ECE 645 Spring 2021
Problem Set 2
Due Monday February 15, 2021

1. [Kay Detection Book P3.14] Consider the hypothesis testing problem

$$\boldsymbol{X} \sim \begin{cases} \mathcal{N}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) & \text{under } H_0 \\ \\ \mathcal{N}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) & \text{under } H_1 \end{cases}$$

where $\boldsymbol{X} = [x[0],\, x[1]]^T$ is observed. Find the NP test statistic (do not evaluate the threshold) and explain what happens if $\rho = 0$.

2. [Kay Detection Book P3.15] Consider the detection of a signal $s[n]$ embedded in white Gaussian noise with variance σ^2 based on the observed samples $x[n]$ for $n = 0,\, 1,\, \ldots,\, 2N-1$. The signal is given by

$$s[n] = \begin{cases} A & n = 0,\, 1,\, \ldots,\, N-1 \\ 0 & n = N,\, N+1,\, \ldots,\, 2N-1 \end{cases}$$

under H_0 and by

$$s[n] = \begin{cases} A & n = 0,\, 1,\, \ldots,\, N-1 \\ 2A & n = N,\, N+1,\, \ldots,\, 2N-1 \end{cases}$$

under H_1. Assume that $A > 0$ and find the NP detector as well as its detection performance. Explain the operation of the detector.

3. [Kay Detection Book P3.17] Assume that we wish to distinguish between the hypotheses $H_0 : \boldsymbol{x} \sim \mathcal{N}(\mathbf{0},\, \sigma^2\boldsymbol{I})$ and $H_1 : \boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu},\, \sigma^2\boldsymbol{I})$ based on $\boldsymbol{x} = [x[0],\, x[1]]^T$. If $\pi_0 = \pi_1 = 0.5$, find the decision regions that minimize probability of error. Show that the decision region boundary is a line that is the perpendicular bisector of the line segment joining $\mathbf{0}$ and $\boldsymbol{\mu}$.

4. Say $X \sim C(a, b)$ if it has the Cauchy density

$$\frac{b}{\pi}\frac{1}{b^2 + (x-a)^2}.$$

   (a) Let X_i $(i = 1,\, 2)$ be independently distributed according to the Cauchy densities $C(a_i, b_i)$. Prove that $X_1 + X_2$ is distributed as $C(a_1 + a_2, b_1 + b_2)$.

   (b) Prove that if $X_1,\, \ldots,\, X_n$ are iid and distributed as $C(a, b)$ then the distribution of the sample mean is again $C(a, b)$. What does this say about the sample mean as an estimator in the Cauchy case?

5. Suppose that Y is a random variable which, under hypothesis H_0, has pdf

$$f_0(y) = \begin{cases} (2/3)(y+1) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and, under hypothesis H_1, has pdf

$$f_1(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} .$$

(a) Find the Bayes rule and minimum Bayes risk for testing H_0 versus H_1 with uniform costs and equal priors.

(b) Find the minimax rule and minimax risk for uniform costs.

6. Consider a pattern classification problem where, in addition to deciding among a certain number of classes, we may decide to reject the pattern if it is not recognizable.

Let observations be denoted by the random vector X. In the case of two possible pattern classes there are two hypotheses:

$$\begin{array}{ll} H_0 : & X \sim f_0 \\ \text{versus} & \\ H_1 : & X \sim f_1 \end{array}$$

with prior probabilities $\pi_0 + \pi_1 = 1$. Instead of making one of two decisions we now allow three decisions

$$d_0 = \text{accept } H_0, \quad d_1 = \text{accept } H_1, \quad \text{or} \quad d_2 = \text{reject}.$$

If we wish to solve this problem in a Bayesian framework we will need costs associated with the three possible decisions and the two hypotheses. Denote these by c_{ij} with the interpretation that this is the cost of making decision d_i when H_j is true and assume that

$$c_{ij} = \begin{cases} 0 & i = j,\ i,j = 0,1 \\ \lambda_m & i \neq j,\ i,j = 0,1 \\ \lambda_r & i = 2 \end{cases} .$$

(a) Since there are three possible decisions a decision rule δ will amount to a partition of the observation space into three regions, each corresponding to a different decision[1] Find the two condional risks

$$R_j(\delta) \stackrel{\text{def}}{=} \text{E}\{\ \text{Cost}\ | H_j \text{ is true }\}$$

for $j = 0,\ 1$.

[1]Note that we do not need to consider randomized decision rules for this problem.

(b) Derive the Bayes decision rule for this problem by specifying the regions of the observation space which should correspond to each of the three decisions.

(c) Simplify the decision rule from part (b) by writing it in terms of the posterior probabilities of the hypotheses given the observations, i.e., in terms of

$$\pi_0(x) \stackrel{\text{def}}{=} P(H_0 \text{ is true } | X = x), \quad \pi_1(x) \stackrel{\text{def}}{=} P(H_1 \text{ is true } | X = x),$$

and the misclassification cost λ_m, and the rejection cost λ_r.

(d) Describe the behavior of the test in the two cases: 1) where the rejection cost λ_r is large, and 2) where it is small, relative to the misclassification cost λ_m.

7. Let $\{Y_i : 1 \le i \le M\}$ be i.i.d. with distribution function F_Y and density f_Y and let $\{Z_i : 1 \le i \le M\}$ be the order statistics of the sample. That is

$$\begin{aligned} Z_1 &= \text{smallest of } Y_1, Y_2, \ldots, Y_M \\ Z_2 &= \text{second smallest of } Y_1, Y_2, \ldots, Y_M \\ &\vdots \\ Z_j &= j\text{–th smallest of } Y_1, Y_2, \ldots, Y_M \\ &\vdots \\ Z_M &= \text{largest of } Y_1, Y_2, \ldots, Y_M \end{aligned}$$

Prove the following formulas:

$$\begin{aligned} F_{Z_r}(z_r) &= \sum_{j=r}^{M} \binom{M}{j} [F_Y(z_r)]^j \, [1 - F_Y(z_r)]^{M-j} \\ f_{Z_r}(z_r) &= \frac{M!}{(r-1)!(M-r)!} [F_Y(z_r)]^{r-1} [1 - F_Y(z_r)]^{M-r} f_Y(z_r) \\ f_{Z_r Z_s}(z_r, z_s) &= \frac{M!}{(r-1)!(s-r-1)!(M-s)!} [F_Y(z_r)]^{r-1} [F_Y(z_s) - F_Y(z_r)]^{s-r-1} \\ &\quad \times [1 - F_Y(z_s)]^{M-s} f_Y(z_r) f_Y(z_s), \quad 1 \le r < s \le M. \end{aligned}$$

8. Consider the hypothesis pair

$$\begin{aligned} H_0 &: \; Y = N \\ H_1 &: \; Y = N + S \end{aligned}$$

where N and S are independent random variables each having pdf

$$f(x) = \begin{cases} e^{-x} & x \ge 0 \\ 0 & x < 0 \end{cases}.$$

(a) Find the likelihood ratio between H_0 and H_1.

(b) Find the threshold and detection probability for α–level Neyman–Pearson testing of H_0 vs. H_1.

Now consider the hypothesis pair

$$\begin{aligned} H_0 &: \quad Y_k = N_k, \quad k = 1, 2, \ldots, n \\ H_1 &: \quad Y_k = N_k + S, \quad k = 1, 2, \ldots, n \end{aligned}$$

where $n > 1$ and $N_1, \ldots, N_n$ and S are independent random variables each having the pdf $f(x)$ above.

(c) Find the likelihood ratio between H_0 and H_1.

(d) Find the threshold and detection probability for α–level Neyman–Pearson testing of H_0 vs. H_1.