots y_n) \geq \eta \iff \sum_{k=1}^{n} y_k^2 \geq \eta'$$

and setting the threshold for any α-level test is no different for any $\sigma^2 > 1$. Hence that test must be U.M.P.