ECE 645 Background Material: LTI Systems, Probability, and Random Processes

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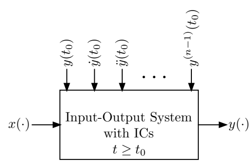
Contents

1 LTI Systems [1, 2]

1.1 Continuous Time (CT) – Time Domain



+ CT dynamical systems often characterized by differential equations of the form:

$$\text{(operations on outputs)} = \text{(operations on inputs)}$$

which, with initial conditions, uniquely specify an output given an input.
+ As an example, for input $x(\cdot)$ and output $y(\cdot)$:

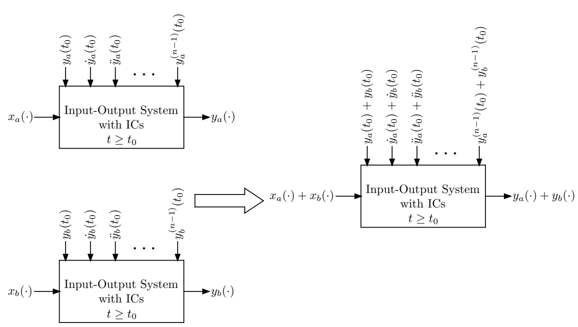
$$\sum_{i=0}^{n} \alpha_i y^{(i)}(t) = \sum_{j=0}^{m} \beta_j x^{(j)}(t)$$

with ICs $y(t_0)$, $y'(t_0)$, …, $y^{(n-1)}(t_0)$.
+ But the form most useful for design is:

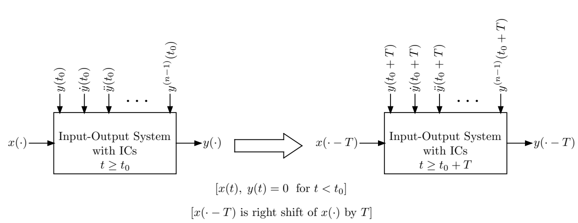
$$\text{outputs} = \text{(operations on inputs)}$$

+ Solving the ODE for a particular input yields a particular output. Integrating the ODE really gives the form we want for design.

1.1.1 Superposition Principle



1.1.2 Time Invariance Principle

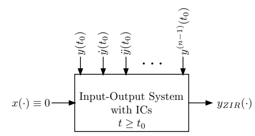


1.1.3 ZIR and ZSR

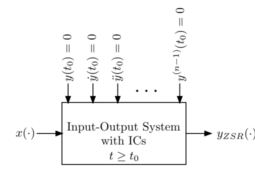
+ General solution for $t \geq t_0$ to system with input $x(\cdot)$ and ICs at t_0 written as a sum of zero-input response (ZIR) and a zero-state response (ZSR):

$$y(t) = y_{ZIR}(t) + y_{ZSR}(t)$$

where



and



1.1.4 ZIR

+ With $x(t) \equiv 0$, the homogeneous equation is:

$$\sum_{i=0}^{n} \alpha_i y^{(i)}(t) = 0$$

with ICs $y(t_0)$, $y'(t_0)$, …, $y^{(n-1)}(t_0)$.
+ Assume solution $y(t) = e^{st}$ and substituting into homogeneous equation yields characteristic equation

$$\sum_{i=0}^{n} \alpha_i s^i = 0.$$

+ Then for solving the ODE, if its characteristic equation has a root s_0 of multiplicity m and if

1. s_0 is real, then linearly independent solutions of the homogeneous ODE are
$$e^{s_0 t},\ te^{s_0 t},\ \ldots,\ t^{m-1}e^{s_0 t}.$$
2. s_0 is complex and $s_0 = a_0 + jb_0$, then linearly independent solutions of the homogeneous ODE are
$$e^{a_0 t}\sin(b_0 t),\ te^{a_0 t}\sin(b_0 t),\ \ldots,\ t^{m-1}e^{a_0 t}\sin(b_0 t)$$
$$e^{a_0 t}\cos(b_0 t),\ te^{a_0 t}\cos(b_0 t),\ \ldots,\ t^{m-1}e^{a_0 t}\cos(b_0 t)$$

+ The weightings in the linear combination are found by applying the IC constraints.

1.1.5 ZSR

+ Let $t_0 = 0$. Then the ZSR corresponding to an LTI differential equation can be written as a convolution integral

$$y_{ZSR}(t) = \int_0^t x(\tau)h(t-\tau)\mathrm{d}\tau = \int_0^t h(\lambda)x(t-\lambda)\mathrm{d}\lambda.$$

+ Above is special case of convolution. For CT signals a and b:

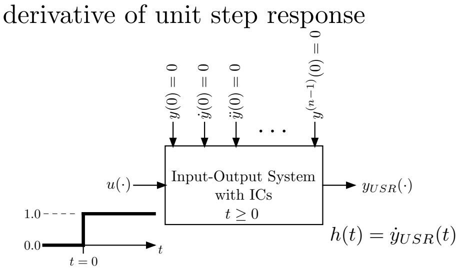
$$a * b(t) = \int_{-\infty}^{+\infty} a(\tau)b(t-\tau)\mathrm{d}\tau.$$

+ Convolution kernel $h(t)$ called the impulse response.

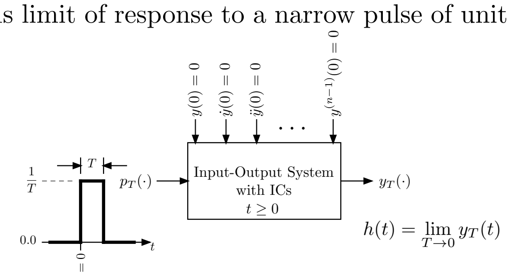
+ LTI systems governed by standard ODEs are causal systems by definition.

1.1.6 To Find Convolution Kernel from ODE

+ As derivative of unit step response



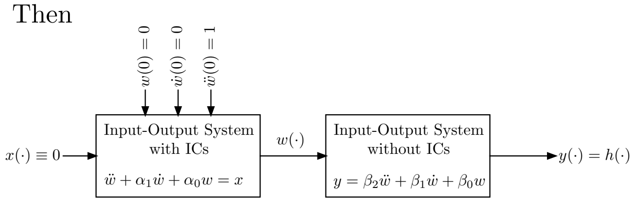
+ As limit of response to a narrow pulse of unit area



+ Directly from solving an ODE with certain initial conditions. For example

$$\ddot{y}(t) + \alpha_1 \dot{y}(t) + \alpha_0 y(t) = \beta_2 \ddot{x}(t) + \beta_1 \dot{x}(t) + \beta_0 x(t)$$

Then



+ That is we solve a homogeneous ODE with certain special ICs and then filter that response with a linear combination of differentiators system.

1.1.7 The Convolution Kernel can be Generalized

+ Does not have to be causal.

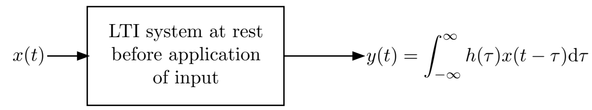
+ Does not have to be a solution to a linear ODE with constant coefficients

- a sliding window integrator

$$h(t) = \frac{1}{T} \left(u(t) - u(t - T) \right).$$

- a multipath channel: $h(t) = \sum_{k=0}^{K} \delta(t - \tau_k)$.

1.2 CT – Frequency Domain



+ Suppose input a complex exponential signal: $x(t) = e^{st}$ where $t \in \mathcal{R}$ and $s \in \mathcal{C}$.

+ Then $y(t) = e^{st} H(s)$, i.e.,

$$\text{output signal} = \text{complex scale factor} \times \text{input signal}.$$

+ Signals e^{st} are the eigenfunctions of LTI systems.

+ Complex scale factors $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \mathrm{d}\tau$ are eigenvalues.

+ Thus if can decompose general signals as superpositions of complex exponential signals, then output is superposition of such responses.

1.2.1 CT Fourier Series

+ If $x(t) = x(t + T_0)$ for all t then

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \\ &\updownarrow \\ X_n &= \frac{1}{T} \int_0^T x(t) e^{-j2\pi n f_0 t} \mathrm{d}t \end{aligned}$$

where $f_0 = 1/T_0$.

+ For a finite energy signal $x(t)$ the FS converges in mean-square. Stronger assumptions can also ensure stronger convergence properties.

+ CTFS Properties are given in Fig. 1.

1.2.2 CT Fourier Transform

+ $x(t) \leftrightarrow X(f)$ if and only if

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \mathrm{d}f \\ &\updownarrow \\ X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \mathrm{d}t \end{aligned}$$

+ CTFT properties in Figure 2.

+ CTFT pairs in Figure 3.

+ $x(t)$, periodic in time with Fourier Series X_k, has CTFT which is a weighted impulse train in frequency: $X(f) = \sum_k X_k \delta(f - kf_0)$.

Property	**Periodic Signal**	**FS Coefficients**
Linearity	$ax(t) + by(t)$	$aX_k + bY_k$
Time Shifting	$x(t-\tau)$	$X_k e^{-j2\pi f_0 \tau k}$
Frequency Shifting	$x(t)e^{j2\pi f_0 M t}$	X_{k-M}
Conjugation	$x^*(t)$	X^*_{-k}
Time Reversal	$x(-t)$	X_{-k}
Periodic Convolution	$\int_0^{T_0} x(\tau)y(t-\tau)\mathrm{d}\tau$	$T_0 X_k Y_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} X_l Y_{k-l}$
Differentiation	$\dot{x}(t)$	$(j2\pi f_0 k)X_k$
Integration (if $X_0 = 0$)	$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau$	$\left(\frac{1}{j2\pi f_0 k}\right) X_k$
Conjugate Symmetry	$x(t)$ real-valued	$X_k = X^*_{-k}$
Real and Even	$x(t)$ real-valued and even	X_k real-valued and even
Real and Odd	$x(t)$ real-valued and odd	X_k pure imaginary and odd
Even Part ($x(t)$ real-valued)	$x_e(t) = \frac{x(t)+x(-t)}{2}$	$\Re\{X_k\}$
Odd Part ($x(t)$ real-valued)	$x_o(t) = \frac{x(t)-x(-t)}{2}$	$j\Im\{X_k\}$

Inner Product $$\frac{1}{T_0}\int_0^{T_0} x(t)y^*(t)\mathrm{d}t = \sum_{k=-\infty}^{\infty} X_k Y_k^*$$

Parseval $$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2\mathrm{d}t = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Figure 1: Some Properties of the DFT. (From [1])

Table XIII.2—Properties of Fourier Transforms

	$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df$	$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$
Conjugate Symmetry	$\Im m[x(t)] = 0$ (i.e., $x(t)$ is real)	$X(f) = X^*(-f)$ (i.e., $\Re e[X(f)] = \Re e[X(-f)]$, $\Im m[X(f)] = -\Im m[X(-f)]$)
Even Symmetry	$x(t) = x(-t)$	$X(f) = X(-f)$
Odd Symmetry	$x(t) = -x(-t)$	$X(f) = -X(-f)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(f) + bX_2(f)$
Duality	$X(t)$	$x(-f)$
Scale Change	$x(at)$	$\frac{1}{\lvert a \rvert} X(f/a)$
Time Delay	$x(t - t_0)$	$e^{-j2\pi f t_0} X(f)$
Times $e^{j2\pi f_0 t}$	$e^{j2\pi f_0 t} x(t)$	$X(f - f_0)$
Differentiation	$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
Times t	$tx(t)$	$\frac{1}{-j2\pi} \frac{dX(f)}{df}$
Convolution	$\int_{-\infty}^{\infty} w(\tau) v(t - \tau)\, d\tau$	$W(f)V(f)$
Product	$w(t)v(t)$	$\int_{-\infty}^{\infty} W(\nu) V(f - \nu)\, d\nu$
Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(f)}{j2\pi f} + \frac{X(0)\delta(f)}{2}$

Other formulas:

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt; \qquad x(0) = \int_{-\infty}^{\infty} X(f)\, df$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df \qquad \text{(Parseval)}$$

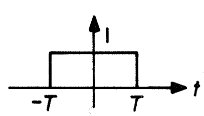
$$\int_{-\infty}^{\infty} x(t) y^*(t + \tau) e^{-j2\pi\nu t}\, dt = \int_{-\infty}^{\infty} X(f + \nu) Y^*(f) e^{-j2\pi f\tau}\, df$$

Figure 2: Some Properties of Fourier Transforms. (From [1])

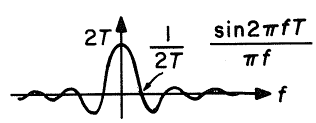
Table XIII.1—Short Table of Fourier Transforms

	$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df$		$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$
a)	$\delta(t)$	$\Longleftrightarrow$	1
b)	1	$\Longleftrightarrow$	$\delta(f)$
c)	$e^{-\pi(t/\tau)^2}$	$\Longleftrightarrow$	$\tau e^{-\pi(f\tau)^2}$
d)	$e^{-\alpha t} u(t)$	$\Longleftrightarrow$	$\dfrac{1}{\alpha + j2\pi f}, \quad \alpha > 0$
e)	$u(t)$	$\Longleftrightarrow$	$\dfrac{\delta(f)}{2} + \dfrac{1}{j2\pi f}$
f)	$\operatorname{sgn} t = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\Longleftrightarrow$	$\dfrac{1}{j\pi f}$
g)	$\dfrac{1}{\pi t}$	$\Longleftrightarrow$	$-j \operatorname{sgn} f$
h)	$e^{-\alpha\lvert t\rvert}$	$\Longleftrightarrow$	$\dfrac{2\alpha}{\alpha^2 + (2\pi f)^2}$
i)	$e^{j2\pi f_0 t}$	$\Longleftrightarrow$	$\delta(f - f_0)$
j)	$\sin 2\pi f_0 t$	$\Longleftrightarrow$	$\dfrac{\delta(f - f_0) - \delta(f + f_0)}{2j}$
k)	$\cos 2\pi f_0 t$	$\Longleftrightarrow$	$\dfrac{\delta(f - f_0) + \delta(f + f_0)}{2}$
l)	$e^{j2\pi f_0 t} u(t)$	$\Longleftrightarrow$	$\dfrac{\delta(f - f_0)}{2} + \dfrac{1}{j2\pi}\left[\dfrac{1}{f - f_0}\right]$
m)	$\sin 2\pi f_0 t\, u(t)$	$\Longleftrightarrow$	$\dfrac{\delta(f - f_0) - \delta(f + f_0)}{4j} + \dfrac{1}{2\pi}\left[\dfrac{f_0}{f_0^2 - f^2}\right]$
n)	$\cos 2\pi f_0 t\, u(t)$	$\Longleftrightarrow$	$\dfrac{\delta(f - f_0) + \delta(f + f_0)}{4} + \dfrac{1}{j2\pi}\left[\dfrac{f}{f^2 - f_0^2}\right]$
o)	$\dot{\delta}(t)$	$\Longleftrightarrow$	$j2\pi f$
p)	$te^{-\alpha t} u(t), \quad a > 0$	$\Longleftrightarrow$	$\dfrac{1}{(\alpha + j2\pi f)^2}$
q)	t	$\Longleftrightarrow$	$\dfrac{j\dot{\delta}(f)}{2\pi}$

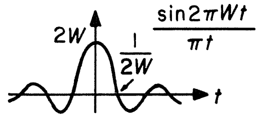
r)



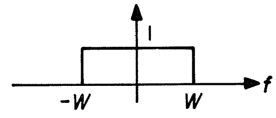
$\Longleftrightarrow$



s)



$\Longleftrightarrow$



t)

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \Longleftrightarrow \quad \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$

Figure 3: Some Fourier Transform Pairs (From [1])

1.3 Discrete Time (DT) – Time Domain

+ DT dynamical systems are characterized by difference equations rather than differential equations. Most concepts from CT have a direct and obvious analogy in DT

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j].$$

+ Linear DT systems satisfy the analogous superposition principal.
+ Time-invariant DT systems satisfy the analogous time invariance principal.

1.3.1 ZIR and ZSR

+ General solution for $n \geq t_0$ to system with input $x[\cdot]$ and ICs at n_0 written as a sum of zero-input response (ZIR) and a zero-state response (ZSR):

$$y[n] = y_{ZIR}[n] + y_{ZSR}[n].$$

1.3.2 ZIR

+ With $x[n] \equiv 0$

$$\sum_{i=0}^{N} a_i y[n-i] = 0.$$

+ Assume solution $y[n] = z^n$ and substituting into homogeneous equation yields characteristic equation

$$\sum_{i=0}^{N} a_i r^{N-i} = 0.$$

+ The solution to the homogeneous equation is a linear combination of terms of the form:

- $\{r^n,\, nr^n,\, \ldots,\, n^{m-1}r^n\}$ for each simple real root $z = r$ of multiplicity m.
- $\{\rho^n \cos(n\phi),\, \rho^n \sin(n\phi),\, n\rho^n \cos(n\phi), \ldots$ $\ldots,\, n^{m-1}\rho^n \cos(n\phi),\, n^{m-1}\rho^n \sin(n\phi)\}$ for each pair of complex conjugate roots $z = a \pm jb = \rho e^{\pm j\phi}$ of multiplicity m.

+ The weightings in the linear combination are found by applying the IC constraints.

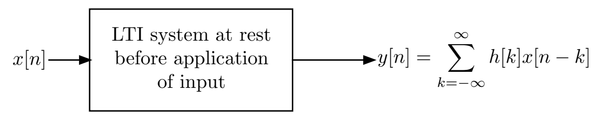
1.3.3 To Find Convolution Kernel from ODE

+ Trivial analogy.

1.3.4 The Convolution Kernel can be Generalized

+ Trivial analogy.

1.4 DT – Frequency Domain



+ Suppose input a complex exponential signal: $x[n] = z^n$ where $n \in \mathcal{Z}$ and $z \in \mathcal{C}$.
+ Then $y[n] = z^n H(z)$, i.e.,

$$\text{output signal} = \text{complex scale factor} \times \text{input signal}.$$

+ Signals z^n are the eigenfunctions of DT LTI systems.
+ Complex scale factors $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$ are eigenvalues.
+ Thus if can decompose general signals as superpositions of complex exponential signals, then output is superposition of such responses.

1.4.1 Case of Interest $z = e^{j\lambda} = e^{j2\pi\nu}$

+ A DT complex sinusoid $x[n] = e^{j\lambda_0 n}$ is periodic $\Leftrightarrow \lambda_0/(2\pi) =$ a rational number.
+ If fix period N and solve for all DT complex sinusoids with that period, find there are exactly N of them: $\phi_l[\cdot]$ for $l = 0, 1, \ldots, N-1$, where

$$\phi_l[n] = \left(e^{j2\pi l/N}\right)^n = e^{j2\pi ln/N}.$$

+ The N DT signals $\phi_l[\cdot]$ have fundamental frequencies which are multiples of $2\pi/N$, i.e., they are harmonically related.

1.4.2 DT Fourier Series

+ If $x[n] = x[n+N]$ for all n, then

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

$$\updownarrow$$

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

where we note that the sequence X_k of DTFS coefficients is also periodic with period N.

+ Sometimes write DTFS as a matrix operation: $\boldsymbol{X} = \boldsymbol{W}\boldsymbol{x}$ where the vector coefficients and signal are formed by stacking,

$$\boldsymbol{W} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

and $W_N = e^{-j2\pi/N}$ is the so-called twiddle factor.

+ The DFT matrix $\boldsymbol{W}$ is a Vandermonde matrix.

+ It is invertible with $\boldsymbol{W}^{-1} =$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & & & & \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

+ DTFS/DFT Properties are given in Fig. 4.

1.4.3 DT Fourier Transform

+ In terms of radian frequency variable λ (radians per sample) unique over an interval of length 2π:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda})e^{j\lambda n}\mathrm{d}\lambda \\ &\updownarrow \\ X(e^{j\lambda}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\lambda n} \end{aligned}$$

+ In terms of Hertzian frequency variable ν (cycles per sample) unique over an interval of length 1 ($2\pi\nu = \lambda$):

$$\begin{aligned} x[n] &= \int_{-1/2}^{1/2} X(e^{j2\pi\nu})e^{j2\pi\nu n}\mathrm{d}\nu \\ &\updownarrow \\ X(e^{j2\pi\nu}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi\nu n} \end{aligned}$$

+ DTFT Properties are given in Fig. 5.

+ DTFT Pairs are given in Fig. **??**.

1.4.4 DT Fourier Transform for Periodic Signals

1.5 Ideal Sampling of CT Signals

1.5.1 Poisson Sum Formula

+ Sampling pulse $g(t) \leftrightarrow G(f)$.

+ Periodic pulse train used to sample: $p(t) = \sum_{n=-\infty}^{\infty} g(t - nT)$.

+ Two ways to write $P(f)$:

$$\begin{aligned} P(f) &= G(f)\sum_{n=-\infty}^{\infty} e^{-j2\pi nTf} \\ &= G(f)\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta(f-k/T) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} G(k/T)\delta(f-k/T) \end{aligned}$$

1.5.2 Modulation Property of CTFT

+ $x(t) \leftrightarrow X(f)$ as the signal to be sampled. Then

$$\begin{aligned} x(t)p(t) &= \sum_{n=-\infty}^{\infty} x(t)g(t-nT) \\ &\updownarrow \\ X * P(f) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} G(k/T)[X * \delta(\cdot - k/T)](f) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} G(k/T)X(f-k/T) \end{aligned}$$

+ Periodic sampling at sample rate $1/T$ produces replications in frequency spaced by $1/T$ Hz. Spectral replications weighted by samples of the spectrum of the sample pulse.

1.5.3 Example Sampling Pulses

+ $g(t) = \delta(t) \leftrightarrow G(f) = 1$

+ $g(t) = \begin{cases} 1/\Delta & |t| \le \Delta/2 \\ 0 & |t| > \Delta/2 \end{cases} \leftrightarrow G(f) = \frac{\sin(\pi\Delta f)}{\pi\Delta f}$

+ $g(t) = \frac{1}{\Delta}e^{-\pi(t/\Delta)^2} \leftrightarrow G(f) = e^{-\pi(f\Delta)^2}$

+ Typically $0 < \Delta \ll T$ (where T is the sampling interval)

+ See Fig. 7.

- Definition:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \leftrightarrow X_k = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}.$$

- Similarly for $y[n] \longleftrightarrow Y_k$.

Property	**Time Domain**	**Frequency Domain**
Linearity	$ax[n] + by[n]$	$aX_k + bY_k$
Symmetry	$\frac{1}{N}X_n$	$x[-k]$
Time Shifting	$x[n-m]$	$X_k e^{-j2\pi km/N}$
Frequency Shifting	$x[n]e^{j2\pi ln/N}$	X_{k-l}
Conjugation	$x^*[n]$	X^*_{-k}
Time Reversal	$x[-n]$	X_{-k}
Periodic Convolution	$\sum_{l=0}^{N-1} x[l]y[n-l]$	$X_k Y_k$
Multiplication	$x[n]y[n]$	$\frac{1}{N}\sum_{l=0}^{N-1} X_l Y_{k-l}$
Conjugate Symmetry	$x[n]$ real-valued	$X_k = X^*_{-k}$
Real and Even	$x[n]$ real-valued and even	X_k real-valued and even
Real and Odd	$x[n]$ real-valued and odd	X_k pure imaginary and odd
Even Part ($x[n]$ real-valued)	$x_e[n] = \frac{x[n]+x[-n]}{2}$	$\Re\{X_k\}$
Odd Part ($x[n]$ real-valued)	$x_o[n] = \frac{x[n]-x[-n]}{2}$	$j\Im\{X_k\}$

Inner Product	$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k Y_k^*$
Parseval	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X_k\rvert^2$

Figure 4: Some Properties of the DTFS/DFT. (From [1])

- Definition:
$$x[n] = \int_{-1/2}^{1/2} X(e^{j2\pi\nu}) e^{j2\pi\nu n} \mathrm{d}\nu \leftrightarrow X(e^{j2\pi\nu}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi\nu n}.$$
- Similarly for $y[n] \longleftrightarrow Y(e^{j2\pi\nu})$.

Property	**DT Signal**	**DTFT**
Linearity	$ax[n] + by[n]$	$aX(e^{j2\pi\nu}) + bY(e^{j2\pi\nu})$
Time Shifting	$x[n-n_0]$	$e^{-j2\pi\nu n_0} X(e^{j2\pi\nu})$
Frequency Shifting	$x[n]e^{j2\pi\nu_0 n}$	$X(e^{j2\pi(\nu-\nu_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j2\pi\nu})$
Time Reversal	$x[-n]$	$X(e^{-j2\pi\nu})$
Convolution	$\sum_{k=-\infty}^{\infty} x[k]y[n-k]$	$X(e^{j2\pi\nu})Y(e^{j2\pi\nu})$
Multiplication	$x[n]y[n]$	$\int_{-1/2}^{+1/2} X(e^{j2\pi\mu})Y(e^{j2\pi(\nu-\mu)})\mathrm{d}\mu$
Differencing	$x[n] - x[n-1]$	$(1-e^{-j2\pi\nu})X(e^{j2\pi\nu})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j2\pi\nu}} X(e^{j2\pi\nu})$
Differentiation in Frequency	$nx[n]$	$-\frac{1}{j2\pi}\dot{X}(e^{j2\pi\nu})$
Conjugate Symmetry	$x[n]$ real-valued	$X(e^{j2\pi\nu}) = X^*(e^{-j2\pi\nu})$
Real and Even	$x[n]$ real-valued and even	$X(e^{j2\pi\nu})$ real-valued and even
Real and Odd	$x[n]$ real-valued and odd	$X(e^{j2\pi\nu})$ pure imaginary and odd
Even Part ($x[n]$ real-valued)	$x_e[n] = \frac{x[n]+x[-n]}{2}$	$\Re\{X(e^{j2\pi\nu})\}$
Odd Part ($x[n]$ real-valued)	$x_o[n] = \frac{x[n]-x[-n]}{2}$	$j\Im\{X(e^{j2\pi\nu})\}$

Inner Product	$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \int_{-1/2}^{+1/2} X(e^{j2\pi\nu})Y^*(e^{j2\pi\nu})\mathrm{d}\nu$
Parseval	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \int_{-1/2}^{+1/2} \lvert X(e^{j2\pi\nu})\rvert^2 \mathrm{d}\nu$

Figure 5: Some Properties of the DTFT. (From [1])

$$a^n u[n],\ |a|<1 \quad \longleftrightarrow \quad \frac{1}{1-ae^{-j2\pi\nu}}$$

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases} \quad \longleftrightarrow \quad \frac{\sin(2\pi\nu(N_1+1/2))}{\sin(\pi\nu)}$$

$$\frac{\sin(Wn)}{\pi n},\ 0 < W < \pi \quad \longleftrightarrow \quad X(e^{j2\pi\nu}) = \begin{cases} 1 & 0 \le |\nu| \le W/2\pi \\ 0 & W/2\pi < |\nu| \le 1/2 \\ \text{periodic} & |\nu| > 1/2 \end{cases}$$

$$\delta[n] \quad \longleftrightarrow \quad 1$$

$$\delta[n-n_0] \quad \longleftrightarrow \quad e^{-j2\pi\nu n_0}$$

$$u[n] \quad \longleftrightarrow \quad \frac{1}{1-e^{-j2\pi\nu}} + \frac{1}{2}\sum_{k=-\infty}^{\infty} \delta(\nu - k)$$

$$(n+1)a^n u[n],\ |a|<1 \quad \longleftrightarrow \quad \frac{1}{(1-ae^{-j2\pi\nu})^2}$$

Figure 6: Some DTFT Pairs. (From [1])

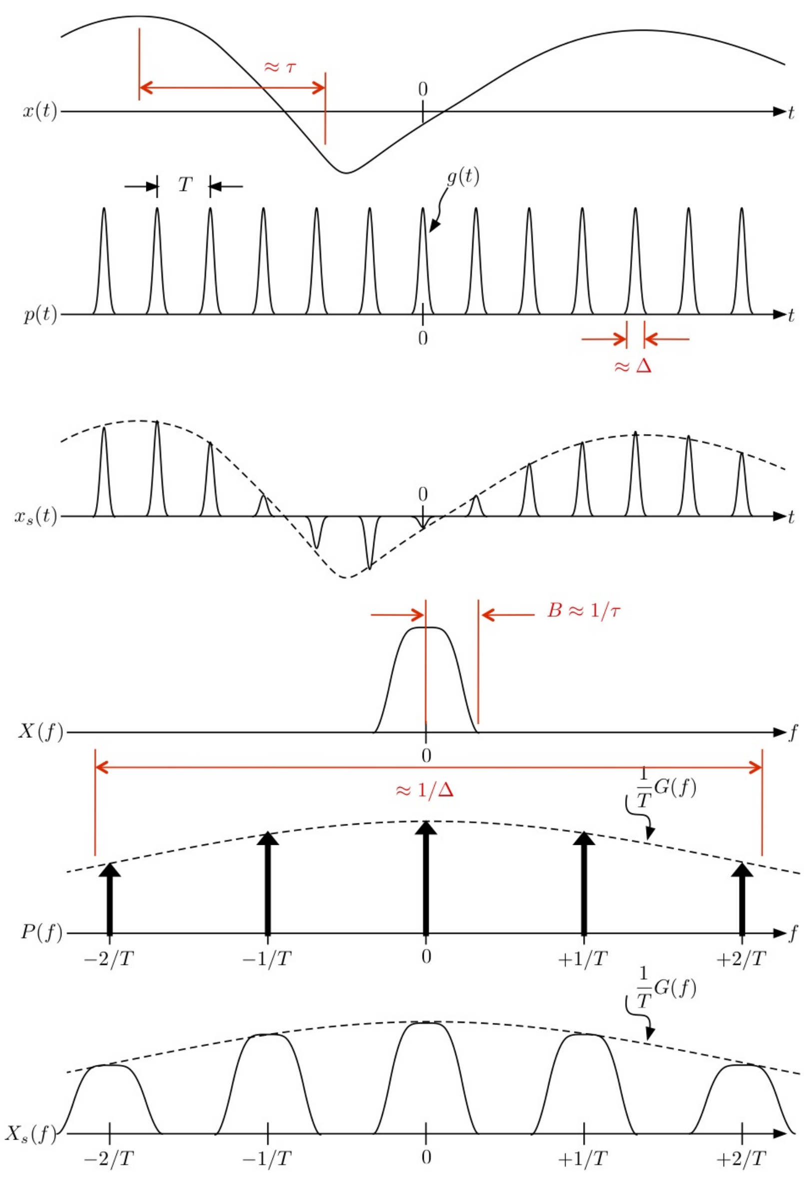


Figure 7: Example Sampling with Gaussian Pulse.

1.5.4 Observations from Example Sampling

+ From Fig. 7:

$$\Delta \ll T \ll \underbrace{\text{Signal time constant}}_{\tau}$$

which is $\Longleftrightarrow$

$$\underbrace{\text{Signal BW}}_{B\approx 1/\tau} \ll 1/T \ll \underbrace{\text{Pulse BW}}_{\approx 1/\Delta}$$

+ See that signal spectrum replications in $X_s(f) = P * X(f)$ are undistorted. Therefore, information carried by signal is still present in sampled signal $x_s(t)$.

1.5.5 Ideal Case

+ Take the limit $\Delta \to 0$ to have ideal impulse sampling

$$\begin{aligned} x_s(t) &= \sum_n x(nT)\delta(t - nT) \\ &\updownarrow \\ X_s(f) &= \frac{1}{T}\sum_k X(f - k/T) \end{aligned}$$

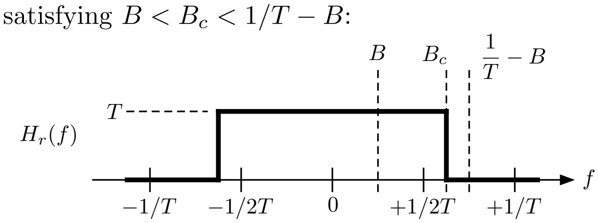
+ See Fig. 8 where we observe:

Spectral copies do not overlap $\Leftrightarrow 1/T > 2B$

1.5.6 Nyquist Sampling Theorem

> If $x(t)$ is bandlimited to B Hz and is sampled at rate $1/T$ via ideal impulse sampling then it may be exactly recovered from its samples via lowpass filtering provided that $1/T > 2B$.

+ Proof is to apply the LPF shown with cutoff B_c satisfying $B < B_c < 1/T - B$:



+ Impulse response corresponding to LPF is

$$h_r(t) = 2B_cT\frac{\sin(2\pi B_c t)}{2\pi B_c t}$$

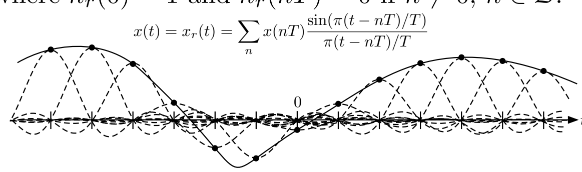
which results in the interpolation formula

$$\begin{aligned} x_r(t) &= x_s * h_r(t) \\ &= 2B_cT\sum_n x(nT)\frac{\sin(2\pi B_c(t - nT))}{2\pi B_c(t - nT)} \end{aligned}$$

+ If Nyquist is satisfied: $x(t) = x_r(t)$.
+ If cutoff chosen equal to the foldover frequency $B_c = 1/(2T)$ then interpolator form is very simple

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

where $h_r(0) = 1$ and $h_r(nT) = 0$ if $n \neq 0$, $n \in \mathcal{Z}$.



1.5.7 The 2WT Dimensionality Theorem

+ Consider a linear vector space of signals approximately bandlimited to $-W_0 < f < W_0$ and approximately time-limited to $0 < t < T_0$.
+ Assuming sampling at the Nyquist rate we have

$$\frac{1}{T} = 2W_0.$$

+ The the approximate number of samples needed to specify an arbitrary member of the space is

$$N = T_0/T = 2W_0T_0.$$

+ The representation of such approximate band- and time-limited signal space is given by a linear combination of sinc-function basis functions:

$$x(t|a_0, a_1, \ldots, a_{N-1}) = \sum_{n=0}^{N-1} a_n \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}.$$

1.6 Sampling: CTFT and DTFT

+ Ideal impulse train sampled signal

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

has CTFT

$$X_s(f) = \sum_n x(nT)e^{-j2\pi fnT} = \frac{1}{T}\sum_n X(f - k/T).$$

+ Second part of the equation is previously found formula for the sampled spectrum.

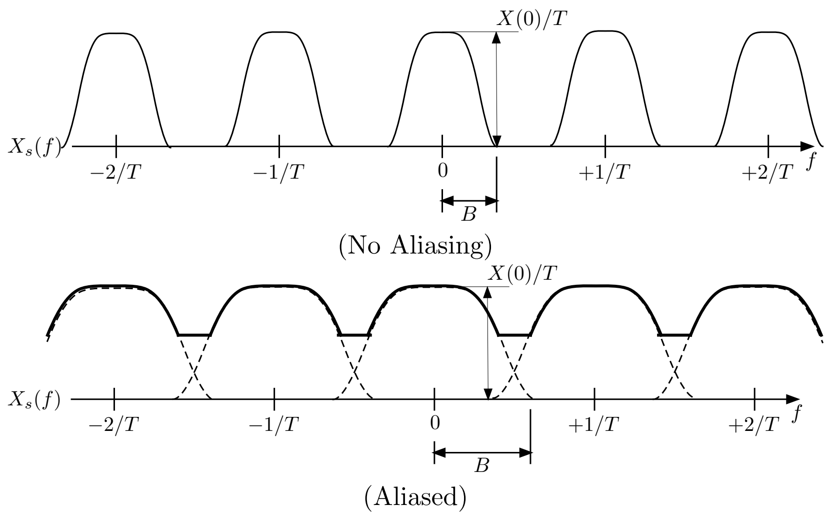


Figure 8: Ideal Impulse Sampling (Aliasing or Not).

+ But the first part is actually the DTFT of the samples themselves evaluated at $\nu = fT$.
+ That is if $x[n] = x(nT)$ and if $x[n] \leftrightarrow X(e^{j2\pi\nu})$ is a DTFT pair then

$$X_s(f) = X(e^{j2\pi\nu})\Big|_{\nu = fT}.$$

2 Probability [3]

2.1 Discrete Distributions

- Bernoulli: A random variable (r.v.) X is said to be a Bernoulli r.v. with parameter p ($0 \leq p \leq 1$) if it only takes two values 0 and 1 and its probability mass function (pmf) is of the form

$$\begin{aligned} p_X(1) &= \Pr(X=1) = p \\ p_X(0) &= \Pr(X=0) = 1-p. \end{aligned}$$

- Binomial: A r.v. X is said to be a Binomial r.v. with parameters (N, p) where N is a positive integer and $0 \leq p \leq 1$ if its pmf is of the form

$$p_X(k) = \left(\begin{array}{c} N \\ k \end{array} \right) p^k (1-p)^{N-k}$$

for $k = 0, 1, 2, \ldots, N$. For such a r.v. X

$$\begin{aligned} \mathrm{E}(X) &= Np \\ \mathrm{Var}(X) &= Np(1-p). \end{aligned}$$

- Poisson: A r.v. X is said to be a Poisson r.v. with parameter $\lambda > 0$ if its pmf is of the form

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{\lambda!}$$

for $k = 0, 1, 2, \ldots$. For such a r.v. X

$$\begin{aligned} \mathrm{E}(X) &= \lambda \\ \mathrm{Var}(X) &= \lambda. \end{aligned}$$

2.2 Continuous Distributions

- Uniform: A r.v. X is said to be uniform on an interval $a \leq x \leq b$ if its probability density function (pdf) is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for} \quad a \leq x \leq b \\ 0 & \text{for} \quad x < a \text{ or } x > b \end{cases}.$$

For such a r.v.

$$\begin{aligned} \mathrm{E}(X) &= (a+b)/2 \\ \mathrm{Var}(X) &= (b-a)^2/12. \end{aligned}$$

- Exponential: A r.v. X is said to be an exponential r.v. with parameter $\lambda > 0$ if its pdf is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases}.$$

For such a r.v.

$$\begin{aligned} \mathrm{E}(X) &= 1/\lambda \\ \mathrm{Var}(X) &= 1/\lambda^2. \end{aligned}$$

- Rayleigh: A r.v. R is said to be Rayleigh distributed with parameter σ if its pdf is

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} & \text{for} \quad r \geq 0 \\ 0 & \text{for} \quad r < 0 \end{cases}$$

For such a r.v.

$$\begin{aligned} \mathrm{E}(R) &= \sqrt{\frac{\pi}{2}}\sigma \\ \mathrm{Var}(R) &= \frac{1}{2}(4-\pi)\sigma^2. \end{aligned}$$

- Gaussian (single variate): A r.v. X is said to be a normal (or Gaussian) r.v. with parameters (μ, σ^2), written $X \sim \mathcal{N}(\mu, \sigma^2)$, if its pdf is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

For such a r.v.

$$\begin{aligned} \mathrm{E}(X) &= \mu \\ \mathrm{Var}(X) &= \sigma^2. \end{aligned}$$

The Gaussian Q function gives the tail probability of a $\mathcal{N}(0,1)$ r.v.,

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} \mathrm{d}z.$$

Note that $Q(-x) = 1 - Q(x)$. A table of values of the Q function is given on the next page.

- Gaussian (two variable): Two r.v.s X and Y are said to be bivariate normal (or Gaussian) if their joint pdf is

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \\ &\times \quad \exp\left\{-\frac{1}{2(1-\rho^2)}F(x,y)\right\} \end{aligned}$$

where $F(x,y)$ is the quadratic form:

$$\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}.$$

For such r.v.s

$$\begin{aligned} \mathrm{E}(X) &= \mu_x \\ \mathrm{E}(Y) &= \mu_y \\ \mathrm{Var}(X) &= \sigma_x^2 \\ \mathrm{Var}(Y) &= \sigma_y^2 \\ \mathrm{Cov}(X,Y) &= \rho\sigma_x\sigma_y \end{aligned}$$

where $-1 \leq \rho \leq +1$. (The bivariate normal distribution can be generalized to an arbitrary number of random variables. Such are called jointly Gaussian r.v.s.)

Table G.1 A Short Table of *Q*-Function Values

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0	0.5	1.5	0.066807	3.0	0.0013499
0.1	0.46017	1.6	0.054799	3.1	0.00096760
0.2	0.42074	1.7	0.044565	3.2	0.00068714
0.3	0.38209	1.8	0.035930	3.3	0.00048342
0.4	0.34458	1.9	0.028717	3.4	0.00033693
0.5	0.30854	2.0	0.022750	3.5	0.00023263
0.6	0.27425	2.1	0.017864	3.6	0.00015911
0.7	0.24196	2.2	0.013903	3.7	0.00010780
0.8	0.21186	2.3	0.010724	3.8	7.2348×10^{-5}
0.9	0.18406	2.4	0.0081975	3.9	4.8096×10^{-5}
1.0	0.15866	2.5	0.0062097	4.0	3.1671×10^{-5}
1.1	0.13567	2.6	0.0046612	4.1	2.0658×10^{-5}
1.2	0.11507	2.7	0.0034670	4.2	1.3346×10^{-5}
1.3	0.096800	2.8	0.0025551	4.3	8.5399×10^{-6}
1.4	0.080757	2.9	0.0018658	4.4	5.4125×10^{-6}

2.3 Gaussian Properties

- Jointly Gaussian r.v.s X and Y are statistically independent if and only if (iff) they are uncorrelated, i.e., $\rho = 0$.

- A linear combination of an arbitrary number of jointly Gaussian r.v.s is a Gaussian r.v.

- Conditional Gaussian: Let r.v.s X and Y be jointly Gaussian with the pdf given in the previous bullet. Then the conditional pdf of X given $Y = y$ is a single variable Gaussian pdf with

$$\begin{aligned} \mathrm{E}(X|Y=y) &= \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y) \\ \mathrm{Var}(X|Y=y) &= \sigma_x^2(1-\rho^2). \end{aligned}$$

- Gaussian Moments: Let X be a Gaussian random variable with mean zero and variance σ^2, i.e., $\mathcal{N}(0, \sigma^2)$. Then

$$\mathrm{E}(X^{2n}) = 1 \times 3 \times \cdots \times (2n-1)\sigma^{2n}$$

and

$$\mathrm{E}(X^{2n-1}) = 0$$

for $n = 1, 2, 3, \ldots$.

- Connection between Gaussian and Rayleigh: Let X and Y be zero mean jointly Gaussian r.v.s with equal variances σ^2 and $\rho = 0$ (i.e., they are statistically independent). Then the derived r.v.s $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan(Y/X)$ (four quadrant inverse tangent) are themselves statistically independent and R is Rayleigh with parameter σ and Θ is uniform on $[0, 2\pi)$.

2.4 Useful Theorems

- Markov's Inequalitiy: X a r.v. taking only nonnegative values. Then for any $a > 0$

$$\Pr\{X \geq a\} \leq \frac{\mathrm{E}[X]}{a}.$$

- Chebyshev's Inequalitiy: X a r.v. with finite mean μ and variance σ^2, then for any value $k > 0$

$$\Pr\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$

- Weak Law of Large Numbers: $X_1, X_2, \ldots$ a sequence of independent and identically distributed (i.i.d.) r.v.s, each having a finite mean $\mathrm{E}[X_i] = \mu$. Then, for any $\epsilon > 0$

$$\Pr\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right\} \to 0$$

as $n \to \infty$. (Sample mean converges to true mean in probability.)

- Central Limit Theorem: $X_1, X_2, \ldots$ a sequence of i.i.d. r.v.s, each having mean μ and variance σ^2. Then the cdf of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the cdf of the standard unit normal as $n \to \infty$. (Convergence in distribution.)

- Strong Law of Large Numbers: $X_1, X_2, \ldots$ a sequence of independent and identically distributed (i.i.d.) r.v.s, each having a finite mean $\mathrm{E}[X_i] = \mu$. Then

$$\Pr\left\{\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu\right\} = 1$$

(i.e., the sample mean converges to the true mean with probability one.)

3 Random Processes [4]

3.1 Second Order RPs

Assume all signals, impulse responses, and random processes $X(t)$, $Y(t)$ are real-valued in this section. Assume that all random variables have finite variance (hence also have finite means). Define moment functions:

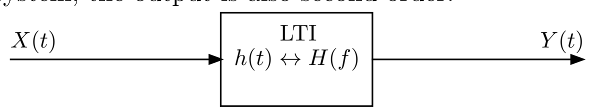
- Mean: $\mu_X(t) = \mathrm{E}[X(t)]$.
- Cross-Correlation: $R_{X,Y}(t,s) = \mathrm{E}[X(t)Y(s)]$.
- Cross-covariance

  $$C_{X,Y}(t,s) = R_{X,Y}(t,s) - \mu_X(t)\mu_Y(s).$$

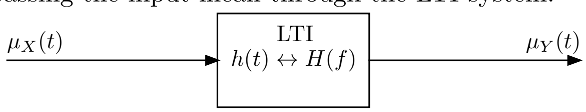
- We get auto-correlation $R_{X,X}(t,s)$ and auto-covariance $C_{X,X}(t,s)$ when $Y \equiv X$ in the definitions above.

3.2 And LTI Systems . . .

Let the impulse response of an LTI system be BIBO stable. Then if a second order rp is input to the system, the output is also second order:



The mean of the output rp is equal to the result of passing the input mean through the LTI system:



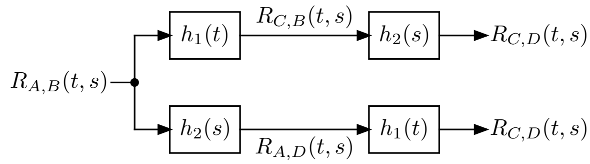
The cross-correlation of input and output and the auto-correlation of the output can be computed via application of the LTI filter as well. First, we give a general lemma.

3.2.1 Special Lemma on Correlation

Let $A(t)$ and $B(t)$ be 2nd order rps. Let $h_1(t)$ and $h_2(t)$ be BIBO stable impulse responses. Generate two additonal rps via:

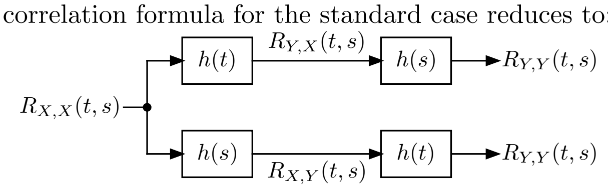
$$\begin{aligned} C(t) &= h_1 * A(t) \\ D(t) &= h_2 * B(t). \end{aligned}$$

Then the cross-correlation of the outputs is obtained via:



3.2.2 The Standard Correlation Formula

Let $A \equiv B \stackrel{\text{def}}{=} X$ and $h_1 \equiv h_2 \stackrel{\text{def}}{=} h$. Then the correlation formula for the standard case reduces to:



3.3 Wide Sense Stationary RPs

To the assumption of finite variance used in the previous sections we here add the assumption that the mean functions are independent of time and that correlations and cross correlations depend only upon the time difference or time lag. A single process with this property is called wide sense stationary (WSS); for a pair of rps we use the term jointly wide sense stationary (JWSS).

In symbols, rps $X(\cdot)$, $Y(\cdot)$ are JWSS if $\mu_X(t) \equiv \mu_X$, $\mu_Y(t) \equiv \mu_Y$ for all times $t \in \mathcal{R}$ and

$$R_{X,Y}(t,s) = R_{X,Y}(t+\lambda, s+\lambda)$$

for all times t, s, $\lambda \in \mathcal{R}$. This means that the auto-correlation function really only depends upon the time lag $\tau = s - t$ between the two time samples.

When JWSS one typically redefines the notation as shown below:

- Mean: $\mu_X = \mathrm{E}[X(t)]$.
- Autocorrelation: $R_X(\tau) = \mathrm{E}[X(t)X(t+\tau)]$.
- Autocovariance: $C_X(\tau) = R_X(\tau) - \mu_X^2$.
- Cross-correlation $R_{X,Y}(\tau) = \mathrm{E}[X(t)Y(t+\tau)]$.
- Cross-covariance $C_{X,Y}(\tau) = R_{X,Y}(\tau) - \mu_X\mu_Y$.

Then the following definitions make sense:

- Power:

$$\begin{aligned}\text{power}[X(t)] &\stackrel{\text{def}}{=} R_X(0)\\ &= C_X(0)+\mu_X^2\\ &= \text{ ac power } + \text{ dc power}\end{aligned}$$

- Power spectral density ($S_X(f)$):

$$R_X(\tau) \leftrightarrow S_X(f);$$

$$\text{power}[X(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\mathrm{d}f.$$

3.4 WSS and LTI Systems

WSS $X(t)$ input to LTI system with $h(t) \leftrightarrow H(f)$. Then output $Y(t) = X * h(t)$ is WSS, $X(t)$ and $Y(t)$ are jointly WSS, and:

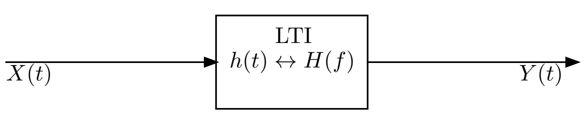
- $\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)\mathrm{d}t = \mu_X H(0)$.
- Crosscorrelation / cross spectral density

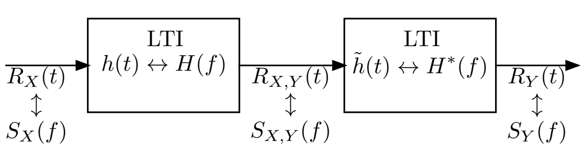
$$\begin{aligned}R_{X,Y}(\tau) &= h * R_X(\tau)\\ &\updownarrow\\ S_{X,Y}(f) &= H(f)S_X(f).\end{aligned}$$

- Autocorrelation / psds:

$$\begin{aligned}R_Y(\tau) &= \tilde{h} * h * R_X(\tau)\\ &\updownarrow\\ S_Y(f) &= |H(f)|^2 S_X(f).\end{aligned}$$

where $\tilde{h}(t) \stackrel{\text{def}}{=} h(-t)$.





3.5 Theorem on Modulation

$A(t)$, $B(t)$ jointly WSS. Θ r.v. uniform on $[0, 2\pi)$, statistically independent of $A(t)$, $B(t)$:

Thm Part A: Then $X(t) = A(t)\cos(2\pi f_c t + \Theta)$ is WSS with $\mu_X = 0$ and

$$\begin{aligned}R_X(\tau) &= 0.5R_A(\tau)\cos(2\pi f_c\tau)\\ &\updownarrow\\ S_X(f) &= 0.25[S_A(f-f_c)+S_A(f+f_c)]\end{aligned}$$

Thm Part B: If $R_A(\tau) = R_B(\tau)$ and $R_{A,B}(\tau) = -R_{B,A}(\tau)$, then

$$X(t) = A(t)\cos(2\pi f_c t) - B(t)\sin(2\pi f_c t)$$

has

$$\begin{aligned}R_X(\tau) &= R_A(\tau)\cos(2\pi f_c\tau)\\ &\qquad - R_{A,B}(\tau)\sin(2\pi f_c\tau)\\ &\updownarrow\\ S_X(f) &= 0.5[S_A(f-f_c)+S_A(f+f_c)]\\ &\quad + j0.5[S_{A,B}(f-f_c)-S_{A,B}(f+f_c)].\end{aligned}$$

Moreover, if $A(t)$, $B(t)$ are zero mean, then $X(t)$ has zero mean and is WSS.

Thm Part C: Then

$$X(t) = A(t)\cos(2\pi f_c t + \Theta) - B(t)\sin(2\pi f_c t + \Theta)$$

is zero mean, WSS with

$$\begin{aligned}R_X(\tau) &= 0.5[R_A(\tau)+R_B(\tau)]\cos(2\pi f_c\tau)\\ &\quad - 0.5[R_{A,B}(\tau)-R_{A,B}(-\tau)]\sin(2\pi f_c\tau)\\ &\updownarrow\\ S_X(f) &= 0.25[S_A(f-f_c)+S_B(f-f_c)\\ &\quad + S_A(f+f_c)+S_B(f+f_c)]\\ &\quad + j0.25[S_{A,B}(f-f_c)-S_{A,B}(f+f_c)\\ &\quad - S_{A,B}(-f+f_c)+S_{A,B}(-f-f_c)]\end{aligned}$$

3.5.1 Gaussian RPs

- $X(t)$ is a Gaussian r.p. if any finite collection of time samples from the process: $X(t_1)$, $X(t_2)$, …, $X(t_N)$ is a set of jointly Gaussian random variables.
- If input to LTI system is WSS Gaussian r.p., then output is WSS Gaussian. Moreover, input and output are jointly Gaussian r.p.s.
- $X(t)$ WSS and Gaussian. If $C_X(\tau_*) = 0$, then $X(t)$ and $X(t+\tau_*)$ are statistically independent for any t.
- $X(t)$, $Y(t)$ jointly WSS and jointly Gaussian. If $C_{X,Y}(\tau_*) = 0$, then $X(t)$ and $Y(t+\tau_*)$ are statistically independent for any t.

3.5.2 AWGN

- WSS Gaussian r.p. $N(t)$ with zero mean and autocorrelation / psd

$$R_N(\tau) = \frac{N_0}{2}\delta(\tau)$$

$$\updownarrow$$

$$S_N(f) = \frac{N_0}{2} \quad \text{for } -\infty < f < \infty$$

said to be a white Gaussian noise (WGN). When $N(t)$ appears in a problem added to a desired signal, we call it additive white Gaussian noise (AWGN).

A Basic Math

A.1 Trig. Identities

$$
\begin{aligned}
e^{j\alpha} &= \cos(\alpha) + j\sin(\alpha) \\
\cos(\alpha) &= \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}) \\
\sin(\alpha) &= \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}) \\
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\
\sin(\alpha)\sin(\beta) &= \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta) \\
\cos(\alpha)\cos(\beta) &= \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta) \\
\sin(\alpha)\cos(\beta) &= \frac{1}{2}\sin(\alpha-\beta) + \frac{1}{2}\sin(\alpha+\beta) \\
\sin^2(\alpha) &= \frac{1}{2}[1 - \cos(2\alpha)] \\
\cos^2(\alpha) &= \frac{1}{2}[1 + \cos(2\alpha)]
\end{aligned}
$$

A.2 Expansions/Sums

$$
\begin{aligned}
\exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\sum_{n=0}^{\infty} a^n &= \frac{1}{1-a} \text{ if } |a| < 1 \\
\sum_{n=0}^{N} a^n &= \frac{1-a^{N+1}}{1-a} \text{ if } a \neq 1 \\
\sum_{k=0}^{N} k &= \frac{N(N+1)}{2}
\end{aligned}
$$

A.3 Taylor Series

$$
\begin{aligned}
f(x) = {} & f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\
& \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots
\end{aligned}
$$

A.4 Integration by Parts

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

A.5 Partial Fraction Expansions

A.5.1 Method Suitable for Inverse CTFT

+ Assume that $G(v)$ is a strictly proper rational function, i.e., its numerator degree is less than its denominator degree. Suppose the denominator of $G(v)$ factors into distinct roots with multiplicities shown:

$$\text{denominator} = (v - v_1)^{m_1}(v - v_2)^{m_2} \cdots (v - v_r)^{m_r}.$$

+ Then

$$G(v) = \sum_{i=1}^{r} \sum_{k=1}^{m_i} \frac{A_{ik}}{(v - v_i)^k}$$

where

$$A_{ik} = \frac{1}{(m_i - k)!} \left[\frac{\mathrm{d}^{m_i - k}}{\mathrm{d}v^{m_i - k}} \left\{ (v - v_i)^{m_i} G(v) \right\} \right] \bigg|_{v = v_i}.$$

+ The simplest case is when a root has multiplicity one, say $m_1 = 1$, then the partial fraction expansion for $G(v)$ contains a term

$$\frac{A}{v - v_1}$$

where

$$A = (v - v_1)G(v)|_{v=v_1}.$$

A.5.2 Method Suitable for Inverse DTFT

+ The partial fraction expansion formula given previously will also work for inverting discrete-time Fourier transforms. However, in order to make the final result correspond to the discrete-time transform tables, the terms in the expansion should be rewritten as

$$\frac{A}{v - v_1} = \frac{-v_1^{-1}A}{1 - v_1^{-1}v}.$$

B Deterministic Autocorrelation and Power Spectral Density

B.1 Energy Signals

- $x(t) \leftrightarrow X(f)$ of finite energy.
- Autocorrelation: $R_x(\tau) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)x(t+\tau)\mathrm{d}t$.
- Energy Density Spectrum: $S_x(f) = |X(f)|^2$.
- Fact: $R_x(\tau) \leftrightarrow S_x(f)$.

B.2 Power Signals

- If CTFT exists denote it: $x(t) \leftrightarrow X(f)$.
- Time Average Operator: For an arbitrary function $f(t)$

$$\langle f(t) \rangle \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)\mathrm{d}t$$

  ($\langle f(t) \rangle$ is not a function of t, notation is used only to show the averaging variable).

- Autocorrelation: $R_x(\tau) \stackrel{\text{def}}{=} \langle x(t)x(t+\tau) \rangle$.
- Properties of autocorrelation:
  * $R_x(0) \geq |R_x(\tau)|$.
  * $R_x(\tau) = R_x(-\tau)$.
  * $\lim_{|\tau|\to\infty} R_x(\tau) = \langle x(t) \rangle^2$ if $x(t)$ does not contain periodic components.
  * If $x(t)$ is periodic with period T_0 then so is $R_x(\tau)$.
  * CTFT of $R_x(\tau)$ is non-negative for all f.
- Power Density Spectrum: Defined to be the CTFT of the autocorrelation:

$$R_x(\tau) \leftrightarrow S_x(f).$$

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