# ECE 645 – Discrete Time Signal Detection

J. V. Krogmeier

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1
1 Models and Architectures

Have in mind the scenario where an observed continuous-time waveform consists of one of two possible signals corrupted by additive noise. The objective is to decide which of the two signals is present by processing $n$ samples of the observed waveform\(^1\).

- **Observation space:** $(\Gamma, \mathcal{G}) = (\mathbb{R}^n, \mathcal{B}^n)$.

- **Hypothesis pair:**
  \[
  H_0 : \quad Y_k = N_k + S_{0k} \quad 1 \leq k \leq n
  \]
  vs.
  \[
  H_1 : \quad Y_k = N_k + S_{1k} \quad 1 \leq k \leq n
  \]
  where
  \[
  * \quad Y = [Y_1, \ldots, Y_n]^T \text{ is observation vector consisting of samples of observed waveform.}
  \]
  \[
  * \quad N = [N_1, \ldots, N_n]^T \text{ vector of noise samples.}
  \]
  \[
  * \quad S_0 = [S_{01}, \ldots, S_{0n}]^T \text{ and } S_1 = [S_{11}, \ldots, S_{1n}]^T \text{ are vectors of samples of the two possible signals.}
  \]

\(^1\)We do not restrict to cases where the samples are taken from a time waveform, samples could also be a spatial snapshot, filter bank output, etc.
• **Signal characterization** is usually one of the three types:

  1. $S_0$ and $S_1$ deterministic and known.
  2. $S_0$ and $S_1$ known except for a set of unknown and possibly random parameters.
  3. $S_0$ and $S_1$ completely random and specified only by their probability distributions.

In addition, for some problems (e.g., radar, sonar) one of the signals $S_0 \equiv 0$ and then the testing problem is one of detecting a signal in noise.

• **Noise characterization**

  1. Noise independent of the signals under each hypothesis.
  2. Probability distribution of the noise is the same for both hypotheses (e.g., noise is signal independent\(^2\)).
  3. Noise distribution given by a density (discrete or continuous) $f_N(\cdot)$ on $\mathcal{R}^n$.

• **Likelihood ratio**

  1. Under $H_0$ the conditional pdf of $Y$ given $S_0 = s_0$ is

  $f_0(y|S_0 = s_0) = f_N(y - s_0)$.

\(^2\)In some problems the noise does depend on the signal, e.g., speckle, clutter, multipath.
Averaging over the distribution of $S_0$ we have $f_0(y) = \mathbb{E}\{f_N(y - S_0)\}$. In the same way, under $H_1$, we have $f_1(y) = \mathbb{E}\{f_N(y - S_1)\}$.

2. In the above the expectations are wrt the random signals $S_0$, $S_1$ according to the case. We have used the independence of signal and noise and the fact that the noise has the same distribution under $H_0$ as under $H_1$.

3. Then the general likelihood ratio is

$$L(y) = \frac{f_1(y)}{f_0(y)} = \frac{\mathbb{E}\{f_N(y - S_1)\}}{\mathbb{E}\{f_N(y - S_0)\}}.$$ 

Therefore, optimal tests for this problem are based on computing $L(y)$ and finding appropriate probabilities of the resulting critical regions.

2 Examples

In the examples to follow we use the notation

$$\tau = \frac{\pi_0(c_{10} - c_{00})}{\pi_1(c_{01} - c_{11})}$$

to denote the standard threshold for the binary Bayesian hypothesis test. We will use $\eta$ to denote a generic threshold as usually set in NP testing to meet a desired false alarm rate.
2.1 Deterministic Signals in Independent Noise

- Signals deterministic and known. Therefore, likelihood ratio expressed as
  \[ L(y) = \frac{f_N(y - s_1)}{f_N(y - s_0)} = \frac{f_N(y_1 - s_{11}, y_2 - s_{12}, \ldots, y_n - s_{1n})}{f_N(y_1 - s_{01}, y_2 - s_{02}, \ldots, y_n - s_{0n})} \]

- Since the noise components \( N_1, N_2, \ldots, N_n \) are statistically independent the joint pdf factors as \( f_N(n) = \prod_{k=1}^n f_{N_k}(n_k) \). Similarly for the likelihood ratio
  \[ L(y) = \prod_{k=1}^n L_k(y_k) \quad \text{where} \quad L_k(y_k) = \frac{f_{N_k}(y_k - s_{1k})}{f_{N_k}(y_k - s_{0k})}. \]

- Taking the logarithm we see that optimal tests have the form shown in the block diagram below.
2.2 Coherent Detection in IID Gaussian Noise

- Suppose \( N_k \sim \mathcal{N}(0, \sigma^2) \) and i.i.d.
- In addition, say that \( s_0 = 0 \) and define \( s_1 = s \) to simplify the notation.
- Then \( \ln L_k(y_k) = s_k(y_k - s_k/2)/\sigma^2 \) and the receiver architecture is of the form shown below.

![Block diagram](image)

- The centering correction in the figure above (i.e., the subtraction of \( s_k/2 \)) can more through the block diagram and be incorporated into the threshold. Thus, an equivalent block diagram is shown below. It is the classical correlator.
• As is well known the correlator above can be re-drawn as a matched filter followed by a sampler as shown. The impulse response of the matched filter is

\[ h_k = \begin{cases} 
  s_{n-k} & \text{for } 0 \leq k \leq n - 1 \\
  0 & \text{otherwise}
\end{cases} \]
2.3 Coherent Detection in IID Laplacian Noise

• Same setup as before with $s_0 = 0$, $s_1 = s$, and i.i.d. noise of Laplacian marginal density
  
  $$f_{N_k}(n_k) = \frac{\alpha}{2}e^{-\alpha|n_k|}, \quad n_k \in \mathcal{R}, \quad k = 1, 2, \ldots, n$$

  where $\alpha > 0$ is a scale factor and assumed known.

• This is sometimes used as a model for impulsive noise in receivers because it has longer tails than the Gaussian representing higher probabilities of large observations.

• Take the ratio of pdfs and cancel common terms
  
  $$L_k(y_k) = \frac{f_{N_k}(y_k - s_k)}{f_{N_k}(y_k)} = \exp\{\alpha(|y_k| - |y_k - s_k|)\}.$$

• Taking the natural log and exploring some cases shows that the non-linearity term in the standard i.i.d. architecture can be written

  $$\ln L_k(y_k) = \alpha(|y_k| - |y_k - s_k|)$$

  
  $$= \begin{cases} 
  -\alpha|s_k| & \text{sgn}(s_k)y_k \leq 0 \\
  \alpha|2y_k - s_k| & \text{if } 0 < \text{sgn}(s_k)y_k < |s_k| \\
  \alpha|s_k| & \text{sgn}(s_k)y_k \geq |s_k| 
  \end{cases}$$

  where $\text{sgn}(x) = 1$ for $x > 0$, = 0 for $x = 0$, and = −1 for $x < 0$.  


DT Signal Detection
Interestingly it can be seen that the test statistic, amounting to the accumulated values of $\ln L_k(y_k)$ can be rewritten in such a fashion that

$$
\delta(y) = \begin{cases} 
1 & \text{if } \sum_{k=1}^{n} \text{sgn}(s_k) l_k(y_k - s_k/2) = \tau' = \ln \tau/(2\alpha) \\
\gamma & < \\
0 & > 
\end{cases}
$$

where $l_k(\cdot)$ is the soft-limiter characteristic shown in the below figure.

- Compare with the architecture for i.i.d. Gaussian noise and note that soft-limiting reduces the effect that large observations have upon the sum.
2.4 Deterministic Signals in Gaussian Noise

(To fill in yet.)