ECE 645 Spring 2021
Problem Set 1
Due February 3, 2021

1. How many different 5 card poker hands are there? What is the probability of a hand containing:
   (a) Five different face values.
   (b) One pair. Two pair. Three of a kind. Four of a kind.
   (c) A full house.
   (d) A straight. A flush.
   (e) A straight-flush. A royal straight-flush.

2. You are trapped in a funhouse at a place containing three doors. The first door leads to the outside after 5 minutes of travel. The second door leads to a twisty tunnel that comes back to your starting point after 20 minutes. The third door is similarly twisty turny and leads back to the same spot after 30 minutes. If you are equally likely to choose any one of the three doors whenever presented with the choice (i.e., you can't remember which door you just tried assuming you come around again), what is the expected length of time until you escape?

3. Let X be a random variable with probability density function
$$f_X(x) = \begin{cases} 1/2 & -1 \leq x \leq 1 \\ 0 & \text{else} \end{cases} .$$
   Let $\mathcal{M}$ be the event $\{X \leq -1/2\} \cup \{X \geq 1/2\}$.
   (a) Find and sketch the conditional probability density function $f_{X|\mathcal{M}}(x|\mathcal{M})$.
   (b) Find and sketch the conditional probability distribution function $F_{X|\mathcal{M}}(x|\mathcal{M})$.

4. A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. If 0.5 percent of the population actually has the disease, what is the probability a person has the disease given that his/her test result is positive?

5. Find the $n \times 1$ vector $\hat{x}$ that minimizes $\|x - \hat{x}\|^2 = (x - \hat{x})^T(x - \hat{x})$ for the two cases below.
   (a) Under the constraint $\hat{x} = H\theta$ where H is $n \times p$ and of rank p.
   (b) Under the constraint $A^T x = 0$ with A an $n \times p$ matrix of rank p.

6. Let $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ denote a bivariate normal random vector. Assume that
$$\mathrm{E}X = 0 \quad \text{and} \quad \mathrm{E}XX^T = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$
   Define $Y_1 = X_1 + X_2$ and $Y_2 = -X_1 + X_2$.

(a) Find the joint distribution of Y_1 and Y_2; find the marginal distributions of Y_1 and Y_2.

(b) Find the conditional density of X_1 given Y_1; find the conditional density of X_1 given Y_2.

(c) Find the conditional mean and variance of X_1 given Y_1; find the conditional mean and variance of X_1 given Y_2.

7. If X_1, X_2, X_3, and X_4 are zero mean and jointly Gaussian random variables, prove the following moment factoring property

$$\mathrm{E}\{X_1X_2X_3X_4\} = \mathrm{E}\{X_1X_2\}\mathrm{E}\{X_3X_4\}+\mathrm{E}\{X_1X_3\}\mathrm{E}\{X_2X_4\}+\mathrm{E}\{X_2X_3\}\mathrm{E}\{X_1X_4\}.$$

Use the moment generating property of the characteristic function.

8. Let X_1, X_2, X_3, ... be an infinite sequence of real-valued random variables. Then for any $n > 1$ define the sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Note that both $\bar{X}$ and S^2 depend on the value chosen for n but for simplicity we suppress this dependence in the notation. The goal of this problem is to learn about the joint distribution of the sample mean and variance.

(a) Assume that the random variables $\{X_i\}$ are i.i.d. with mean μ and variance σ^2. Show that

i. $\mathrm{E}[\bar{X}] = \mu$.

ii. $\mathrm{Var}[\bar{X}] = \sigma^2/n$.

iii. $\mathrm{E}[S^2] = \sigma^2$.

(b) Add to the previous assumption that the marginal distributions of the rvs $\{X_i\}$ are Gaussian. Show that

i. $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$

ii. $\bar{X}$ and S^2 are statistically independent.

(c) Based on the preceeding results to complete the description of the joint statistics of $\bar{X}$ and S^2 we only need the marginal distribution of S^2.

i. To this end demonstrate that

$$\frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2$$

ii. Recall that the sum of squares of m i.i.d. $\mathcal{N}(0,1)$ random variables has a chi-squared distribution with m degrees of freedom. Use characteristic functions and the independence of $\bar{X}$ and S^2 to show that

$$\frac{(n-1)S^2}{\sigma^2}$$

is chi-squared with $n-1$ degrees of freedom.

(d) With the preceeding assumptions what is the variance of the sample variance?