ECE 645 Spring 2021
Problem Set 3
Due Wednesday March 3, 2021

1. Consider the composite hypothesis testing problem:

$$\begin{array}{ll} H_0: & Y \text{ has density } p_0(y) = (1/2)e^{-|y|},\ y \in \mathcal{R} \\ H_1: & Y \text{ has density } p_\theta(y) = (1/2)e^{-|y-\theta|},\ y \in \mathcal{R},\ \theta > 0 \end{array}.$$

   (a) Describe the locally most powerful α–level test and derive its power function.

   (b) Does a uniformly most powerful test exist? If so, find it and derive its power function. If not, find the generalized likelihood ratio test for H_0 versus H_1.

2. Consider the following pair of hypotheses concerning a sequence $Y_1, Y_2, \ldots, Y_n$ of random variables

$$\begin{array}{ll} H_0: & Y_k \sim \mathcal{N}(\mu_0, \sigma_0^2),\ k = 1, 2, \ldots, n \\ H_1: & Y_k \sim \mathcal{N}(\mu_1, \sigma_1^2),\ k = 1, 2, \ldots, n \end{array}$$

   where μ_0, μ_1, σ_0^2, and σ_1^2 are known constants.

   (a) Show that the likelihood ratio can be expressed as a function of the parameters μ_0, μ_1, σ_0^2, and σ_1^2, and the quantities $\sum_{k=1}^n Y_k^2$ and $\sum_{k=1}^n Y_k$.

   (b) Describe the Neyman–Pearson test for the two cases ($\mu_0 = \mu_1$, $\sigma_1^2 > \sigma_0^2$) and ($\sigma_0^2 = \sigma_1^2$, $\mu_1 > \mu_0$).

   (c) Find the threshold and ROCs for the case $\mu_0 = \mu_1$, $\sigma_1^2 > \sigma_0^2$ with $n = 1$.

3. Consider the hypotheses of Problem 2 with $\mu = \mu_1 > \mu_0 = 0$ and $\sigma^2 = \sigma_0^2 = \sigma_1^2 > 0$. Does there exist a uniformly most powerful test of these hypotheses under the assumption that μ is known and σ^2 is not? If so, find it and show that it is UMP. If not, show why and find the generalized likelihood ratio test.

4. Derivation of some important pdfs.

   (a) Let $Z_1, Z_2, \ldots, Z_n$ be i.i.d. and $\mathcal{N}(0, 1)$. Then

$$Y \stackrel{\text{def}}{=} \sum_{i=1}^{n} Z_i^2$$

   is said to be a <u>chi-squared random variable with n degrees of freedom</u> (χ_n^2). Show that the pdf is

$$f_Y(y) = \frac{e^{y/2} y^{(n/2)-1}}{2^{n/2}\Gamma(n/2)}$$

   for $y > 0$ (and zero for $y < 0$). Note $\Gamma(\cdot)$ is the gamma function.

(b) Let $X \sim \mathcal{N}(0,1)$, $Y \sim \chi^2_n$ and X and Y statistically independent. Then we say (William Gosset, 1908)

$$T = \frac{X}{\sqrt{Y/n}}$$

has <u>"Student's" T distribution with n degrees of freedom</u>. Show that the pdf is

$$f_T(t) = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{\pi n}} \frac{1}{(1+t^2/n)^{(n+1)/2}}$$

for $t \in \mathcal{R}$.

Hint: A workable approach to the solution involves using the pdf change of variable formula with the following invertible transformation from (x,y): $x \in \mathcal{R}$, $y > 0$ to (t,u): $t \in \mathcal{R}$, $u > 0$

$$\begin{aligned} t &= \frac{x}{\sqrt{y/n}} \\ u &= y \end{aligned}$$

to get the joint pdf of T and U. Then one integrates over the variable $U = u$ to get the marginal pdf for T.

5. [Kay Detection Book P4.4] Assume that we wish to detect a known signal $s[n]$ which is nonzero only for $n = 0, 1, \ldots, N-1$ in white Gaussian noise with variance σ^2. We now allow our observation interval to be infinite, i.e., we observe $x[n]$ for $-\infty < n < \infty$. Let $h[n]$ be the impulse response of a linear shift invariant filter. Then the output $y[n]$ corresponding to an input $x[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k].$$

If the filter output is sampled at time $n = N-1$ the output SNR can be defined as

$$\eta = \frac{\left(\sum_{k=-\infty}^{\infty} h[N-1-k]s[k]\right)^2}{\mathrm{E}\left[\left(\sum_{k=-\infty}^{\infty} h[N-1-k]w[k]\right)^2\right]}.$$

Show that η is maximized by choosing $h[n]$ as the matched filter or $h[n] = s[N-1-n]$ for $n = 0, 1, \ldots, N-1$ and $h[n] = 0$ otherwise. Thus, the use of noise samples outside of the signal interval does not improve the detection performance.

6. [Kay Detection Book P4.5] In Problem 5 it was shown that the matched filter was optimum even for an infinite observation interval. In this problem we show that this is not generally the case when the noice is correlated. Repeat the problem but now assume that the noise samples $\{w[0], w[1], \ldots, w[N-1]\}$ are white Gaussian noise and that outside the $[0, N-1]$ interval the noise is

periodic. Thus $w[n] = w[n+N]$. Find the output SNR η (use the expression in the previous problem) if the LSI filter is

$$h[n] = \begin{cases} 1 & \text{for } n = 0,\, 1,\, \ldots,\, N-1 \\ -1 & \text{for } n = N,\, N+1,\, \ldots,\, 2N-1 \\ 0 & \text{otherwise} \end{cases}$$

and the output is sampled at $n = N-1$. Explain your results.

7. [Kay Detection Book P4.9] Consider the detection of $s[n] = A\cos 2\pi f_0 n$ for $n = 0, 1 \ldots, N-1$ in the presence of white Gaussian noise of variance σ^2. Define the input SNR as the average power of a signal sample to the noise power. This is approximately $\eta_{in} = (A^2/2)/\sigma^2$. Find the output SNR of the matched filter and hence the processing gain (defined as $\mathrm{PG} = 10\log_{10}(\eta_{out}/\eta_{in})$). Next determine the frequency response of the matched filter and plot its magnitude as N increases. Explain why the matched filter improves the detectability of a sinusoid. Assume that $0 < f_0 < 1/2$ and N is large. Hint: You will need the result that
$$\sum_{n=0}^{N-1} \exp(j\alpha n) = \exp[j(N-1)\alpha/2]\frac{\sin N\alpha/2}{\sin \alpha/2}.$$

8. In this problem we determine the optimal linear filter for detecting a known signal in colored wide sense stationary Gaussian noise based on an infinite observation interval. Thus, we use all the noise samples from outside the signal interval. The detection problem then becomes
$$\begin{aligned} H_0: \; x[n] &= w[n] \text{ for } -\infty < n < \infty \\ H_1: \; x[n] &= \begin{cases} s[n] + w[n] & 0 \le n \le N-1 \\ w[n] & \text{otherwise} \end{cases} \end{aligned}$$
where $w[n]$ is Gaussian noise with a power spectral density $P_{ww}(f)$. The output SNR of an LSI filter at time $n = N-1$ is
$$\eta = \frac{\left(\sum_{k=-\infty}^{\infty} h[k]s[N-1-k]\right)^2}{\mathrm{E}\left[\left(\sum_{k=-\infty}^{\infty} h[k]w[N-1-k]\right)^2\right]}.$$
Note that since the filter processes data over the infinite observation interval $-\infty < n < \infty$, it will be noncausal in general. Show that η may be written in the frequency domain as
$$\eta = \frac{\left(\int_{-1/2}^{1/2} H(f)S(f)\exp[j2\pi f(N-1)]\mathrm{d}f\right)^2}{\int_{-1/2}^{1/2} |H(f)|^2 P_{ww}(f)\mathrm{d}f}.$$
Next, use the Cauchy-Schwarz inequality to prove that η is maximized for
$$H(f) = \frac{S^*(f)\exp[-j2\pi f(N-1)]}{P_{ww}(f)}.$$