Previously on ECE 645 ...

Bayesian Composite Binary Test where Costs are Constant over H_0 and H_1:

- Random parameter $\Theta = \theta \in \Lambda = \Lambda_0 \cup \Lambda_1$; $\Lambda_0 \cap \Lambda_1 = \phi$
- $H_0 : \Theta \in \Lambda_0$ vs. $H_1 : \Theta \in \Lambda_1$
- When $\Theta = \theta$ is drawn (i.e., H_0 vs. H_1 determined) an observation is made according to

  $Y \sim P_\theta$ $\quad$ $Y = y \in \Gamma$

  P_θ is conditional distribution of Y given $\Theta = \theta$

- Cost function : $C(i,\theta)$ $\quad i = 0,1 \quad \theta \in \Lambda$

  Assume $C(i,\theta) = c_{ij}$ if $\theta \in \Lambda_j$

- Find $\delta(y)$ a 0,1 function of $y \in \Gamma$ st. $\overset{r(\delta)}{\cancel{\delta(y)}} = E\{C(\delta(Y),\Theta)\}$ is minimized.

Equivalent Formulations of Bayes Rule

$$\delta_B(y) = \begin{cases} 1 & \text{if } E\{C(1,\Theta) \mid Y=y\} < E\{C(0,\Theta) \mid Y=y\} \\ 0 \text{ or } 1 & \text{if } E\{C(1,\Theta) \mid Y=y\} = E\{C(0,\Theta) \mid Y=y\} \\ 0 & \text{if } E\{C(1,\Theta) \mid Y=y\} > E\{C(0,\Theta) \mid Y=y\} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \dfrac{P(\Theta \in \Lambda_1 \mid Y=y)}{P(\Theta \in \Lambda_0 \mid Y=y)} > \dfrac{c_{10} - c_{00}}{c_{01} - c_{11}} \\ 0 \text{ or } 1 & \text{if } \dfrac{P(\Theta \in \Lambda_1 \mid Y=y)}{P(\Theta \in \Lambda_0 \mid Y=y)} = \dfrac{c_{10} - c_{00}}{c_{01} - c_{11}} \\ 0 & \text{if } \dfrac{P(\Theta \in \Lambda_1 \mid Y=y)}{P(\Theta \in \Lambda_0 \mid Y=y)} < \dfrac{c_{10} - c_{00}}{c_{01} - c_{11}} \end{cases}$$

Assume $c_{00} < c_{10}$, $c_{11} < c_{01}$

$$= \begin{cases} 1 & \text{if } \underbrace{\dfrac{f_Y(y \mid \Theta \in \Lambda_1)}{f_Y(y \mid \Theta \in \Lambda_0)}}_{\text{define as } L(y)} > \dfrac{\pi_0 (c_{10} - c_{00})}{\pi_1 (c_{01} - c_{11})} \\ 0 \text{ or } 1 & \text{if } \dfrac{f_Y(y \mid \Theta \in \Lambda_1)}{f_Y(y \mid \Theta \in \Lambda_0)} = \dfrac{\pi_0 (c_{10} - c_{00})}{\pi_1 (c_{01} - c_{11})} \\ 0 & \text{if } \dfrac{f_Y(y \mid \Theta \in \Lambda_1)}{f_Y(y \mid \Theta \in \Lambda_0)} < \dfrac{\pi_0 (c_{10} - c_{00})}{\pi_1 (c_{01} - c_{11})} \end{cases}$$

$\pi_0 = P(\Theta \in \Lambda_0)$
$\pi_1 = P(\Theta \in \Lambda_1)$

<u>Note:</u> To go from δ_B version #2 to δ_B version #3 use Bayes Formula

$$P(\Theta \in \Lambda_1 \mid Y = y) = \frac{f_Y(y \mid \Theta \in \Lambda_1)\, P(\Theta \in \Lambda_1)}{f_Y(y \mid \Theta \in \Lambda_0)\, P(\Theta \in \Lambda_0) + f_Y(y \mid \Theta \in \Lambda_1)\, P(\Theta \in \Lambda_1)}$$

⋮

Similarly for

$$P(\Theta \in \Lambda_0 \mid Y = y) = \cdots$$

600: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$\cancel{P(\Theta \in \Lambda_1)}$

Re: Testing on Radius of Point in $\mathbb{R}^2$

$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathbb{R}^2$

$H_0 : Y_1 = \varepsilon_1, \; Y_2 = \varepsilon_2$

vs.

$H_1 : Y_1 = A\cos\Psi + \varepsilon_1, \; Y_2 = A\sin\Psi + \varepsilon_2$

$\varepsilon_1, \varepsilon_2 \sim N(0, \sigma^2)$

$A > 0$

Ψ unif. $[0, 2\pi)$

$\varepsilon_1 \perp\!\!\!\perp \varepsilon_2 \perp\!\!\!\perp \Psi \perp\!\!\!\perp$ hypoth.

Setting up the Bayesian Model more carefully:

$\Theta = \begin{pmatrix} A \\ \Psi \end{pmatrix}$ where this is a mixed discrete - continuous random vector made up of

$$A \perp\!\!\!\perp \Psi$$

A: discrete rv. taking values $0, a$ with

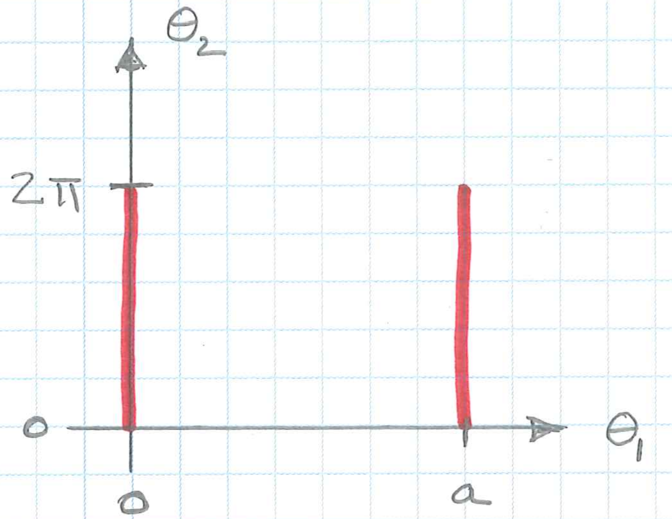
$P(A = a) = p$

$P(A = 0) = q$ $\qquad p + q = 1$

Ψ: cont. rv. unif on $[0, 2\pi)$ i.e.

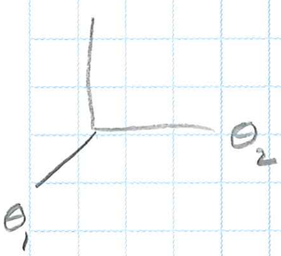
$$f_\Psi(\varphi) = \begin{cases} \frac{1}{2\pi} & 0 \le \varphi < 2\pi \\ 0 & \text{else} \end{cases}$$

Then the r.vec. Θ takes its values in the following subset of $\mathbb{R}^2$:



There is no continuous pdf on $\mathbb{R}^2$ that can model the r.vec. Θ.

However, allowing delta functions we can model this mixed situation as ...



$$f_{\Theta}(\theta_1, \theta_2) = \begin{cases} q \frac{1}{2\pi} \delta(\theta_1) + p \frac{1}{2\pi} \delta(\theta_1 - a) & 0 \le \theta_2 < 2\pi \\ 0 & \text{else} \end{cases}$$

Then

$H_0: A = 0$

vs.

$H_1: A = a$

with $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = A \begin{pmatrix} \cos \Psi \\ \sin \Psi \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$

To compute $L(y)$ need $f(y \mid \Theta \in \Lambda_j)$ $\quad y \in \mathbb{R}^2 \quad j=0,1$

"Lambda"

What is $f(y \mid \Theta \in \Lambda_j)$ $\qquad \dfrac{P(Y=y, \Theta \in \Lambda_j)}{P(\Theta \in \Lambda_j)}$??

$F_{Y \mid \{\Theta \in \Lambda_j\}}(y_1, y_2)$ $\qquad P\cancel{(y+\Delta y \le Y \le y \mid \Lambda_j)}$

$$= P(Y_1 \le y_1, Y_2 \le y_2 \mid \Theta \in \Lambda_j)$$

$$= \frac{P(Y_1 \le y_1, Y_2 \le y_2, \Theta \in \Lambda_j)}{P(\Theta \in \Lambda_j)}$$

$$\int_{-\infty}^{y_2} \int_{-\infty}^{y_1} \int_{\Lambda_j} \underbrace{f_{Y_1 Y_2 \Theta_1 \Theta_2}(y_1\, y_2\, \theta_1\, \theta_2)}_{f_{Y_1 Y_2 \mid \Theta_1 \Theta_2}(y_1\, y_2 \mid \theta_1\, \theta_2)\, f_{\Theta}(\theta_1\, \theta_2)} \, dy_1\, dy_2\, d\theta_1\, d\theta_2$$

$$f_{\Theta}(\theta_1\, \theta_2) \to q \frac{1}{2\pi}\delta(\theta_1) + p\frac{1}{2\pi}\delta(\theta_1 - a)$$

Uniformly Most Powerful Tests

θ is modeled as deterministic, but unknown.

$\Lambda = \Lambda_0 \cup \Lambda_1 \quad \Lambda_0 \cap \Lambda_1 = \phi$

θ (Λ_0): H_0 θ (Λ_1): H_1.

$\delta(y)$ the dec. rule is a number between 0 and 1

$$\delta : \Gamma \rightarrow [0, 1]$$

False alarm: $P_F(\delta; \theta) = E_\theta\{\delta(Y)\}$ for $\theta \in \Lambda_0$

Det. Prob. $P_D(\delta; \theta) = E_\theta\{\delta(Y)\}$ for $\theta \in \Lambda_1$

A <u>UMP</u> test of level α is one that maximizes

$$P_D(\delta; \theta)$$

for every $\theta \in \Lambda_1$ subject to

$$P_F(\delta; \theta) \le \alpha \quad \text{for all } \theta \in \Lambda_0.$$

But UMP Test Don't Always Exist

$\Lambda = \Lambda_0 \cup \Lambda_1$ Say H_0 is simple ie $\Lambda_0 = \{\theta_0\}$

Say P_θ has a density $f_\theta(\cdot)$ for each $\theta \in \Lambda$.

$$H_0 : Y \sim P_{\theta_0}$$

vs

$$H_\theta : Y \sim P_\theta$$

for some fixed $\theta \in \Lambda_1$.

↳ a simple H.T.

Know from NPL that $\exists$ a most powerful α-level test for θ vs. θ_0

$$\Gamma_\theta = \{ y \in \Gamma : f_\theta(y) > \tau f_{\theta_0}(y) \}$$

where τ and a poss. randomization are chosen to give size α.

Say have two distinct params $\theta', \theta'' \in \Lambda_1$

$$\Gamma_{\theta'} \longleftarrow \theta' \qquad \theta'' \longrightarrow \Gamma_{\theta''}$$

Test with $\Gamma_{\theta'}$ will have $P_D(\delta'; \theta'') < P_D(\delta''; \theta'')$ unless $\Gamma_{\theta'}$ and $\Gamma_{\theta''}$ are <u>the same</u>.

Fact A UMP test for H_{θ_0} vs $H_\theta : Y \sim P_\theta \quad \theta \in \Lambda_1$ exists $\Longleftrightarrow$ the critical region Γ_θ is same for all θ.

$\Longrightarrow$