ECE 645 **Exam 1** **2/24/14**

Academic Honesty Statement: I am aware of the course policies concerning academic honesty for Professor Krogmeier's section of ECE 645 and for this exam. Furthermore, I promise that the work I am submitting with this exam is my own work.

Signature: _____

Name Printed: _____

General Instructions:

- **You have 120 minutes to complete the exam.**
- Write your name or initials on every page of the exam.
- Do not write on the backs of the pages. If you need more paper, it will be provided to you upon request.
- The exam is open classnotes.
- Calculators are allowed.
- Your work must be explained to receive full credit.
- Point values for each problem are as indicated. The exam totals 100 points.
- All plots must be carefully drawn with axes labeled.

Do not open the exam until you are told to begin.

Name or Initials: _____

Problem 1. [1 part, 20 pts. total] Suppose that $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$, that $\{V_k\}_{k=1}^N$ is i.i.d. with $V_k \sim \mathcal{N}(0, \sigma_v^2)$, and that the V_k are independent of X. Then given

$$Y_k = X + V_k,\ 1 \leq k \leq N,$$

specify the posterior density $f(x|y_1, y_2, \ldots, y_N)$ by specifying its form and deriving the values of the parameters characterizing it.

Problem 1. (cont'd.) Name or Initials: _____

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Problem 2. [3 parts, 35 pts. total] Under H_0 a random variable X has probability density

$$f_0(x) = \begin{cases} A(a^2 - x^2) & |x| < a \\ 0 & |x| \geq a \end{cases}$$

and under H_1 its probability density is

$$f_1(x) = \begin{cases} B(b^2 - x^2) & |x| < b \\ 0 & |x| \geq b \end{cases}$$

where $b > a$.

(a) [10 pts.] Calculate the constants A and B.

Problem 2. (cont'd.) Name or Initials: _____

(b) [15 pts.] Determine the optimum Neyman-Pearson decision rule for H_0 vs. H_1 at level α ($0 < \alpha < 1$). You only need to find the equation one solves for the threshold given α, you do **not** need a closed form giving threshold as a function of α.

Problem 2. (cont'd.) Name or Initials: _____

(c) [10 pts.] For $b = 2a$, find the detection probability corresponding to level α and plot the ROC. Here it is easier to parameterize the ROC in terms of the threshold rather than as an explicit function of α.

Problem 3. [4 parts, 45 pts. total] Consider a pattern classification problem where, in addition to deciding among a certain number of classes, we may decide to reject the pattern if it is not recognizable.

Let observations be denoted by the random vector X. In the case of two possible pattern classes there are two hypotheses:

$$\begin{array}{ll} H_0: & X \sim f_0 \\ \text{versus} & \\ H_1: & X \sim f_1 \end{array}$$

with prior probabilities $\pi_0 + \pi_1 = 1$. Instead of making one of two decisions we now allow three decisions

$$d_0 = \text{accept } H_0, \quad d_1 = \text{accept } H_1, \quad \text{or} \quad d_2 = \text{reject}.$$

If we wish to solve this problem in a Bayesian framework we will need costs associated with the three possible decisions and the two hypotheses. Denote these by c_{ij} with the interpretation that this is the cost of making decision d_i when H_j is true and assume that

$$c_{ij} = \begin{cases} 0 & i = j,\ i, j = 0, 1 \\ \lambda_m & i \neq j,\ i, j = 0, 1 \\ \lambda_r & i = 2 \end{cases} .$$

(a) [10 pts.] Since there are three possible decisions a decision rule δ will amount to a partition of the observation space into three regions, each corresponding to a different decision[1] Find the two condional risks

$$R_j(\delta) \stackrel{\text{def}}{=} \text{E}\{ \text{ Cost } | H_j \text{ is true } \}$$

for $j = 0,\ 1$.

[1]Note that we do not need to consider randomized decision rules for this problem.

Problem 3. (cont'd.) Name or Initials: _____

(b) [20 pts.] Derive the Bayes decision rule for this problem by specifying the regions of the observation space which should correspond to each of the three decisions.

Problem 3. (cont'd.) Name or Initials: _____

(c) [10 pts.] Simplify the decision rule from part (b) by writing it in terms of the posterior probabilities of the hypotheses given the observations, i.e., in terms of

$$\pi_0(x) \stackrel{\text{def}}{=} P(H_0 \text{ is true } | X = x), \quad \pi_1(x) \stackrel{\text{def}}{=} P(H_1 \text{ is true } | X = x),$$

and the misclassification cost λ_m, and the rejection cost λ_r.

Problem 3. (cont'd.) Name or Initials: _____

(d) [5 pts.] Describe the behavior of the test in the two cases: 1) where the rejection cost λ_r is large, and 2) where it is small, relative to the misclassification cost λ_m.

Problem 3. (cont'd.) Name or Initials: _____