1. Consider a sequence \( \{X_k\}_{k=0}^{\infty} \) of binary random variables, each taking on the values 0 or 1. Suppose that this sequence has the Markov property

\[
P\{X_k = x_k | X_0 = x_0, \ldots, X_{k-1} = x_{k-1}\} = P\{X_k = x_k | X_{k-1} = x_{k-1}\} \overset{\text{def}}{=} p_{x_k, x_{k-1}}
\]

for all integers \( k \geq 1 \), and for all binary sequences \( \{x_k\}_{k=0}^{\infty} \). Consider the observation model

\[
Y_k = X_k + N_k, \quad k = 0, 1, 2, \ldots
\]

where \( \{N_k\}_{k=0}^{\infty} \) is an i.i.d. sequence, independent of \( \{X_k\}_{k=0}^{\infty} \), and having the common marginal probability density function \( f \). For each integer \( k \geq 0 \), let \( \hat{X}_{k/k} \) denote the MMSE estimate of \( X_k \) given measurements \( \{Y_0, Y_1, \ldots, Y_k\} \), and let \( \hat{X}_{k/k-1} \) denote the MMSE estimate of \( X_k \) given measurements \( \{Y_0, Y_1, \ldots, Y_{k-1}\} \). Define

\[
\hat{X}_{0/-1} = E\{X_0\} = P\{X_0 = 1\}.
\]

Show that \( \hat{X}_{k/k} \) and \( \hat{X}_{k/k-1} \) satisfy the joint recursions

\[
\hat{X}_{k/k} = \frac{\hat{X}_{k/k-1} f(y_k - 1)}{\hat{X}_{k/k-1} f(y_k - 1) + (1 - \hat{X}_{k/k-1}) f(y_k)}
\]

\[
\hat{X}_{k+1/k} = p_{1,1} \hat{X}_{k/k} + p_{1,0} (1 - \hat{X}_{k/k})
\]

for \( k \geq 0 \).

2. Suppose that the state equation in the Kalman filter model is modified as follows:

\[
X_{k+1} = F_k X_k + G_k W_k + \Gamma_k s_k \quad k \geq 0
\]

where \( s_k \) is a known sequence of vectors (a control input) and \( \Gamma_k \) is a known sequence of matrices of appropriate dimension.

(a) Find the appropriate modification of the Kalman recursions.

(b) Repeat where each \( s_k \) is allowed to be a function of the past measurement, i.e. of \( Z(k) \) (feedback control).

3. Let \( L(\cdot) \) be a scalar function with \( L(0) = 0, L(y) \geq L(z) \) for \( \|y\| \geq \|z\| \), \( L(y) = L(-y) \), and with \( L(\cdot) \) convex. Let \( p_{X|Y}(x|y) \) be symmetric about \( \hat{x} = E\{X|Y = y\} \). Prove that for all \( z \)

\[
E\{L(X - \hat{x})|Y = y\} \leq E\{L(X - z)|Y = y\}.
\]

4. With \( P_k = E\{X_k X_k^T\} \) and assuming the standard Kalman filter setup from class show that

\[
P_{k+1} - \Sigma_{k+1|k} \geq 0
\]

and interpret this result.

5. Let \( P_k \) be a sequence of nonnegative definite matrices such that for some nonnegative definite matrix \( P \) and for all \( k \)

\[
P \geq P_{k+1} \geq P_k.
\]

Show that the limit \( \lim_{k \to \infty} P_k \) exists.
6. Suppose that $X \sim \mathcal{N}(\mu_x, \sigma^2_x)$, that $\{V_k\}_{k=1}^N$ is i.i.d. with $V_k \sim \mathcal{N}(0, \sigma^2_v)$, and that the $V_k$ are independent of $X$. Then given

$$Y_k = X + V_k, \; 1 \leq k \leq N,$$

find the posterior density $f(x|y_1, y_2, \ldots, y_N)$.

7. Consider the following Bayesian estimation problem. Observations of the following form are made of a random variable $X$

$$Y_k = X + (1 + \alpha D)V_k, \; 1 \leq k \leq N,$$

where $\alpha > 0$, and the statistical assumptions are that:

- $X \sim \mathcal{N}(\mu_x, \sigma^2_x)$ and the measurement noise is i.i.d. with $V_k \sim \mathcal{N}(0, \sigma^2_v)$.
- $D$ is a zero-one random variable with $P\{D = 1\} = q$.
- The random variables $X$, $D$, and $\{V_k\}_{k=1}^N$ are statistically independent.

The model is for a measurement scenario where two sensors are available to make measurements of a quantity. One sensor is “good” (variance $\sigma^2_v$) and the other sensor is “bad” (variance $(1 + \alpha)^2\sigma^2_v$). We do not know apriori which sensor has been used.

(a) The minimum mean-squared error estimator of $X$ given $Y = [Y_1, Y_2, \ldots, Y_N]$ is of the form

$$E\{X|Y = y\} = C_0(y)L_0(y) + C_1(y)L_1(y)$$

where the $L_i(y)$ are linear functions and the coefficients $C_i(y)$ are nonlinear functions.

(a1) Explain how the formula above comes about (Hint: condition on the value of $D$).

(a2) Find explicit formulas for $L_0(y)$ and $L_1(y)$.

(a3) Demonstrate how to get the coefficients $C_0(y)$ and $C_1(y)$ but don’t carry out the calculations.

(b) Derive the complete formula for the minimum mean-squared error linear estimator of $X$ given $Y$. 