ECE 544: Digital Communications (Session 28)

Constellation: \( \{ s_i \in \mathbb{R}^5 : i = 0, 1, \ldots, M-1 \} \) \( M \) 5-dimensional real vectors

\[ E_s = \frac{1}{n} \sum_{i=0}^{M-1} \| s_i \|^2 \] the average energy per transmitted symbol (assuming equal priors)

\[ E_b = \frac{E_s}{\log_2 M} \] the average energy per bit

Antipodal / BPSK

\[ \begin{array}{c|c}
\sqrt{E} & \sqrt{E} \\
\hline
s_1 & s_0
\end{array} \xrightarrow{\phi_0} \begin{array}{c}
\sqrt{E} \\
\hline
s_1 = (0, \sqrt{E})
\end{array} \Rightarrow E_s = E = E_b \]

\[ D_{\text{min}} \triangleq \text{minimum distance} = 2\sqrt{E} \]

Orthogonal

\[ \begin{array}{c|c}
\sqrt{E} & \sqrt{E} \\
\hline
s_1 = (\sqrt{E}, 0) \\
\hline
s_0 = (0, \sqrt{E})
\end{array} \xrightarrow{\phi_0} \]

\[ \begin{array}{c}
\sqrt{E} \\
\hline
s_1 = (\sqrt{E}, 0)
\end{array} \Rightarrow E_s = E = E_b \]

\[ D_{\text{min}} = \sqrt{2E} \]
\[
\begin{align*}
\text{Square / QPSK / 4-OQASK} & \\
\begin{align*}
\Rightarrow \quad \|s_1\|^2 = 2E & \Rightarrow E_s = \frac{1}{4} \cdot 4(2E) = 2E \\
E_b = E \\
D_{\text{min}} = 2\sqrt{E} \\
\Rightarrow \quad \|s_0\|^2 = 4E = \|s_2\|^2 \\
\|s_3\|^2 = 0 \quad \|s_5\|^2 = 4E + 4E = 8E \\
\therefore E_s = \frac{16E}{4} = 4E & \quad \text{Same } D_{\text{min}} \text{ as square} \\
E_b = 2E \\
D_{\text{min}} = 2\sqrt{E} \\
\Rightarrow \text{ same error perf in AWGN} \\
\Rightarrow \text{ takes twice the SIR energy}
\end{align*}
\end{align*}
\]
General Fact: so $s_1, \ldots, s_{M-1}$ a constellation

Consider translating it by a vector $t$, i.e., new constellation $s_0 - t, s_1 - t, \ldots, s_{M-1} - t$

$$E_s = \frac{1}{M} \sum_{i=0}^{M-1} ||s_i - t||^2$$ average energy per symbol for new constellation

* $D_{\text{min}}$ invariant to translation $t$

* To minimize $E_s$ over $t$, choose $t = \frac{1}{M} \sum_{i=0}^{M-1} s_i$ (centroid)
Prob. of Symbol Error Calculations for ML Receivers

Consider case of two sigs \( s_0 \) and \( s_1 \) separated by dist.

\[ D = \| s_0 - s_1 \| \]

Property of AWGN

Its projection onto any unit norm axis is a Gaussian \( N(0, \frac{N_0}{2}) \) rv.

1. Rotate axes until one axis points in direction of line connecting \( s_0 \) and \( s_1 \); axes rotation does not change signal energies.
3. Two signal error probability:

\[ P_2(1|0) = \text{prob. that ML receiver chooses } S_1 \text{ given } S_0 \]

\[ = P(N > D/2) = P\left(\frac{N}{\sqrt{N_0/2}} > \frac{D/2}{\sqrt{N_0/2}}\right) \]

\[ = \Phi\left(\frac{D}{\sqrt{2N_0}}\right) \]

5. Let \( N \) be the Gaussian r.v. resulting from projection of AWGN along direction of line joining \( S_0 \) and \( S_1 \).
Notes

1. By symmetry \( p_2(110) = p_2(011) \)

2. If constellation has only two signals then these pairwise conditional error probabilities tell the whole story.

3. Fact useful for bounds:

\[ Q(x) \leq \frac{1}{2} \exp(-\frac{1}{2}x^2) \]

\( \therefore \) as \( x \to \infty \) the Gaussian tail prob.
function tends to zero as \( e^{-x^2/2} \)
Many-Signal Error Probabilities/Bounds

So, \( s_1 \), \( \ldots \), \( s_{m-1} \) AWGN, ML receiver, observation \( y \)

\[
P_e = \sum_{k=0}^{M-1} P(\text{error} | s_k \text{ sent}) P_k = \frac{1}{M} \sum_{k=0}^{M-1} P(y \notin \Gamma_k | s_k \text{ sent})
\]

Look at terms

\[
P(y \notin \Gamma_k | s_k \text{ sent}) = P(y \in \bigcup_{i=0}^{M-1} \Gamma_i | s_k \text{ sent})
\]

\[
= \sum_{i=0}^{M-1} P(y \in \Gamma_i | s_k \text{ sent})
\]

Let \( p_2(i | k) = \) prob. that obs. closer to \( s_i \) than to \( s_k \) given \( s_k \) sent.
In general \[ P(y \in \Pi_i | s_k \text{ sent}) = p_2(i|k) \]

\[ P(y \notin \Pi_i | s_k \text{ sent}) = \sum_{i=0}^{M-1} p_2(i|k) = \sum_{i=0}^{M-1} \Phi \left( \frac{\|s_i - s_k\|}{\sqrt{2N_0}} \right) \]

\[ \Rightarrow P_e \leq \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i\neq k} \Phi \left( \frac{\|s_i - s_k\|}{\sqrt{2N_0}} \right) \]

sum has \( M(M-1) \) terms

each term decays like

\[ \frac{1}{2} \exp \left( -\frac{1}{2} \frac{\|s_i - s_k\|^2}{2N_0} \right) \]

for large \( \|s_i - s_k\| \).

Therefore, the terms with least distance will dominate the sum.

Let \( K \doteq \# \) of signal pairs whose distance is equal to the min. distance.
The double sum counts these pairs twice so

\[ P_e \leq \frac{2K}{M} Q \left( \frac{D_{\min}}{\sqrt{2N_0}} \right) \]

\[ \Rightarrow P_e \approx \frac{2K}{M} Q \left( \sqrt{\frac{d_{\min}^2 E_b}{N_0}} \right) \]

\[ d_{\min} = \frac{D_{\min}}{\sqrt{2E_b}} \]

Writing this ways to separate effect of bit energy \( E_b \) from constellation geometry parameters \( K, M, d_{\min} \).