ECE 645 Spring 2021
Problem Set 4
Due Wednesday April 12, 2021

1. Suppose that Θ is a random parameter with prior density

$$w(\theta) = \begin{cases} \alpha e^{-\alpha\theta} & \theta \geq 0 \\ 0 & \theta < 0 \end{cases}$$

where $\alpha > 0$ is known. Suppose our observation Y is a Poisson random variable with rate Θ. That is

$$p_\theta(y) = P(Y = y | \Theta = \theta) = \frac{\theta^y e^{-\theta}}{y!}$$

for $y = 0, 1, 2, \ldots$. Find the MMSE and MAP estimates of Θ based upon Y. How would you find the MMAE estimate?

2. Suppose that Θ is a random parameter uniformly distributed on the interval $[0, 1]$. Noisy observations

$$Y_k = X_k + \Theta, \quad k = 1, 2, \ldots, n,$$

are made where the $\{X_k\}$ are i.i.d. and independent of Θ. The common marginal pdf of the $\{X_k\}$ is

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

   (a) Find the minimum mean-squared error estimate of Θ.
   (b) Find the maximum aposteriori probability estimate of Θ.
   (c) Find the minimum absolute error estimate of Θ.
   (d) What happens to these estimators as $n \to \infty$?

3. Suppose that $\theta > 0$ is a parameter of interest and that given θ, $\{Y_k : 1 \leq k \leq n\}$ is a set of iid observations with marginal distribution function

$$F_\theta(y) = [F(y)]^{1/\theta}, \quad y \in \mathcal{R}$$

where F is a known distribution function with pdf f.

   (a) Show that

$$\hat{\theta}_{MV}(\boldsymbol{y}) = -\frac{1}{n} \sum_{k=1}^{n} \log F(y_k)$$

   is a minimum variance unbiased estimate of θ.

(b) Suppose now that θ is replaced by a random variable Θ drawn at random using the prior density

$$w(\theta) = c^m \frac{e^{-c/\theta}}{\Gamma(m)\theta^{m+1}} \quad \theta > 0$$

where $c > 0$ and $m > 1$. Use the fact that $\mathrm{E}\{\Theta\} = c/(m-1)$ to show that the MMSE estimator of Θ from $\boldsymbol{Y}$ is

$$\hat{\theta}_{MMSE}(\boldsymbol{y}) = \frac{c - \sum_{k=1}^{n} \log F(y_k)}{m + n - 1}.$$

(c) Compare $\hat{\theta}_{MV}$ and $\hat{\theta}_{MMSE}$ with regard to the role of prior information.

4. Consider the observation model

$$Y_k = \sqrt{\theta} s_k R_k + N_k \quad 1 \le k \le n$$

where $\{s_k : 1 \le k \le n\}$ is a known signal, the N_k and R_k are iid $\mathcal{N}(0, 1)$ random variables, and $\theta \ge 0$ is an unknown parameter.

(a) Find the likelihood equation for estimating θ from $\{Y_k : 1 \le k \le n\}$.

(b) Find the Cramer–Rao lower bound on the variance of unbiased estimates of θ.

(c) Suppose that s_k is a sequence of ± 1's. Find the MLE of θ explicitly.

(d) Compute the bias and variance of the estimate from (c) and compare with the CRB.

5. Consider the following questions concerning unbiased estimators in the classical setting, i.e., with a deterministic parameter.

(a) **Unbiased estimators don't always exist.** Let X be a binomial random variable with parameter θ ($0 \le \theta \le 1$), i.e., the pmf of X assuming θ is the true parameter is

$$p_\theta(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad x = 0, 1, \ldots, n.$$

Let $g(\theta) = \theta^p$. Show that there exists an unbiased estimator of $g(\theta)$ if p is an integer such that $0 \le p \le n$ and there does not if p is an integer such that $p > n$. In the first case, you do not need to solve for the unbiased estimator to prove its existence.

(b) **Unbiased estimators don't always make sense.** Let X be a discrete random variable taking values on the positive integers with the truncated Poisson distribution ($\theta > 0$)

$$p_\theta(x) = \frac{e^{-\theta}\theta^x}{(1-e^{-\theta})x!} \quad x = 1, 2, 3, \ldots$$

Show that an unbiased estimator of $g(\theta) = 1 - e^{-\theta}$ must be such that it equals 0 if x is odd, and it equals 2 if x is even.

6. Prove the factorization criterion for a sufficient statistic in the case of discrete distributions. Attempt to generalize your proof to the case of continuous distributions assuming whatever regularity conditions you need.

7. Suppose that $Z_1, Z_2, \ldots, Z_m$ are independent, identically-distributed samples from a uniform distribution on the interval $(0, \theta)$, $0 < \theta < \infty$. Let $Y_1, Y_2, \ldots, Y_m$ be the order statistics for the sample. Prove that Y_m is a sufficient statistic.

8. The random variables $Z_1, Z_2, \ldots, Z_m$ are i.i.d. samples from a Laplace distribution

$$f(z_i|\sigma) = \frac{1}{2\sigma} \exp\left\{-\frac{|z_i|}{\sigma}\right\}$$

where $-\infty < z_i < \infty$ and $0 < \sigma < \infty$. Does

$$t_m = \sum_{i=1}^{m} |z_i|$$

represent a sufficient statistic?

9. Consider the observation model

$$Z = \frac{1}{\theta} + V$$

where V is a Gaussian random variable with zero mean and variance equal to one. Let $\psi = g(\theta) = 1/\theta$. In this problem we consider the estimation of the unknown parameters θ and ψ.

   (a) Assume that θ is a nonrandom parameter.

      (i) Find the maximum likelihood estimator $\hat{\psi}_{ML}$ of ψ.

      (ii) Find the maximum likelihood estimator $\hat{\theta}_{ML}$ of θ.

   (b) Assume that θ is a realization of a random parameter Θ with probability density

$$p_\Theta(\theta) = \frac{1}{\theta^2\sqrt{2\pi}} \exp\left\{-\frac{1}{2\theta^2}\right\} \quad \theta \neq 0.$$

   Assume also that Θ and V are independent.

      (i) Find the maximum a posteriori estimator $\hat{\Psi}_{MAP}$ of $\Psi = g(\Theta)$.

      (ii) Find the maximum a posteriori estimator $\hat{\Theta}_{MAP}$ of Θ.

10. Let $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}^T$ be a random vector where the individual components are i.i.d. Poisson random variables with parameter θ.

   (a) Show that $T(\mathbf{Y}) = Y_1 + Y_2 + \cdots + Y_n$ is a complete sufficient statistic for θ. You must explain why it is sufficient and why it is complete.

   (b) Show that $T(\mathbf{Y})$ is also Poisson. Start with $n = 2$ and use induction. What is the parameter for the distribution of $T(\mathbf{Y})$?

(c) For any (fixed) integer $k \geq 0$ find a minimum variance unbiased estimate (MVUE) of the probability

$$P_\theta\{Y_1 = k\}.$$

(d) For any (fixed) integer $k \geq 0$ find the maximum likelihood (ML) estimator of the probability

$$P_\theta\{Y_1 = k\}.$$

Is this ML estimator biased?

11. This problem refers back to the previous problem. Recall that $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}^T$ was a random vector where the individual components were i.i.d. Poisson random variables with parameter θ. Previously, you showed that $T(\mathbf{Y}) = Y_1 + Y_2 + \cdots + Y_n$ is a complete sufficient statistic for θ, that $T(\mathbf{Y})$ is Poisson with parameter $n\theta$, and you found the MVUE and ML estimators for

$$\phi = g(\theta) = P_\theta\{Y_1 = 0\} = e^{-\theta}.$$

(actually, a slightly more general result was found before). These estimators were (for $n > 1$)

$$\begin{aligned} \hat{\phi}_{MVUE}(\mathbf{y}) &= \left(\frac{n-1}{n}\right)^{T(\mathbf{y})} \\ \hat{\phi}_{ML}(\mathbf{y}) &= e^{-T(\mathbf{y})/n}. \end{aligned}$$

(a) Find the CRLB for estimating θ based upon the n observations.

(b) The ML estimate for θ is $\hat{\theta}_{ML}(\mathbf{y}) = T(\mathbf{y})/n$. Is $\hat{\theta}_{ML}(\mathbf{y})$ unbiased? Is it efficient? Is it MVUE?

(c) Now consider estimation of $\phi = e^{-\theta}$. Why can we say that $\hat{\phi}_{ML}$ is biased without calculation? Calculate the bias directly. From your result show that the ML estimator is asymptotically (as $n \to \infty$) unbiased.

(d) Directly calculate the variance of $\hat{\phi}_{ML}$ as a function of θ and n and argue that the estimator is consistent.

(e) Find the CRLB for estimating $\phi = e^{-\theta}$ based upon the n observations and express it as a function of θ.

(f) Does the unbiased estimator $\hat{\phi}_{MVUE}$ meet the CRLB for finite n? Answer the question without calculation.

(g) Now calculate the variance of $\hat{\phi}_{MVUE}$ expressed as a function of θ and n. Form the ratio

$$\frac{\mathrm{Var}_\theta\{\hat{\phi}_{MVUE}(\mathbf{Y})\}}{\text{the CRLB for } \phi}$$

and directly explore the efficiency question of (f).