Recall: We're working on $M$-ary signalling with orthogonal signals

$$\begin{align*}
H_i : Y &= S_i + N \\
0 \leq i \leq M-1 \\
S_i^T S_j &= \begin{cases} 
0 & i \neq j \\
|S_i|^2 = |S_j|^2 & i = j
\end{cases}
\end{align*}$$

For uniform costs and equal priors, the Bayes receiver is the ML receiver and it can be characterized by decision regions

$$\Pi_i = \left\{ y : \| y - S_i \|^2 = \min_k \| y - S_k \|^2 \right\}$$

$$= \left\{ y : S_i^T y = \max_k S_k^T y \right\}$$
\[ P_e = P(\{ \text{error occurs} \}) = \frac{1}{M} \sum_{i=0}^{M-1} P_i(\gamma \in \Pi_i) \]

Then \[ P_i(\gamma \in \Pi_i) = 1 - P_i(\gamma \notin \Pi_i) \]

\[ = 1 - P_i(s_i^T y > \max_{k \neq i} s_k^T y) \]

**Fact:** Under \( H_i \):

\[ s_0^T y, s_1^T y, \ldots, s_{M-1}^T y \]

are independent, Gaussian with

\[ s_i^T y \sim \begin{cases} N(0, \sigma^2 \|s_i\|) & j \neq i \\ N(\|s_i\|^2, \sigma^2 \|s_i\|^2) & j = i \end{cases} \]
Lemma $\mathbf{U} \perp \mathbf{V}$

$$P(U < V) = \int_{-\infty}^{\infty} P(U < r) f_V(r) \, dr$$

Re:

$$P_i \left( \max_{k \neq i} s_{tk}^T y < s_i^T y \right)$$

$$= \int_{-\infty}^{\infty} P_i \left( \max_{k \neq i} s_{tk}^T y < z \right) \frac{1}{\sqrt{2\pi} \sigma_1 s_{11}^2} e^{-\frac{(y - \mu s_{11}^2)^2}{2\sigma_1^2 \sigma_{11}^2}} \, dz$$

$$= P_i \left( \bigcap_{k \neq i} \{ s_{tk}^T y < z \} \right) = \prod_{k \neq i} P_i (s_{tk}^T y < z) = \left[ \Phi \left( \frac{3}{\sigma_1 s_{11}} \right) \right]^{n-1}$$
Note that $P_i(Y \notin \Gamma_i)$ does not depend on $i$. Thus sub into formula for $P_e$ to get:

$$P_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \Phi(x) \right]^{M-1} e^{-(x-d)^2/2} \, dx$$

made a change of variables

$$x = \frac{3}{d} \| s_0 \|; \quad d = \| s_0 \| / \theta$$

Equivalent to MBP Eq. (6.71).
Students Should Read

Sec. 6.7 "Offset 2D Mode"
6.8 "BW Comparisons"
6.9 "Continuous Phase Modulation"
MBP Ch 7 Non-Coherent Communications

- Phase of received sig. not known to receiver and no attempt is made to estimate it.
- Some signals are useless in this context—BBSK, QPSK, 16 QAM, 8-PSK.
- Start with binary

\[ s_0(t) = \sqrt{2} A a_0(t) \cos \left( \omega t + \Theta_0(t) + \Phi_0 \right) p_T(t) \]

\[ s_1(t) = \sqrt{2} A a_1(t) \cos \left( \omega t + \Theta_1(t) + \Phi_1 \right) p_T(t) \]

BB amplitude mod.  
BB phase mod.

receiver has no knowledge. Often model as realizations of indep unif. \([0,2\pi]\) r.v.s.
Example 1 Binary Freq. Shift Keying (BFSK)

\[ a_0(t) = a_1(t) = 1 \]

\[ \Theta_0(t) = (\omega_0 - \omega_c)t \quad \rightarrow \quad s_0(t) = \sqrt{2} A \cos[\omega_0 t + \phi_0] p_T(t) \]

\[ \Theta_1(t) = (\omega_1 - \omega_c)t \quad \rightarrow \quad s_1(t) = \sqrt{2} A \cos[\omega_1 t + \phi_1] p_T(t) \]

Fact: \( \omega_i T \) mult. of \( 2\pi \), \( i = 0,1 \) then the BFSK signals are orthogonal & have equal energy.

\[ \langle s_0, s_1 \rangle = 0 \]

\[ \langle s_0, s_0 \rangle = \langle s_1, s_1 \rangle = A^2 T \]

\[ \|s_0\|^2 = \|s_1\|^2 \]
Example 2  Orthog. AM

\[ \theta_0(t) = \theta_1(t) = 0 \]

For \( a_0(t) \):

- Sine wave labeled \( a_0(t) \) with peaks at 0, \( \frac{T}{4} \), \( \frac{T}{2} \), \( \frac{3T}{4} \), and \( T \).

For \( a_1(t) \):

- Sine wave labeled \( a_1(t) \) with a peak at \( \frac{T}{4} \) and troughs at 0, \( \frac{T}{2} \), \( \frac{3T}{4} \), and \( T \).
Front End for Any Optimal Receiver for a Binary Non-coherent Signalling Problem

\[ Y(t) \]

\[ \sqrt{2} a_0(t) \cos [\omega t + \theta_0(t)] \]
\[ \int_0^T (\cdot) \, dt \]
\[ U_0 \]

\[ \sqrt{2} a_0(t) \sin [\omega t + \theta_0(t)] \]
\[ V_0 \]

\[ \sqrt{2} a_1(t) \cos [\omega t + \theta_1(t)] \]
\[ U_1 \]

\[ \sqrt{2} a_1(t) \sin [\omega t + \theta_1(t)] \]
\[ V_1 \]

* Seems that non-coherent case needs 4 correlators whereas only 2 (or even 1) needed for coherent.

* Why?
What we really need to show: That the r.v.s. \( U_0, V_0, U_1, V_1 \) are a suff. statistic for binary test (and are they minimal?).

Sufficiency: All signals we could ever possibly see are in span of the 4 mixer inputs in the correlator bank. It's trivial actually →

\[
S_i(t) = \sqrt{2} A a_i(t) \cos \left[ \omega_c t + \Theta_i(t) + \phi_i \right] \\
= \sqrt{2} A a_i(t) \left\{ \cos \left[ \omega_c t + \Theta_i(t) \right] \cos \phi_i - \sin \left[ \omega_c t + \Theta_i(t) \right] \sin \phi_i \right\} \\
= (A \cos \phi_i) \sqrt{2} a_i(t) \cos \left[ \omega_c t + \Theta_i(t) \right] + (-A \sin \phi_i) \sqrt{2} a_i(t) \sin \left[ \omega_c t + \Theta_i(t) \right]
\]
Note

1. Do not require
\[ \sqrt{2} a_0(t) \cos [\omega t + \theta_0(t)] , \]
\[ \sqrt{2} a_1(t) \sin [\omega t + \theta_1(t)] \]
to be orthog. or linearly indep.

2. Possible to use G.S. Procedure to find a smaller dim. basis.

**Statistical Characterization of Correlator Outputs**

\[ Y(t) = S_i(t) + X(t) \quad i = 0, 1 \]
\[ \Rightarrow \text{AWGN, } N_0/2 \]

For either hypothesis \( U_0, V_0, U_1, V_1 \) are jointly Gauss.
so char. by 1) mean vector, 2) 4x4 Cov. matrix.
\[ E \{ U_i \mid H_i, \text{true} \} = \]
\[ = \left( 2A \cos \phi_i \right) \int_0^T a_i(t) a_j(t) \cos \left[ w_0 t + \Theta_i(t) \right] \cos \left[ w_1 t + \Theta_j(t) \right] \, dt \]
\[ - \left( 2A \sin \phi_i \right) \int_0^T a_i(t) a_j(t) \sin \left[ w_0 t + \Theta_i(t) \right] \cos \left[ w_1 t + \Theta_j(t) \right] \, dt \]
\[ \quad \text{for } i = 0, 1, j = 0, 1 \]

\[ E \{ V_i \mid H_i, \text{true} \} = \]
\[ = \left( 2A \cos \phi_i \right) \int_0^T a_i(t) a_j(t) \cos \left[ w_0 t + \Theta_i(t) \right] \sin \left[ w_1 t + \Theta_j(t) \right] \, dt \]
\[ - \left( 2A \sin \phi_i \right) \int_0^T a_i(t) a_j(t) \sin \left[ w_0 t + \Theta_i(t) \right] \sin \left[ w_1 t + \Theta_j(t) \right] \, dt \]
\[ \quad \text{for } i = 0, 1 \]

Total = 8 numbers; 4 for each of the two hypotheses.