Feb 10, 2021 – LEC 10

Composite Hyp. Testing (Both Bayesian & Non-Bayesian Approaches).

$\{P_\theta : \theta \in \Lambda\}$ a family of prob. dists. on Γ

Say $\Lambda = \Lambda_0 \cup \Lambda_1$ (a disjoint union)

$$H_0 : Y \sim P_\theta \quad \theta \in \Lambda_0$$

vs

$$H_1 : Y \sim P_\theta \quad \theta \in \Lambda_1$$

Start with Bayesian

- Assume param θ is a realization of a rv Θ taking values in Λ.
- Binary dec. about $\Theta = \theta \in \Lambda_0$ or Λ_1.
- Cost Function: $C(i, \theta)$ $\quad i = 0, 1$, $\theta \in \Lambda$
- Conditional risk for a decision rule δ: $R_\theta(\delta) = E\{C(\delta(Y), \Theta) \mid \Theta = \theta\}$

Average or Bayes Risk : $r(\delta) = E\{R_{\Theta}(\delta)\}$

$$= E\{C(\delta(Y), \Theta)\}$$

A **Bayes rule** is one that minimizes $r(\delta)$.

$$r(\delta) = E\{C(\delta(Y), \Theta)\}$$
$$= E\{E\{C(\delta(Y), \Theta) \mid \Theta\}\}$$
$$= E\{E\{C(\delta(Y), \Theta) \mid Y\}\}$$

$\rightarrow$ look at this version since goal is to pick $\delta(y)$ for each $y \in \Gamma$

We can do this by minimizing posterior cost

$$E\{C(\delta(y), \Theta) \mid Y = y\}$$

$\swarrow$

$\delta(y) = 0, 1$

So we must compare

$$E\{C(1,\Theta)\,|\,Y=y\} \;\overset{\text{say } \delta(y)=1}{\underset{\text{say } \delta(y)=0}{\gtreqless}}\; E\{C(0,\Theta)\,|\,Y=y\}.$$

The Bayes δ_B. rule is the above.

Now if the cost function is constant over the sets Λ_0 and Λ_1 ...

$$C(i,\theta) = c_{i,j} \quad \text{if } \theta \in \Lambda_j \quad j=0,1.$$

Suppose this holds.

$$E\{C(1,\Theta)\,|\,Y=y\} = \int_{\Lambda=\Lambda_0\cup\Lambda_1} c(1,\theta)\, f_{\Theta|Y}(\theta|y)\, d\theta$$

$$= \int_{\Lambda_0} \overset{c_{1,0}}{c(1,\theta)}\, f_{\Theta|Y}(\theta|y)\, d\theta + \int_{\Lambda_1} \overbrace{c(1,\theta)}^{c_{1,1}}\, f_{\Theta|Y}(\theta|y)\, d\theta$$

$$= c_{10}\, P(\Theta\in\Lambda_0\,|\,Y=y) + c_{11}\, P(\Theta\in\Lambda_1\,|\,Y=y).$$

Similarly for $E\{C(0,\Theta)\,|\,Y=y\}$

Then

$$\delta_B(y) = \begin{cases} 1 & > \\ x & \text{if } \dfrac{P(\Theta \in \Lambda_1 \mid Y=y)}{P(\Theta \in \Lambda_0 \mid Y=y)} = \dfrac{c_{10}-c_{00}}{c_{01}-c_{11}} \\ 0 & < \end{cases}$$

Lastly if assume Y has conditional pdfs $f_Y(y \mid \Theta \in \Lambda_0)$
$f_Y(y \mid \Theta \in \Lambda_1)$.

$$\Rightarrow \delta_B(y) = \begin{cases} 1 & > \\ 0 \text{ or } 1 & \text{if } L(y) = \dfrac{\pi_0(c_{10}-c_{00})}{\pi_1(c_{01}-c_{11})} \\ 0 & < \end{cases}$$

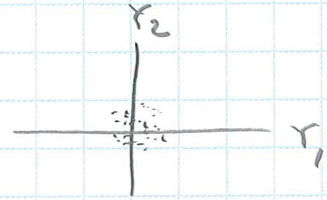
$$\dfrac{f_Y(y \mid \Theta \in \Lambda_1)}{f_Y(y \mid \Theta \in \Lambda_0)}.$$

$\pi_0 = P(\Theta \in \Lambda_0)$
$\pi_1 = P(\Theta \in \Lambda_1)$.

Example (Bayesian Case)

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \qquad \varepsilon_1, \varepsilon_2 \text{ are iid } N(0, \sigma^2)$$

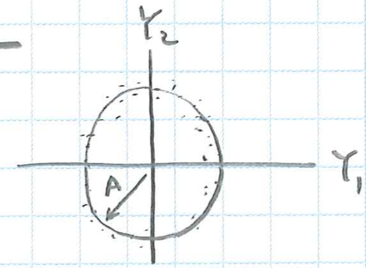
$$H_0 : \quad \begin{aligned} Y_1 &= \varepsilon_1 \\ Y_2 &= \varepsilon_2 \end{aligned}$$



vs.

$$H_1 : \quad \begin{aligned} Y_1 &= A\cos\Psi + \varepsilon_1 \\ Y_2 &= A\sin\Psi + \varepsilon_2 \end{aligned}$$

Ψ is unif. over $[0, 2\pi]$ and indep. of $\varepsilon_1, \varepsilon_2$.

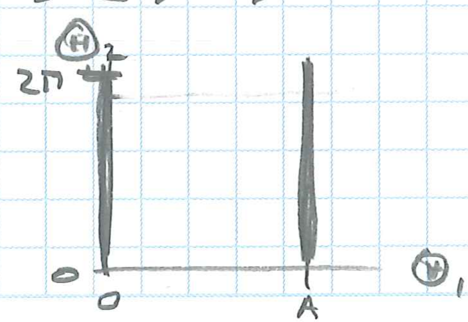


$$\Theta = (\Theta_1, \Theta_2) \qquad \Theta_1 \in \{0, A\} \qquad \Theta_2 \in [0, 2\pi]$$

$$\Lambda = \{0, A\} \times [0, 2\pi]$$

$$\Lambda_0 = \{\theta \in \Lambda : \theta_1 = 0\}$$

$$\Lambda_1 = \{\theta \in \Lambda : \theta_1 = A\}$$



To use results from before, need prob. density of $Y = y$ given $\Theta = \theta$. It is the joint pdf of 2 indep $N(0, \sigma^2)$ rvs with means shifted to

$$\theta_1 \cos\theta_2 \text{ and } \theta_1 \sin\theta_2$$

$$f_\theta(y) = \frac{1}{2\pi\sigma^2} \exp\left\{ \frac{-g(y,\theta)}{2\sigma^2} \right\} \qquad y \in \mathbb{R}^2$$

$$g(y,\theta) = (y_1 - \theta_1 \cos\theta_2)^2 + (y_2 - \theta_1 \sin\theta_2)^2$$

Needed. $$L(y) = \frac{f(y \mid \Theta \in \Lambda_1)}{f(y \mid \Theta \in \Lambda_0)}$$

Can show:

$$f(y \mid \Theta \in \Lambda_0) = f_\theta(y) \Big|_{\theta_1 = 0}$$

$$f(y \mid \Theta \in \Lambda_1) = \frac{1}{2\pi} \int_0^{2\pi} f_\theta(y) \Big|_{\theta_1 = A} d\theta_2.$$

To Do

Test is based on $\sqrt{y_1^2 + y_2^2}$