Example

LEC 15 Wed. Feb 24

Outline NP Detection Problem for:

$$H_0 : \quad f_0(x) = \begin{cases} A(a^2 - x^2) & |x| < a \\ 0 & |x| \geq a \end{cases}$$

vs.

$$H_1 : \quad f_1(x) = \begin{cases} B(b^2 - x^2) & |x| < b \\ 0 & |x| \geq b \end{cases}$$

where $b > a$.

Find A, B $\Rightarrow$ Use fact $\int_{-a}^{a} f_0(x)\,dx = 1 \rightarrow$ solve for A.

Ditto B

$$A = \frac{3}{4a^3} \qquad B = \frac{3}{4b^3}$$

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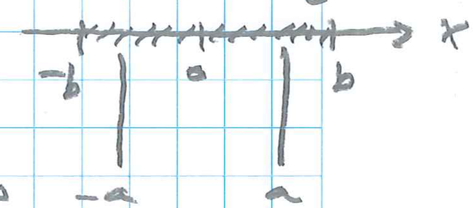
NP Test is a L.R.T. So find $L(x) = \dfrac{f_1(x)}{f_0(x)}$

$f_1(x)$ → nonzero on

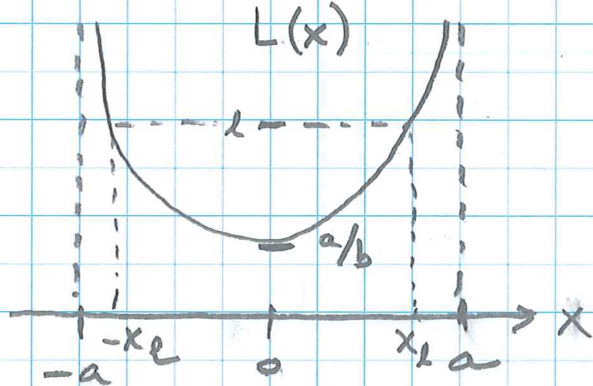
$f_0(x)$ → nonzero on



$$L(x) = \frac{f_1(x)}{f_0(x)} = \begin{cases} \dfrac{a^3}{b^3}\,\dfrac{b^2 - x^2}{a^2 - x^2} & |x| < a \\ +\infty & a \le |x| < b \end{cases}$$

For setting η need comp. cdf of L under H_0. (L finite wp1 under H_0).

First get cdf $F_L(\ell) = P_0(L \le \ell)$



$P_0(L \le \ell) = 0$ if $\ell \le a/b$.

To find $P_0(L \le \ell)$ for $\ell \ge a/b$ need to solve

$L(x) = \ell \rightarrow x_\ell$

$P_0(-x_\ell < X < x_\ell) \longrightarrow$ Can do, maybe should not ...

A better idea ...

Observe that $L(x)$ is montone in $|x|$

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Therefore

$$L(x) \underset{<}{\overset{>}{=}} \tau \quad \begin{matrix}\text{say } H_1 \\ \\ \text{say } H_0\end{matrix} \iff |x| \underset{<}{\overset{>}{=}} \eta \quad \begin{matrix}\text{say } H_1 \\ \\ \text{say } H_0.\end{matrix}$$

So since L or $|X|$ are continuous $\overset{\text{under } H_0}{\vee}$ rvs there is no need to randomize.

$$P_0(|x| > \eta) \overset{\text{Sel.}}{=} \alpha$$

$$1 - \int_{-\eta}^{\eta} f_0(x)\,dx = \alpha = 2\int_{\eta}^{a} \frac{3}{4a^3}(a^2 - x^2)\,dx = 1 - \frac{3}{2a}\eta + \frac{1}{2a^3}\eta^3 \triangleq G(\eta)$$

What I'd do next is find P_D say in case $b = 2a$.

$$P_D = P_1(|X| > \eta) = 1 - \frac{3}{2a}\left(\frac{\eta}{2}\right) + \frac{1}{2a^3}\left(\frac{\eta}{2}\right)^3 = G(\eta/2).$$

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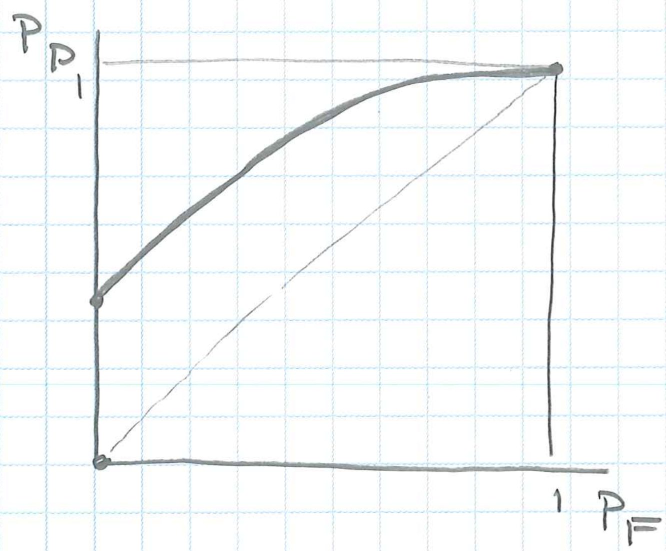
The ROC would be

$$(P_D, P_F) = \left(G(\eta/2), G(\eta)\right)$$ as η varies from 0 to a



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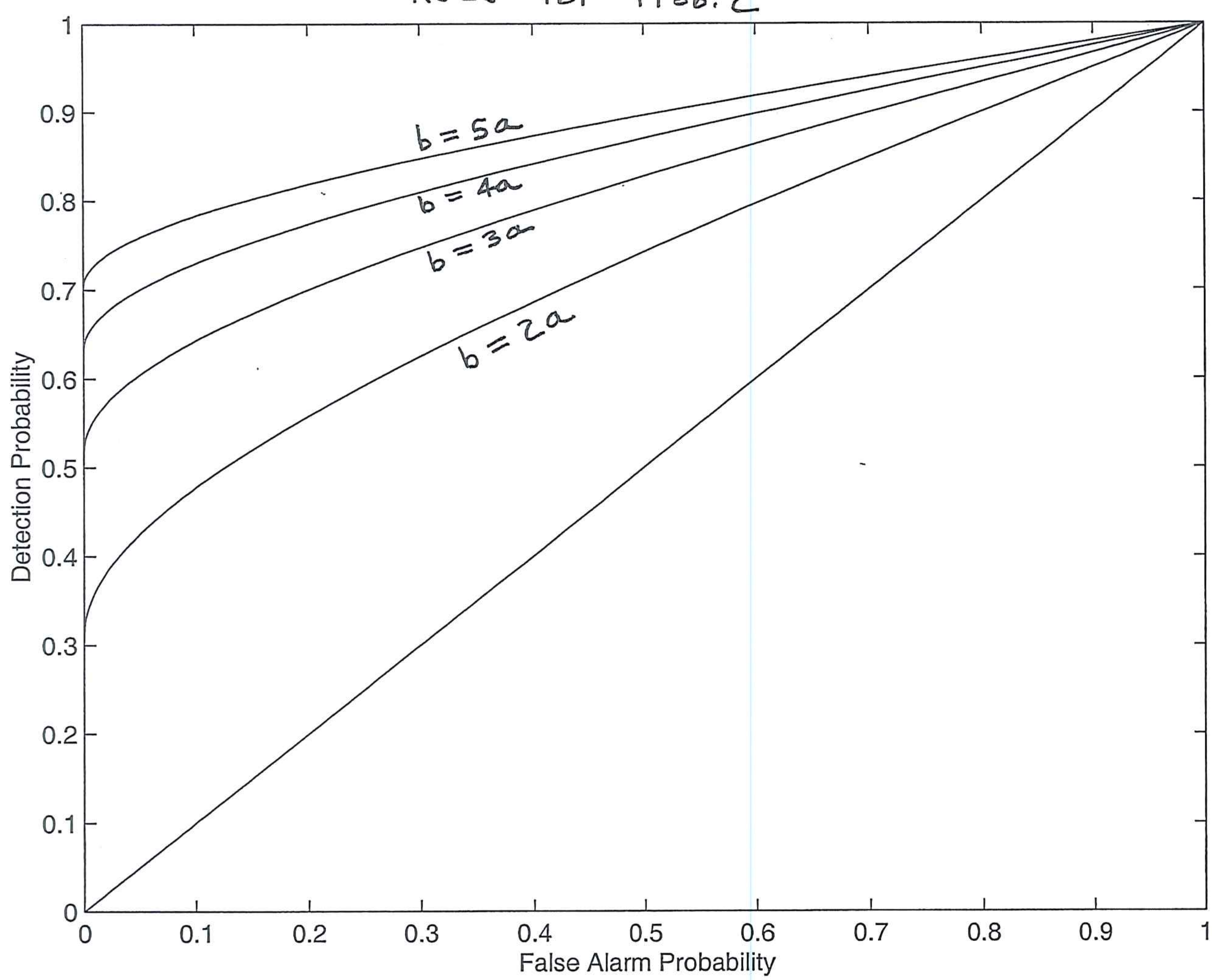
ROCs for Prob. 2



(This part not required for the test).

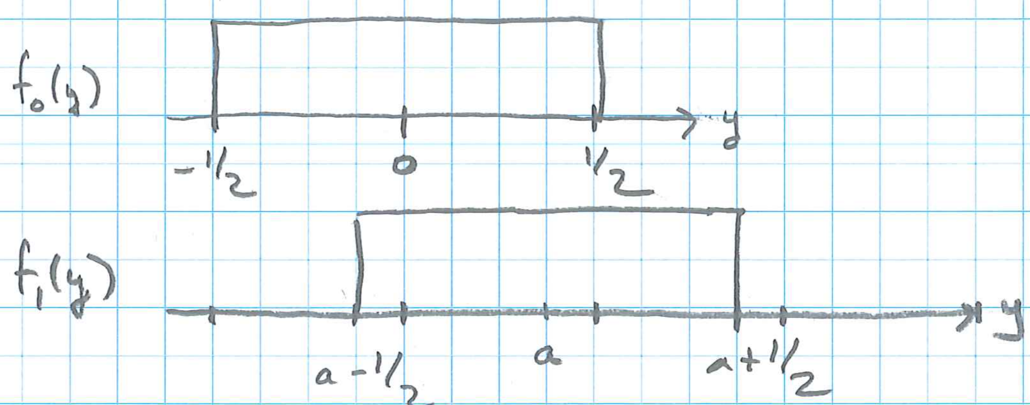
Example:

$H_0 : Y = U$ $\quad U \sim$ unif on $[-1/2, 1/2]$

vs

$H_1 : Y = a + U$ $\quad a$ determ. # $0 < a < 1$

Find NP test for $P_F \le \alpha$



$$L(y) = \begin{cases} +\infty & \frac{1}{2} < y < a + \frac{1}{2} \\ 1 & a - \frac{1}{2} < y < \frac{1}{2} \\ 0 & -\frac{1}{2} < y < a - 1/2 \end{cases}$$

So $L(y)$ takes only 3 values; it is a discrete r.v.

Under H_0

$P_0(L = +\infty) = 0$

$P_0(L = 1) = 1 - a$

$P_0(L = 0) = a$

Under H_1 ...

Think on this ...

Another Example:

$H_0: Y_k \sim N(0,1)$ iid under either. $\sigma^2 > 1$

vs.

$H_1: Y_k \sim N(0,\sigma^2)$

Taking LRT and simplifying (note Gaussian) ...

$$\Gamma_{\sigma^2} = \left\{ y \in \mathbb{R}^n : \sum_{k=1}^{n} y_k^2 \geq \left(\frac{2\sigma^2}{\sigma^2-1}\right)\left(\log\tau + n\log\sigma\right) \right\}$$

$L(y) \geq \tau$ Is there going to be a UMP test?

$$\alpha \overset{\text{set}}{=} P_0\left(\Gamma_{\sigma^2}\right)$$ what happens?.

Next time
... more probs.