Modified
(Delay P5, P7 to Prob. Set 3)

ECE 645 Spring 2014
Problem Set 2
Due February 14, 2014

1. Let \( \{Y_i : 1 \leq i \leq M\} \) be i.i.d. with distribution function \( F_Y \) and density \( f_Y \) and let \( \{Z_i : 1 \leq i \leq M\} \) be the order statistics of the sample. That is
   
   \[
   \begin{align*}
   Z_1 &= \text{smallest of } Y_1, Y_2, \ldots, Y_M \\
   Z_2 &= \text{second smallest of } Y_1, Y_2, \ldots, Y_M \\
   &\vdots \\
   Z_j &= \text{\( j \)-th smallest of } Y_1, Y_2, \ldots, Y_M \\
   &\vdots \\
   Z_M &= \text{largest of } Y_1, Y_2, \ldots, Y_M
   \end{align*}
   \]

   Prove the following formulas:
   
   \[
   \begin{align*}
   F_{Z_r}(x_r) &= \sum_{j=r}^{M} \binom{M}{j} [F_Y(x_r)]^j [1 - F_Y(x_r)]^{M-j} \\
   f_{Z_r}(x_r) &= \frac{M!}{(r-1)!(M-r)!} [F_Y(x_r)]^{r-1} [1 - F_Y(x_r)]^{M-r} f_Y(x_r) \\
   f_{Z_r,Z_s}(x_r,x_s) &= \frac{M!}{(r-1)!(s-r-1)!(M-s)!} [F_Y(x_r)]^{r-1} [F_Y(x_s) - F_Y(x_r)]^{s-r-1} \\
   &\times [1 - F_Y(x_s)]^{M-s} f_Y(x_r)f_Y(x_s), \ 1 \leq r < s \leq M.
   \end{align*}
   \]

2. Say \( X \sim C(a,b) \) if it has the Cauchy density
   
   \[
   f(x) = \frac{1}{\pi b^2 + (x-a)^2}.
   \]

   (a) Let \( X_i (i = 1, 2) \) be independently distributed according to the Cauchy densities \( C(a_i, b_i) \). Prove that \( X_1 + X_2 \) is distributed as \( C(a_1 + a_2, b_1 + b_2) \).

   (b) Prove that if \( X_1, \ldots, X_n \) are iid and distributed as \( C(a,b) \) then the distribution of the sample mean is again \( C(a,b) \). What does this say about the sample mean as an estimator in the Cauchy case?

3. Suppose that \( Y \) is a random variable which, under hypothesis \( H_0 \), has pdf
   
   \[
   f_0(y) = \begin{cases} 
   (2/3)(y+1) & 0 \leq y \leq 1 \\
   0 & \text{otherwise}
   \end{cases}
   \]

   and, under hypothesis \( H_1 \), has pdf
   
   \[
   f_1(y) = \begin{cases} 
   1 & 0 \leq y \leq 1 \\
   0 & \text{otherwise}
   \end{cases}
   \]

   Find the Bayes rule and minimum Bayes risk for testing \( H_0 \) versus \( H_1 \) with uniform costs and equal priors.

4. Consider the hypothesis pair
   
   \[
   \begin{align*}
   H_0 : \ Y &\sim N \\
   H_1 : \ Y &\sim N + S
   \end{align*}
   \]

   where \( N \) and \( S \) are independent random variables each having pdf
   
   \[
   p(x) = \begin{cases} 
   e^{-x} & x \geq 0 \\
   0 & x < 0
   \end{cases}
   \]
(a) Find the likelihood ratio between \( H_0 \) and \( H_1 \).

(b) Find the threshold and detection probability for \( \alpha \)-level Neyman–Pearson testing of \( H_0 \) vs. \( H_1 \).

Now consider the hypothesis pair

\[
H_0 : \quad Y_k = N_k, \quad k = 1, 2, \ldots, n \\
H_1 : \quad Y_k = N_k + S, \quad k = 1, 2, \ldots, n
\]

where \( n > 1 \) and \( N_1, \ldots, N_n \) and \( S \) are independent random variables each having the pdf \( p(x) \) above.

(c) Find the likelihood ratio between \( H_0 \) and \( H_1 \).

(d) Find the threshold and detection probability for \( \alpha \)-level Neyman–Pearson testing of \( H_0 \) vs. \( H_1 \).

5. Consider the composite hypothesis testing problem:

\[
H_0 : \quad Y \text{ has density } p_0(y) = (1/2)e^{-|y|}, \quad y \in \mathbb{R} \\
H_1 : \quad Y \text{ has density } p_0(y) = (1/2)e^{-|y-\theta|}, \quad y \in \mathbb{R}, \quad \theta > 0
\]

(a) Describe the locally most powerful \( \alpha \)-level test and derive its power function.

(b) Does a uniformly most powerful test exist? If so, find it and derive its power function. If not, find the generalized likelihood ratio test for \( H_0 \) versus \( H_1 \).

6. Consider the following pair of hypotheses concerning a sequence \( Y_1, Y_2, \ldots, Y_n \) of random variables

\[
H_0 : \quad Y_k \sim \mathcal{N}(\mu_0, \sigma_0^2), \quad k = 1, 2, \ldots, n \\
H_1 : \quad Y_k \sim \mathcal{N}(\mu_1, \sigma_1^2), \quad k = 1, 2, \ldots, n
\]

where \( \mu_0, \mu_1, \sigma_0^2, \) and \( \sigma_1^2 \) are known constants.

(a) Show that the likelihood ratio can be expressed as a function of the parameters \( \mu_0, \mu_1, \sigma_0^2, \) and \( \sigma_1^2 \), and the quantities \( \sum_{k=1}^n Y_k^2 \) and \( \sum_{k=1}^n Y_k \).

(b) Describe the Neyman–Pearson test for the two cases \( (\mu_0 = \mu_1, \sigma_0^2 < \sigma_1^2) \) and \( (\sigma_0^2 = \sigma_1^2, \mu_1 > \mu_0) \).

(c) Find the threshold and ROCs for the case \( \mu_0 = \mu_1, \sigma_1^2 > \sigma_0^2 \) with \( n = 1 \).

7. Consider the hypotheses of the previous problem with \( \mu = \mu_1 > \mu_0 = 0 \) and \( \sigma^2 = \sigma_0^2 = \sigma_1^2 > 0 \). Does there exist a uniformly most powerful test of these hypotheses under the assumption that \( \mu \) is known and \( \sigma^2 \) is not? If so, find it and show that it is UMP. If not, show why and find the generalized likelihood ratio test.

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Delay to Prob. Set 3 due Feb. 21.
Let \( \{Y_i : 1 \leq i \leq M\} \) be i.i.d. with distribution function \( F_Y \) and density \( f_Y \) and let \( \{Z_i : 1 \leq i \leq M\} \) be the order statistics of the sample. That is
\[
Z_1 = \text{smallest of } Y_1, Y_2, \ldots, Y_M \\
Z_2 = \text{second smallest of } Y_1, Y_2, \ldots, Y_M \\
\vdots \\
Z_j = j\text{-th smallest of } Y_1, Y_2, \ldots, Y_M \\
\vdots \\
Z_M = \text{largest of } Y_1, Y_2, \ldots, Y_M
\]
Prove the following formulas:
\[
F_{Z_r}(z_r) = \sum_{j=r}^{M} \binom{M}{j} [F_Y(z_r)]^j [1 - F_Y(z_r)]^{M-j}
\]
\[
f_{Z_r}(z_r) = \frac{M!}{(r-1)!(M-r)!} [F_Y(z_r)]^{r-1} [1 - F_Y(z_r)]^{M-r} f_Y(z_r)
\]
\[
f_{Z_r,Z_s}(z_r,z_s) = \frac{M!}{(r-1)!(s-r-1)!(M-s)!} [F_Y(z_r)]^{r-1} [F_Y(z_s) - F_Y(z_r)]^{s-r-1} [1 - F_Y(z_s)]^{M-s} f_Y(z_r) f_Y(z_s), \ 1 \leq r < s \leq M.
\]
Order Statistics

\{ Y_i : 1 \leq i \leq M \} iid with dist. funct. F_Y and density f_Y. Define the order statistics \{ Z_i : 1 \leq i \leq M \} by

\[ Z_i = i \text{-th smallest of } Y_1, Y_2 \ldots Y_M \]

Recall

\[ F_{Z_i}(z_i) = P\{ Z_i \leq z_i \} \]

and note

\[ \{ Z_i \leq \alpha \} = \{ i \text{ or more of the } Y_k \text{ are } \leq \alpha \} \]

We can relate this to the number of successes in M Bernoulli trials where for a fixed \( \alpha \) define

\[ E_k = \begin{cases} 1 & \text{if } Y_k \leq \alpha \\ 0 & \text{if } Y_k > \alpha \end{cases} \quad \Rightarrow \quad P\{ E_k = 1 \} = F_Y(\alpha) \]

The number of successes in M independent Bernoulli trials is a Binomial random variable where

\[ P\left\{ \sum_{k=1}^{M} E_k = N \right\} = \binom{M}{N} [F_Y(\alpha)]^N [1 - F_Y(\alpha)]^{M-N} \]

Then

\[ \{ Z_i \leq \alpha \} = \{ \# \text{ of successes} \geq i \} \]

\[ \therefore \quad P\{ Z_i \leq \alpha \} = F_{Z_i}(\alpha) = \sum_{j=i}^{M} \binom{M}{j} [F_Y(\alpha)]^j [1 - F_Y(\alpha)]^{M-j} \quad (\ast) \]

This is the first formula.
To get the density $f_{Z_i}(x)$ note (from definition of derivative)

$$f_{Z_i}(x) = \lim_{h \to 0} \frac{F_{Z_i}(x+h) - F_{Z_i}(x)}{h} = \lim_{h \to 0} \frac{P[x < Z_i \leq x+h]}{h}$$

In the above it is enough to take the one sided limit since we already know $F_{Z_i}$ is differentiable. Note that we are simply trying to find a way to compute $f_{Z_i}$ which is easier than just taking the derivative of $F_{Z_i}$. Let's look more closely at the event

$$\{x < Z_i \leq x+h\} = \{x < Z_i\} \cap \{Z_i \leq x+h\}$$

The order statistics are ordered $Z_1 \leq Z_2 \leq \cdots \leq Z_{i-1} \leq Z_i \leq Z_{i+1} \leq \cdots \leq Z_M$.

- $x < Z_i \iff$ at least $M-i+1$ of the $Y_k$ are $> x$
- $Z_i \leq x+h \iff$ at least $i$ of the $Y_k$ are $\leq x+h$

From counting the number of $Y_k$ which fall into the two sets above we conclude that at least one of them must fall into the interval $(x, x+h]$. There may be more and in fact all of them could fall in $(x, x+h]$ and still be consistent with $\{x < Z_i \leq x+h\}$.

Let $A$ be the event that $i-1$ of the $Y_k$ lie in $(-\infty, x]$, one lies in $(x, x+h]$ and $M-i$ lie in $(x+h, \infty)$. In taking the limit as $h \to 0$ it will be easier to work with $A$. We will see this and then justify our looking at $A$ rather than $\{x < Z_i \leq x+h\}$. 

\[\text{Diagram:}\]

\[\begin{array}{c}
\alpha \\
| \quad | \\
\alpha+h \\
| \quad | \\
M-i+1 \quad \text{or more of } Y_k \\
in \text{ here}
\end{array}\]
Probability of A

For any of the individual random variables $Y_k$

$$P\{Y_k \leq \alpha\} = F_Y(\alpha)$$
$$P\{Y_k > \alpha + h\} = 1 - F_Y(\alpha + h)$$
$$P\{\alpha < Y_k \leq \alpha + h\} = F_Y(\alpha + h) - F_Y(\alpha)$$

To get the prob. of the event A we should be looking at the multinomial distribution. The experiment consists 3 possible outcomes at stage $k$

$$O_1 = \{Y_k \leq \alpha\}$$
$$O_2 = \{Y_k > \alpha + h\}$$
$$O_3 = \{\alpha < Y_k \leq \alpha + h\}$$

The experiment is repeated $M$ times and each run is indep of the others. Then the event A is equivalent to the event

$O_1$ happens $i-1$ times, $O_2$ happens $M-i$ times, and $O_3$ happens once.

From the multinomial distribution

$$P(A) = \frac{M!}{(i-1)! (M-i)!} (F_Y(\alpha))^{i-1} (1-F_Y(\alpha + h))^{M-i} (F_Y(\alpha + h) - F_Y(\alpha))$$

Now notice that

$$P(A) = \frac{M!}{n} \frac{1}{(i-1)! (M-i)!} (F_Y(\alpha))^{i-1} (1-F_Y(\alpha + h))^{M-i} \frac{F_Y(\alpha + h) - F_Y(\alpha)}{h}$$

and if we take the limit as $h \to 0$ of this we obtain

$$\lim_{h \to 0} \frac{P(A)}{n} = \frac{M!}{(i-1)! (M-i)!} (F_Y(\alpha))^{i-1} (1-F_Y(\alpha))^{M-i} F_Y(\alpha)$$

We have obtained the desired formula... however we still need to show

$$\lim_{h \to 0} \frac{P(A)}{n} = \lim_{h \to 0} \frac{P\{\alpha < Z_i \leq \alpha + h\}}{h} \quad \text{(***)}$$
Remember that we argued that for the event \( \{x < Z_i \leq x+h\} \) to happen at least one of the \( Y_k \) must lie in the interval \((x, x+h]\). Hence \( A \subseteq \{x < Z_i \leq x+h\} \) and
\[
P\left(\{x < Z_i \leq x+h\}\right) - P(A) \geq 0
\]

If the event \( \{x < Z_i \leq x+h\}\) \( A \) is to happen, then two or more of the \( Y_k \) must fall into the interval \((x, x+h]\). From binomial distribution argument
\[
0 \leq P\left(\{x < Z_i \leq x+h\}\right) - P(A) = P\left(\{x < Z_i \leq x+h\}\right | A) \leq
\]
\[
\sum_{l=2}^{M} \binom{M}{l} (F_Y(x+h)-F_Y(x))^l (1-F_Y(x+h)+F_Y(x))^{M-l}
\]

Limit of this as \( h \to 0 \) is zero because
\[
\lim_{h \to 0} (F_Y(x+h)-F_Y(x)) = 0
\]
and all exponents \( l \) above are non-zero.

Conclude: we have shown (**).
\[
\therefore \int_{Z_i} f_{Z_i}(x) = \frac{M!}{(i-1)!(M-i)!} (F_Y(x))^{i-1} (1-F_Y(x))^{M-i} f_Y(x)
\]
Direct Differentiation Approach

\[
 f_{Z_i}(x) = \frac{d}{d\alpha} f_{Z_i}(x) = \sum_{j=1}^{M} \binom{M}{j} \frac{d}{d\alpha} \left\{ [F_Y(\alpha)]^j [1 - F_Y(\alpha)]^{M-j} \right\}
\]

If \(1 \leq j < M\)

\[
 \frac{d}{d\alpha} \left\{ [F_Y(\alpha)]^j [1 - F_Y(\alpha)]^{M-j} \right\} = j [F_Y(\alpha)]^{j-1} F_Y(\alpha) [1 - F_Y(\alpha)]^{M-j}
\]

\[
 - (M-j) [F_Y(\alpha)]^j [1 - F_Y(\alpha)]^{M-j-1} F_Y(\alpha)
\]

If \(j = M\)

\[
 \frac{d}{d\alpha} \left[ [F_Y(\alpha)]^M [1 - F_Y(\alpha)]^{M-j} \right] = \frac{d}{d\alpha} [F_Y(\alpha)]^M = M [F_Y(\alpha)]^{M-1} F_Y(\alpha)
\]

Thus

\[
 f_{Z_i}(x) = \sum_{j=1}^{M} \binom{M}{j} \left\{ j [F_Y(\alpha)]^{j-1} [1 - F_Y(\alpha)]^{M-j} - (M-j) [F_Y(\alpha)]^j [1 - F_Y(\alpha)]^{M-j-1} \right\} f_Y(\alpha)
\]

\[
 + \binom{M}{j} M [F_Y(\alpha)]^{M-1} F_Y(\alpha)
\]

We would need some serious combinatorial identities to reduce this to the expected form (or so it seems). For starters let's mess around:

\[
 \binom{M}{j} = \frac{M!}{j! (M-j)!} \Rightarrow \binom{M}{j} j = \frac{M!}{(j-1)! (M-j)!}
\]

\[
 \binom{M}{j} (M-j) = \frac{M!}{j! (M-j-1)!}
\]

Stripping out one of the terms (the one we want)

\[
 f_{Z_i}(x) = \frac{M!}{(i-1)! (M-i)!} [F_Y(\alpha)]^{i-1} [1 - F_Y(\alpha)]^{M-i} f_Y(\alpha)
\]

\[
 + \sum_{j=i+1}^{M-1} \binom{M}{j} \left\{ j [F_Y(\alpha)]^{j-1} [1 - F_Y(\alpha)]^{M-j} - (M-j) [F_Y(\alpha)]^j [1 - F_Y(\alpha)]^{M-j-1} \right\} f_Y(\alpha)
\]

\[
 + \binom{M}{M} M [F_Y(\alpha)]^{M-1} F_Y(\alpha)
\]
So we are done if it can be shown that:

\[- \binom{M}{i} (M-i) \left[ F_Y(x) \right]^i \left[ 1-F_Y(x) \right]^{M-i-1} + \binom{M}{i} M \left[ F_Y(x) \right]^{M-i} \]

\[+ \sum_{j=i+1}^{M-1} \binom{M}{j} \left\{ j \left[ F_Y(x) \right]^{j-1} \left[ 1-F_Y(x) \right]^{M-j} - (M-j) \left[ F_Y(x) \right]^{j} \left[ 1-F_Y(x) \right]^{M-j-1} \right\} = 0 \]

To simplify (should have done long ago) let \( p = F_Y(x) \) and \( q = [1-F_Y(x)] \). Then need to show:

\[- \binom{M}{i} (M-i) \left[ p \right]^i \left[ q \right]^{M-i-1} + \binom{M}{i} M \left[ p \right]^{M-i} \]

\[+ \sum_{j=i+1}^{M-1} \binom{M}{j} \left\{ j \left[ p \right]^{j-1} \left[ q \right]^{M-j} - (M-j) \left[ p \right]^{j} \left[ q \right]^{M-j-1} \right\} = 0 \]

So it's equivalent to showing:

\[\binom{M}{i} M \left[ p \right]^{M-i-1} + \sum_{j=i+1}^{M-1} \binom{M}{j} j \left[ p \right]^{j-1} \left[ q \right]^{M-j} = \binom{M}{i} (M-i) \left[ p \right]^{M-i-1} \]

\[+ \sum_{j=i+1}^{M-1} \binom{M}{j} M \left[ p \right]^{j} \left[ q \right]^{M-j-1} \]

Recall the Binomial Thm

\[(p+q)^M = 1^M = 1 = \sum_{j=0}^{M} \binom{M}{j} p^j q^{M-j} = \sum_{j=0}^{M} \binom{M}{j} p^j (1-p)^{M-j} \]

Note

\[\frac{d}{dp} \left\{ p^j (1-p)^{M-j} \right\} = j p^{j-1} (1-p)^{M-j} - (M-j) p^j (1-p)^{M-j-1} \]

\[= j p^{j-1} q^{M-j} - (M-j) p^j q^{M-j-1} \quad 1 \leq j < M \]

\[= -M (1-p)^{M-1} - M q^{M-1} \quad j = 0 \]

\[= M p^{M-1} \quad j = M \]
Differentiating the Binomial Thm wrt p gives the identity:

\[ 0 = \sum_{j=0}^{M} \binom{M}{j} \frac{d}{dp} \left\{ p^j (1-p)^{M-j} \right\} \]

\[ = -\binom{M}{0} M q^{M-1} + \sum_{j=1}^{M-1} \binom{M}{j} \left\{ j p^{j-1} q^{M-j} - (M-j) p^j q^{M-j-1} \right\} \]

\[ + \binom{M}{M} M p^{M-1} \]

Forget the factorization made in (**) and go back to the original way of writing it.

\[ \binom{M}{M} M p^{M-1} + \sum_{j=iN}^{M-1} \binom{M}{j} j p^{j-1} q^{M-j} \]

\[ = \binom{M}{i} (M-i) p^i q^{M-i-1} + \sum_{j=iN}^{M-1} \binom{M}{j} (M-j) p^j q^{M-j-1} \]

\[ \sum_{j=iN}^{M-1} \binom{M}{j} j p^{j-1} q^{M-j} \]

To Show

\[ \sum_{j=iN}^{M-1} \binom{M}{j} (M-j) p^j q^{M-j-1} \]

The Binomial Thm identity says that

\[ \sum_{j=1}^{M} \binom{M}{j} j p^{j-1} q^{M-j} = \sum_{j=0}^{M-1} \binom{M}{j} (M-j) p^j q^{M-j-1} \]

This indicates to me that if the identity we want to show is true \( 1 \leq i < M \) then we must have for example that

\[ \binom{M}{1} i q^{M-1} = \binom{M}{0} M q^{M-1} \]

which is trivial. This is the big hint. Now go back to what we want to show:

\[ \sum_{j=iN}^{M} \binom{M}{j} j p^{j-1} q^{M-j} = \sum_{j=iN}^{M-1} \binom{M}{j} (M-j) p^j q^{M-j-1} \]

\[ \sum_{j=iN}^{M-1} \binom{M}{j} (M-j) p^j q^{M-j-1} \]
This is also trivial because corresponding terms in the sum are equal:

\[
\begin{align*}
\left( \binom{M}{j} \right) j \frac{p^{j-1} q^{M-j}}{j!} &= \left( \binom{M}{j-1} \right) \frac{(M-j+1)}{M!} \frac{p^{j-1} q^{M-j}}{(j-1)! (M-j)!} \\
\frac{M!}{j! (M-j)!} &= \frac{M!}{(j-1)! (M-j+1)!} \\
\frac{M!}{(j-1)! (M-j)!} &= \frac{M!}{(j-1)! (M-j)!}
\end{align*}
\]
Simple and Nonrigorous Approach

This is a Papoulis-class argument. It is not rigorous but captures the essential idea. The basic observation is that the order statistics \( Z_1, \ldots, Z_M \) will take on values

\[ Z_1 < Z_2 < \cdots < Z_M \]

if and only if for some permutation \((i_1, i_2, \ldots, i_M)\) of \((1, 2, \ldots, M)\) the data \( Y_k \) take these values

\[ Y_1 = Z_{i_1}, \quad Y_2 = Z_{i_2}, \quad \ldots, \quad Y_M = Z_{i_M} \]

But for any such permutation

\[
P\left\{ \frac{\varepsilon}{2} < Y_1 < Z_{i_1} + \frac{\varepsilon}{2}, \ldots, \frac{\varepsilon}{2} < Y_M < Z_{i_M} + \frac{\varepsilon}{2} \right\}
\]

\[ \cong \varepsilon^M f_{Y_1 \cdots Y_M}(Z_{i_1}, Z_{i_2}, \ldots, Z_{i_M}) = \varepsilon^M f_Y(z_{i_1}) \cdots f_Y(z_{i_M}) \]

\[ = \varepsilon^M f_Y(z_1) \cdots f_Y(z_M) \]

Since there are \( M! \) possible permutations of \((1, 2, \ldots, M)\) we have

\[
P\left\{ \frac{\varepsilon}{2} < Z_1 < Z_{i_1} + \frac{\varepsilon}{2}, \ldots, \frac{\varepsilon}{2} < Z_M < Z_{i_M} + \frac{\varepsilon}{2} \right\}
\]

\[ \cong M! \varepsilon^M f_Y(z_1) \cdots f_Y(z_M) \]

Dividing through by \( \varepsilon^M \) and taking the limit \( \varepsilon \to 0 \) gives

\(\star\) \quad \int_{Z_1 \leq Z_2 \leq \cdots \leq Z_M} f_{Z_1 \cdots Z_M}(Z_1, \ldots, Z_M) = M! f_Y(z_1) \cdots f_Y(z_M) \quad \text{for } z_1 \leq z_2 \leq \cdots \leq z_M

\text{Should also note that if the } z_i \text{ are not ordered as }^\uparrow \text{ then the joint density above is zero at that point. The other desired densities can be obtained by integrating the above. For example:}

\[
\int_{z_i \leq z_j} f_{Z_1 \cdots Z_M}(z_i, z_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{Z_1 \cdots Z_M}(z_i, z_j) \, dz_i \, dz_M
\]

\text{dz}_i \text{ and } \text{dz}_j \text{ left out.}
Now must be careful in this integration because $f_{z_1 \ldots z_M}$ is only non-zero on the set (and that is where the formula (4) is valid)

$$\{ z_k \leq k \leq M \mid z_1 \leq z_2 \leq \ldots \leq z_M \}$$

which for $M=2$ is

We want to write the integral for $f_{z_i z_j} (z_i, z_j) \quad (i < j)$ as an iterated integral. Think of $z_i$ and $z_j$ as fixed for the moment and state $z_i \leq z_j$. The order stats statistics are ordered so other variables must be like:

\[ -\infty < z_1 \leq z_2 \leq \ldots \leq z_{i-1} \leq z_i \leq z_{i+1} \leq \ldots \leq z_j \leq z_{j+1} \leq \ldots \leq z_M < \infty \]

provided they remain in given ordering

\[ \text{vary over} \ (-\infty, z_i] \]

\[ \text{vary over} \ [z_j, \infty) \]

\[ \text{fixed subject to ordering} \]

\[ \text{fixed subject to ordering} \]

Thus

\[ f_{z_i z_j} (z_i, z_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{z_1 \ldots z_M} (z_1 \ldots z_M) \, dz_1 \ldots dz_M \]

\[ = H! \int_{z_i}^{\infty} \int_{z_j}^{\infty} \cdots \int_{z_M}^{\infty} f_Y (z_i) \cdots f_Y (z_M) \, dz_1 \ldots dz_M \]
Carrying out the iterated integral

\[
\int_{-\infty}^{z_2} f_Y(z_1) \, dz_1 = F_Y(z_2)
\]

\[
\int_{-\infty}^{z_3} f_Y(z_2) \, F_Y(z_2) \, dz_2 = \frac{1}{2} \left[ F_Y(z_3) \right]^2 \quad \text{(by parts or a change of variable)}
\]

\[
\int_{-\infty}^{z_4} f_Y(z_3) \frac{1}{2} \left[ F_Y(z_3) \right]^2 \, dz_3 = \frac{1}{2} \cdot \frac{1}{3} \left[ F_Y(z_4) \right]^3
\]

\[
\int_{-\infty}^{z_i} f_Y(z_{i-1}) \frac{1}{(i-2)!} \left[ F_Y(z_{i-1}) \right]^{i-2} \, dz_{i-1} = \frac{1}{(i-1)!} \left[ F_Y(z_i) \right]^{i-1}
\]

\[
\int_{-\infty}^{z_{i+2}} f_Y(z_{i+1}) \frac{1}{(i-1)!} \left[ F_Y(z_i) \right]^{i-1} \, dz_{i+1} = \frac{1}{(i-1)!} \left[ F_Y(z_{i+2}) \right]^{i-1} \left\{ F_Y(z_i) - F_Y(z_{i+2}) \right\}
\]

\[
\int_{z_i}^{z_{i+3}} f_Y(z_{i+2}) \frac{1}{(i-1)!} \left[ F_Y(z_i) \right]^{i-1} \left\{ F_Y(z_{i+3}) - F_Y(z_i) \right\} \, dz_{i+2}
\]

\[
= \frac{1}{(i-1)!} \left[ F_Y(z_i) \right]^{i-1} \left\{ \frac{1}{2} \left[ F_Y(z_{i+3}) - F_Y(z_i) \right] - \frac{1}{2} \left[ F_Y(z_{i+3}) - F_Y(z_i) \right] \right\}
\]

\[
= \frac{1}{(i-1)!} \left[ F_Y(z_i) \right]^{i-1} \frac{1}{2} \left[ F_Y(z_{i+3}) - F_Y(z_i) \right]^2
\]

\[
\int_{z_i}^{z_{i+p}} f_Y(z_{i+p-1}) \frac{1}{(i-1)!} \left[ F_Y(z_i) \right]^{i-1} \frac{1}{(p-1)!} \left[ F_Y(z_{i+p-1}) - F_Y(z_i) \right]^{p-2} \, dz_{i+p-1}
\]

\[
= \frac{1}{(i-1)!} \left[ F_Y(z_i) \right]^{i-1} \frac{1}{(p-1)!} \left[ F_Y(z_{i+p}) - F_Y(z_i) \right]^{p-1}
\]
\[
\int_{z_i}^{z_{j+1}} (\text{blah}) \, dz_j = \frac{1}{(i-1)!} \left[ F_Y(z_i) \right]^{-1} \frac{1}{(j-i-1)!} \left[ F_Y(z_j) - F_Y(z_i) \right]^{-j-i-1}
\]

these are now constants for any further integrations ... put back in at end.

\[
\int_{z_j}^{z_{j+2}} f_Y(z_{j+1}) \, dz_{j+1} = F_Y(z_{j+2}) - F_Y(z_j)
\]

\[
\int_{z_j}^{z_{j+3}} f_Y(z_{j+2}) \left[ F_Y(z_{j+2}) - F_Y(z_j) \right] \, dz_{j+2} = \frac{1}{2} \left\{ \left[ F_Y(z_{j+3}) - F_Y(z_j) \right]^2 - \left[ F_Y(z_j) - F_Y(z_j) \right]^2 \right\}
\]

\[
= \frac{1}{2} \left[ F_Y(z_{j+3}) - F_Y(z_j) \right]^2
\]

\[
\ldots
\]

You get the picture?

At last then

\[
\int_{z_i}^{z_j} (z_i \, z_j) = \frac{M!}{(i-1)! \, (j-i-1)! \, (M-j)!} \int f_Y(z_i) \, f_Y(z_j)
\]

Can get other formulas in a similar way.
Say $X \sim C(a, b)$ if it has the Cauchy density
\[
\frac{b}{\pi b^2 + (x-a)^2},
\]

(a) Let $X_i$ $(i = 1, 2)$ be independently distributed according to the Cauchy densities $C(a_i, b_i)$. Prove that $X_1 + X_2$ is distributed as $C(a_1 + a_2, b_1 + b_2)$.

(b) Prove that if $X_1, \ldots, X_n$ are iid and distributed as $C(a, b)$ then the distribution of the sample mean is again $C(a, b)$. What does this say about the sample mean as an estimator in the Cauchy case?
\( X \sim C(a, b) \triangleq \frac{b}{\pi} \frac{1}{b^2 + (x-a)^2} \)

(a) Consider distribution of the sum \( X_1 + X_2 \) where the terms are independent and 
\[ X_i \sim C(a_i, b_i). \]

From a table of Fourier Transforms, the characteristic function of \( X \) above is easily seen to be
\[
\Phi_X(\omega) = E\{e^{i\omega X}\} = e^{-\frac{b}{\omega a} \omega a - b |\omega|}
\]

\[
\Phi_{X_1 + X_2}(\omega) = \Phi_{X_1}(\omega) \Phi_{X_2}(\omega) = e^{i \omega (a_1 + a_2) - (b_1 + b_2) |\omega|} = X_1 + X_2 \sim C(a_1 + a_2, b_1 + b_2).
\]

(b) From induction and the result above we have
\[ X_1 + X_2 + \cdots + X_n \sim C(na, nb). \]

For any random variable
\[
\Phi_{\bar{X}/n}(\omega) = E\{e^{i\omega \bar{X}/n}\} = \Phi_{\bar{X}}(\omega/n)
\]

Therefore, the char. funct. of the sample mean \( \bar{X} = \frac{X_1 + \cdots + X_n}{n} \) is
\[
e^{\frac{i\omega}{n} (na) - nb |\omega|} = e^{i\omega a - b |\omega|}
\]

From uniqueness of char. functions \( \bar{X} \sim C(a, b) \). This tells us that the sample mean contains the same information about \( a, b \) that a single observation does. It is useless as an estimator in this case.
Suppose that $Y$ is a random variable which, under hypothesis $H_0$, has pdf

\[ f_0(y) = \begin{cases} \frac{2}{3}(y + 1) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

and, under hypothesis $H_1$, has pdf

\[ f_1(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

Find the Bayes rule and minimum Bayes risk for testing $H_0$ versus $H_1$ with uniform costs and equal priors.
This is the solution I turned in as a student. As you will see, I made a frontal assault on the problem. Typically I am stupid and brave.

Y is a random variable which under $H_0$ has pdf:

$$p_0(y) = \begin{cases} \frac{2}{3} (y+1) & 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

and under $H_1$, has pdf:

$$p_1(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

Find the Bayes rule and minimum Bayes risk for testing $H_0$ vs. $H_1$ with uniform costs and equal priors.

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{3/2}{y+1}$$

$$\Gamma_1 = \{ y \in \mathbb{R} \mid L(y) > \frac{p_0}{1-p_0} \}$$

For uniform costs: $(c_{00} = c_{11} = 0)$

$$\mathcal{L} = \frac{\pi_0}{\pi_1} = \frac{\pi_0}{1-\pi_0}$$

$$L(y)$$

$$\Gamma_1$$
First, I will solve for general priors.

**Solve for cases:**

\[ \pi_0 \geq \frac{3}{2}, \quad \pi_0 \geq \frac{3}{2}, \quad \pi_0 \geq \frac{3}{2} \]

\[ \pi_0 \geq \frac{3}{2} - \frac{3}{2} \pi_0 \]

\[ \frac{3}{2} \pi_0 \geq \frac{3}{2} \]

\[ \pi_0 \geq \frac{3}{6} \]

\[ \pi_0 \leq \frac{3}{4}, \quad \pi_0 \leq \frac{3}{4} \]

\[ \pi_0 \leq \frac{3}{4} - \frac{3}{4} \pi_0 \]

\[ \frac{3}{4} \pi_0 \leq \frac{3}{4} \]

\[ \pi_0 \leq \frac{3}{7} \]

---

For \( \pi_0 \leq \frac{3}{7} \),

\[ r(\pi_0, \delta \pi_0) = (1 - \pi_0) + \int_{\pi_0}^{\delta \pi_0} p_0(y) - (1 - \pi_0) p_1(y) \, dy \]

\[ = 1 - \pi_0 + \pi_0 - (1 - \pi_0) = \pi_0 \]
\( \frac{3}{7} \leq \pi_0 \leq \frac{3}{5} \)

\[ \gamma (\pi_0, \delta \pi_0) = 1 - \pi_0 + \int_{0}^{\pi_0} \frac{2}{3} (y + 1) - (1 - \pi_0) \, dy \]

\[ = 1 - \pi_0 + \frac{2\pi_0}{3} \left( \frac{3}{\pi_0} - 5 \right) - \frac{1}{2} (1 - \pi_0) \left( \frac{3}{\pi_0} - 5 \right) \]

\[ = 1 - \pi_0 + \frac{2\pi_0}{3} \left[ \frac{1}{2} \left( \frac{3}{\pi_0} - 5 \right)^2 + \frac{3}{2} \left( \frac{3}{\pi_0} - 5 \right) \right] - \frac{1}{2} (1 - \pi_0) \left( \frac{3}{\pi_0} - 5 \right) \]

\[ = 1 - \pi_0 + \frac{\pi_0}{12} \left( \frac{9}{\pi_0^2} - \frac{30}{\pi_0} + 25 \right) + \frac{\pi_0}{3} \left( \frac{3}{\pi_0} - 5 \right) - \frac{1}{2} \left( \frac{3}{\pi_0} - 5 - 3 + 5\pi_0 \right) \]

\[ = 1 - \pi_0 + \frac{3}{4\pi_0} - \frac{5}{2} + \frac{25\pi_0}{12} + 1 - \frac{5\pi_0}{3} - \frac{3}{2\pi_0} + 4 - \frac{5\pi_0}{2} \]

\[ = \frac{7}{2} - \frac{37\pi_0}{12} - \frac{3}{4\pi_0} \]

For equal priors \((\pi_0 = \pi_1 = \frac{1}{2})\) we can summarize by saying:

\( \pi_1 = (-\infty, \frac{1}{2}] \cup (1, \infty) \) i.e. \( y \in \pi_1 \) pick \( H_1 \)

\( \gamma (\frac{1}{2}, \delta_{\frac{1}{2}}) = \frac{11}{24} \)
Consider the hypothesis pair

\[ H_0 : \ Y = N \]
\[ H_1 : \ Y = N + S \]

where \( N \) and \( S \) are independent random variables each having pdf

\[
p(x) = \begin{cases} 
  e^{-x} & x \geq 0 \\
  0 & x < 0 
\end{cases}
\]

(a) Find the likelihood ratio between \( H_0 \) and \( H_1 \).

(b) Find the threshold and detection probability for \( \alpha \)-level Neyman–Pearson testing of \( H_0 \) vs. \( H_1 \).

Now consider the hypothesis pair

\[ H_0 : \ Y_k = N_k, \ k = 1, 2, \ldots, n \]
\[ H_1 : \ Y_k = N_k + S, \ k = 1, 2, \ldots, n \]

where \( n > 1 \) and \( N_1, \ldots, N_n \) and \( S \) are independent random variables each having the pdf \( p(x) \) above.

(c) Find the likelihood ratio between \( H_0 \) and \( H_1 \).

(d) Find the threshold and detection probability for \( \alpha \)-level Neyman–Pearson testing of \( H_0 \) vs. \( H_1 \).
Special Problem: Another student days effort.

H₀ : Y = N

vs.

H₁ : Y = N + S

where N and S are independent random variables each having pdf

\[ p(x) = \begin{cases} 
  e^{-x} & x \geq 0 \\
  0 & x < 0 
\end{cases} \]

Find the likelihood ratio between H₀ and H₁

Since N and S are independent, the density of N+S is the convolution:

\[ p_{1}(y) = \int_{0}^{y} p(x) p(y-x) \, dx \]

\[ = \int_{0}^{y} e^{-x} e^{-(y-x)} \, dx \]

\[ = \int_{0}^{y} e^{-y} \, dx = ye^{-y} \quad y \geq 0 \]

So we have the hypothesis pair:

\[ p_{0}(y) = \begin{cases} 
  e^{-y} & y \geq 0 \\
  0 & y < 0 
\end{cases} \]

\[ p_{1}(y) = \begin{cases} 
  ye^{-y} & y \geq 0 \\
  0 & y < 0 
\end{cases} \]

The likelihood ratio is

\[ L(y) = \frac{p_{1}(y)}{p_{0}(y)} = y \quad y \geq 0 \]
(b) Find the threshold and detection probability for \( \alpha \)-level Neyman-Pearson testing in (a).

Need \( P_o(L(Y) > \eta) = P_o(Y > \eta) = \int_{\eta}^{\infty} e^{-x} \, dx = e^{-\eta} \bigg|_{\eta}^{\infty} = e^{-\eta} \)

\[ P_o(L(Y) > \eta) \]

\[ \eta_0 = -\log x \]

Randomization is arbitrary

\[ e^{-\eta} = x \]
\[ -\eta = \log x \]
\[ \eta = -\log \alpha \]

So the NP rule is

\[ \overset{\sim}{{\delta}}_{NP}(y) = \begin{cases} 
1 & \text{if } y > \eta_0 \quad \text{or } p_o(y) \\
0 & \text{else} 
\end{cases} = \begin{cases} 
1 & y > \eta_0 = -\log \alpha \\
0 & y < \eta_0 
\end{cases} \]

\[ P_D(\overset{\sim}{{\delta}}_{NP}) = E \left[ \overset{\sim}{{\delta}}_{NP}(y) \right] = \int_{-\log \alpha}^{\infty} xe^{-x} \, dx = e^{-x}(x+1) \bigg|_{-\log \alpha}^{\infty} \]
\[ = e^{\log \alpha}(-\log \alpha + 1) \]
\[ = \alpha(-\log \alpha + 1) \]
\[ = \alpha(1-\log \alpha) \]
(C) Consider the hypothesis pair

\[ H_0 : Y_k = N_k \quad k = 1, 2, \ldots, n \]
\[ H_1 : Y_k = N_k + S \quad k = 1, 2, \ldots, n \]

where \( n \geq 1 \) and \( N_1 \ldots N_n \) and \( S \) are independent random variables each having the pdf of (a). Find the likelihood ratio.

We need the density functions \( p_0, p_1 : \mathbb{R}^n \to \mathbb{R} \).

Because \( N_1 \ldots N_n \) are independent we have that

\[
p_0(y_1, \ldots, y_n) = p(y_1) p(y_2) \cdots p(y_n)
= \begin{cases} 
\exp \left\{ - \frac{1}{2} \sum_{i=1}^{n} y_i^2 \right\} & y_k > 0 \quad \forall k \\
0 & \text{else}
\end{cases}
\]

Now the random variables \( Y_k \quad k = 1, 2, \ldots, n \) are not conditionally independent under \( H_1 \). The best way to compute the joint density under \( H_1 \) is by conditioning on \( S = s \) and noting that given \( S = s \) the random variables \( Y_1 \ldots Y_n \) are independent so

\[
p_1(y_1, \ldots, y_n | S = s) = p(y_1 - s) p(y_2 - s) \cdots p(y_n - s)
\]

then

\[
p_1(y_1 \ldots y_n | S = s) \cdot p_1(y_1, \ldots, y_n | S = s) p(s) \]

is the joint of \( Y_1 \ldots Y_n \) and \( S \).

To get the joint density of \( Y_1 \ldots Y_n \) we integrate with respect to \( s \).
\[ p_i(y_1, \ldots, y_n) = \int p(y_i-s) p(y_2-s) \ldots p(y_n-s) p(s) \, ds \]

\[ \begin{align*}
\min_0^{\infty} \text{(-)} & \quad \min_0^{\infty} \text{(-)} \\
= \int_0^\infty \exp \left\{ -\sum_{i=1}^n \frac{y_i}{s} \right\} e^{-s} \, ds & = \int_0^\infty \exp \left\{ -\sum_{i=1}^n y_i \right\} e^{ns} \, ds \\
= \exp \left\{ -\sum_{i=1}^n y_i \right\} \frac{1}{n-1} \left( e^{n-1} - 1 \right) & n > 1
\end{align*} \]

So the likelihood ratio is

\[ L(y) = \frac{p_i(y)}{p_0(y)} = \frac{1}{n-1} \left( e^{(n-1) \min \{y_1, \ldots, y_n\}} - 1 \right) \quad y = (y_1, \ldots, y_n) \quad \text{for } y_i > 0 \, \text{all } i \]

(1) Find the threshold for \( \alpha \)-level Neyman-Pearson testing in (c)

Need to solve for the smallest \( \eta_0 \) such that

\[ P_0(L(Y) > \eta_0) \leq \alpha \quad \text{given some } \alpha \in (0,1). \]

Can we solve for the set of \( y \in \mathbb{R}^n \) s.t. \( L(y) > \eta_0 \)?

\[ \frac{1}{n-1} \left( e^{(n-1) \min \{y_1, \ldots, y_n\}} - 1 \right) = \eta_0 \]

\[ e^{(n-1) \min \{\ldots\}} = (n-1) \eta_0 + 1 \]

\[ (n-1) \min \{\ldots\} = \log((n-1) \eta_0 + 1) \]
\[
\min \{\gamma_1, \ldots, \gamma_n\} \geq \frac{1}{n-1} \log ((n-1) \eta_0 + 1)
\]

So
\[
P_0 (L(Y) > \eta_0) = \int \exp \left( -\frac{1}{n} \sum_{i=1}^{n} y_i \right) \mu(dy)
\]

\[
\text{y st.} \quad \min \{\gamma_1, \ldots, \gamma_n\} \geq \frac{1}{n-1} \log ((n-1) \eta_0 + 1)
\]

with \( a = \frac{1}{n-1} \log \left( (n-1) \eta_0 + 1 \right) \)

\[
P_0 (L(Y) > \eta_0) = \int \ldots \int \frac{1}{a} e^{-\frac{1}{n} \sum_{i=1}^{n} y_i} \, dy_1 \, dy_2 \ldots \, dy_n
\]

\[
= \prod_{i=1}^{n} e^{-a} = e^{-na}
\]

we have \( P_0 (L(Y) > \eta_0) = e^{-na} \) where \( a = \frac{1}{n-1} \log \left( (n-1) \eta_0 + 1 \right) \).

To get the threshold solve

\[
\frac{e^{-na}}{\alpha} = 1
\]

\[
\exp \left\{ -\frac{n}{n-1} \log \left( (n-1) \eta_0 + 1 \right) \right\} = \left( (n-1) \eta_0 + 1 \right)^{-\frac{n}{n-1}} = \alpha
\]

\[
(n-1) \eta_0 + 1 = \alpha^{-\frac{1}{n-1}}
\]

\[
\eta_0 = \frac{1}{n-1} \left[ \alpha^{-\frac{1}{n-1}} - 1 \right]
\]
Consider the following pair of hypotheses concerning a sequence $Y_1, Y_2, \ldots, Y_n$ of random variables

\begin{align*}
H_0 & : \quad Y_k \sim \mathcal{N}(\mu_0, \sigma_0^2), \quad k = 1, 2, \ldots, n \\
H_1 & : \quad Y_k \sim \mathcal{N}(\mu_1, \sigma_1^2), \quad k = 1, 2, \ldots, n
\end{align*}

where $\mu_0$, $\mu_1$, $\sigma_0^2$, and $\sigma_1^2$ are known constants.

(a) Show that the likelihood ratio can be expressed as a function of the parameters $\mu_0$, $\mu_1$, $\sigma_0^2$, and $\sigma_1^2$, and the quantities $\sum_{k=1}^{n} Y_k^2$ and $\sum_{k=1}^{n} Y_k$.

(b) Describe the Neyman–Pearson test for the two cases ($\mu_0 = \mu_1$, $\sigma_1^2 > \sigma_0^2$) and ($\sigma_0^2 = \sigma_1^2$, $\mu_1 > \mu_0$).

(c) Find the threshold and ROCs for the case $\mu_0 = \mu_1$, $\sigma_1^2 > \sigma_0^2$ with $n = 1$. 
\( H_0 : \gamma_k \sim \eta(\mu_0, \sigma_0^2) \)

\( k = 1, 2, \ldots, n \) (iid under each hypothesis)

\( H_1 : \gamma_k \sim \eta(\mu_1, \sigma_1^2) \)

(a) Easy to see that the joint densities are

\[
p_j(y) = \prod_{k=1}^{n} \frac{1}{(2\pi \sigma_j^2)^{1/2}} \exp\left\{-\frac{1}{2} \frac{(y_k - \mu_j)^2}{\sigma_j^2}\right\} = \frac{1}{(2\pi \sigma_j^2)^{n/2}} \exp\left\{ -\frac{1}{2 \sigma_j^2} \sum_{k=1}^{n} (y_k - \mu_j)^2 \right\}
\]

\( j = 0, 1 \)

Thus the likelihood ratio is

\[
L(y) = \frac{p_j(y)}{p_0(y)} = \left( \frac{\sigma_0}{\sigma_1} \right)^n \exp\left\{ -\frac{1}{2 \sigma_0^2} \sum_{k=1}^{n} (y_k - \mu_0)^2 + \frac{1}{2 \sigma_1^2} \sum_{k=1}^{n} (y_k - \mu_1)^2 \right\} \frac{1}{2 \sigma_0^2 \sigma_1^2} \sum_{k=1}^{n} (y_k - \mu_0)^2 \frac{1}{2 \sigma_0^2 \sigma_1^2} \sum_{k=1}^{n} (y_k - \mu_1)^2 \left( \frac{\mu_0^2}{2 \sigma_0^2} - \frac{\mu_1^2}{2 \sigma_1^2} \right)^n \right\}
\]

This shows the structure desired.

(b) Case \( \mu_0 = \mu_1 \neq \mu \neq \sigma_1^2 > \sigma_0^2 \)

In this case it is best to write the likelihood function as

\[
L(y) = \left( \frac{\sigma_0}{\sigma_1} \right)^n \exp\left\{ -\frac{1}{2 \sigma_0^2 \sigma_1^2} \sum_{k=1}^{n} (y_k - \mu)^2 \right\}
\]

Using the fact that log is monotone increasing we see that comparing \( L(y) \) to a threshold \( \tau \) is equivalent to comparing \( \sum_{k=1}^{n} (y_k - \mu)^2 \) to another threshold \( \tau' \) where

\[
\tau' = \left( \frac{2 \sigma_0^2 \sigma_1^2}{\sigma_1^2 - \sigma_0^2} \right) \left[ \log \tau - n \log \left( \frac{\sigma_0}{\sigma_1} \right) \right]
\]

Case \( \sigma_0^2 = \sigma_1^2 = \sigma^2 \), \( \mu > \mu_0 \)

Write the likelihood

\[
L(y) = \exp\left\{ -\frac{1}{2 \sigma^2} \sum_{k=1}^{n} (y_k - \mu)^2 + n \left( \frac{\mu_0^2}{2 \sigma^2} - \frac{\mu_1^2}{2 \sigma^2} \right)^n \right\}
\]
Comparing $L(y)$ to $z$ is equivalent to comparing $\sum_{k=1}^{\eta} y_k$ to

$$z' = \left( \frac{\sigma_0^2}{\mu_1 - \mu_0} \right) \left[ \log z + n \left( \frac{\mu_1^2 - \mu_0^2}{2\sigma_0^2} \right) \right].$$

(c) Consider the case where $\mu_0 = \mu_1 = \mu$ and $\sigma_1^2 > \sigma_0^2$ with $n=1$. From the first case in (b) the proper test statistic is

$$\frac{(y_1 - \mu)^2}{\sigma_1^2}$$

Therefore (randomization is not needed), the form of the NP test is

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } (y_1 - \mu)^2 \geq z' \\ 0 & \text{if } (y_1 - \mu)^2 < z' \end{cases} \quad y_1 = y_1,$$

where $z'$ is an appropriately chosen threshold. To find the threshold for a given size $\alpha$ we compute

$$P_F(\delta_{NP}) = P_{\theta_0} \{(y_1 - \mu)^2 \geq z' \} = 1 - P_{\theta_0} \{(y_1 - \mu)^2 \leq z' \}$$

$$= 1 - P_{\theta_0} \{-\sqrt{z'} \leq y_1 - \mu \leq \sqrt{z'} \} \quad \text{under } H_0 \ Y \sim \eta(\mu_0^2)$$

$$= 1 - \left[ \Phi \left( \frac{\sqrt{z'}}{\sigma_0} \right) - \Phi \left( \frac{-\sqrt{z'}}{\sigma_0} \right) \right] = 2 \left[ 1 - \Phi \left( \frac{\sqrt{z'}}{\sigma_0} \right) \right]$$

To get size $\alpha$ we solve for $z'$ in the equation

$$\alpha = 2 \left[ 1 - \Phi \left( \frac{\sqrt{z'}}{\sigma_0} \right) \right] \Rightarrow z' = \left[ \sigma_0^2 \Phi^{-1}(1-\alpha) \right]^2$$

The detection probability is

$$P_D(\delta_{NP}) = 1 - P_{\theta_1} \{(y_1 - \mu)^2 \leq z' \} = 2 \left[ 1 - \Phi \left( \frac{\sqrt{z'}}{\sigma_1} \right) \right]$$

$$= 2 \left[ 1 - \Phi \left( \frac{\sigma_0^2}{\sigma_1^2} \Phi^{-1}(1-\alpha) \right) \right] \quad 0 < \alpha < 1.$$