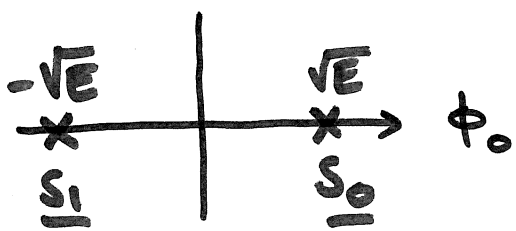
ECE 544: Digital Communications (Session 28)

Constellation: $\{ \underline{s}_i \in \mathbb{R}^J : i = 0, 1, \ldots M-1 \}$ M, J-dimensional real vectors

$$E_s \triangleq \frac{1}{M} \sum_{i=0}^{M-1} \| \underline{s}_i \|^2$$ the average energy per transmitted symbol (assuming equal priors)

$$E_b = E_s / \log_2 M$$ the average energy per bit

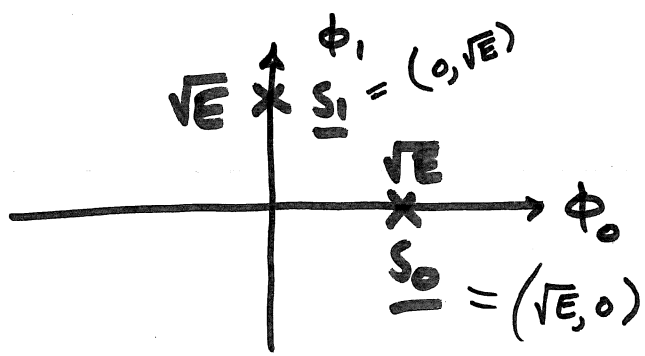
Antipodal / BPSK



$\Rightarrow E_s = E = E_b$

$D_{min} \triangleq$ minimum distance $= 2\sqrt{E}$

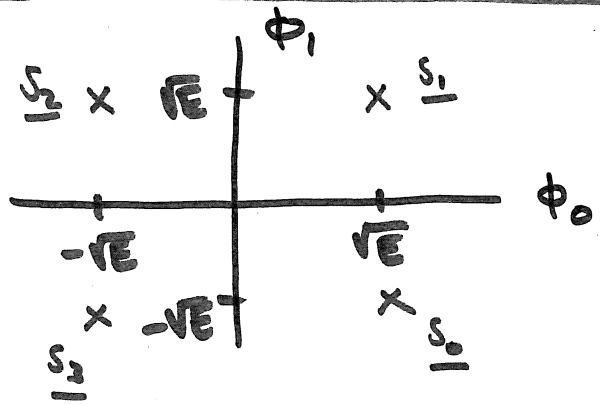
Orthogonal



$\Rightarrow E_s = E = E_b$

$D_{min} = \sqrt{2E}$

Square / QPSK / 4-QASK



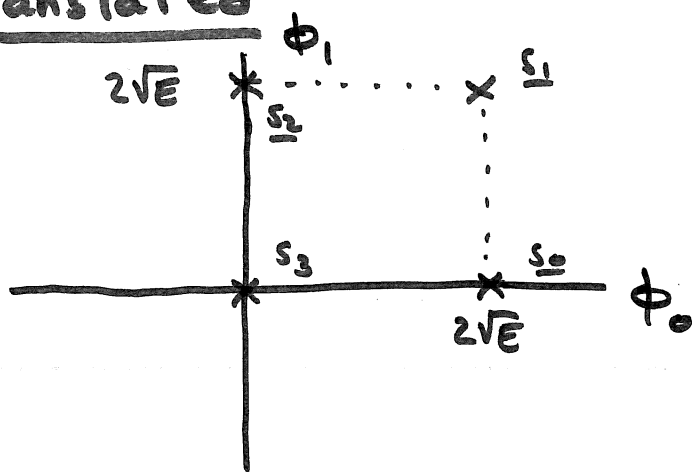
$$\Rightarrow \|\underline{s_1}\|^2 = 2E \Rightarrow E_s = \frac{1}{4}\cdot 4(2E) = 2E$$

$$= (\sqrt{E})^2 + (\sqrt{E})^2$$

$$E_b = E$$

$$D_{min} = 2\sqrt{E}$$

Translated



$$\Rightarrow \|\underline{s_0}\|^2 = 4E = \|\underline{s_2}\|^2$$

$$\|\underline{s_3}\|^2 = 0 \qquad \|\underline{s_1}\|^2 = 4E + 4E = 8E$$

$$\therefore E_s = \frac{16E}{4} = 4E$$

$$E_b = 2E$$

$$D_{min} = 2\sqrt{E}$$

Same D_{min} as square

$\Rightarrow$ same error perf in AWGN

$\Rightarrow$ takes twice the sig. energy

<u>General Fact</u> $\underline{s}_0 \; \underline{s}_1 \; \cdots \; \underline{s}_{M-1}$ a constellation

Consider translating it by a vector $\underline{t}$, i.e., new constellation

$$\underline{s}_0 - \underline{t}, \quad \underline{s}_1 - \underline{t}, \quad \ldots, \quad \underline{s}_{M-1} - \underline{t}$$

$$E_s = \frac{1}{M} \sum_{i=0}^{M-1} \| \underline{s}_i - \underline{t} \|^2$$

average energy per symbol for new constellation

* D_{min} invarient to translation $\underline{t}$

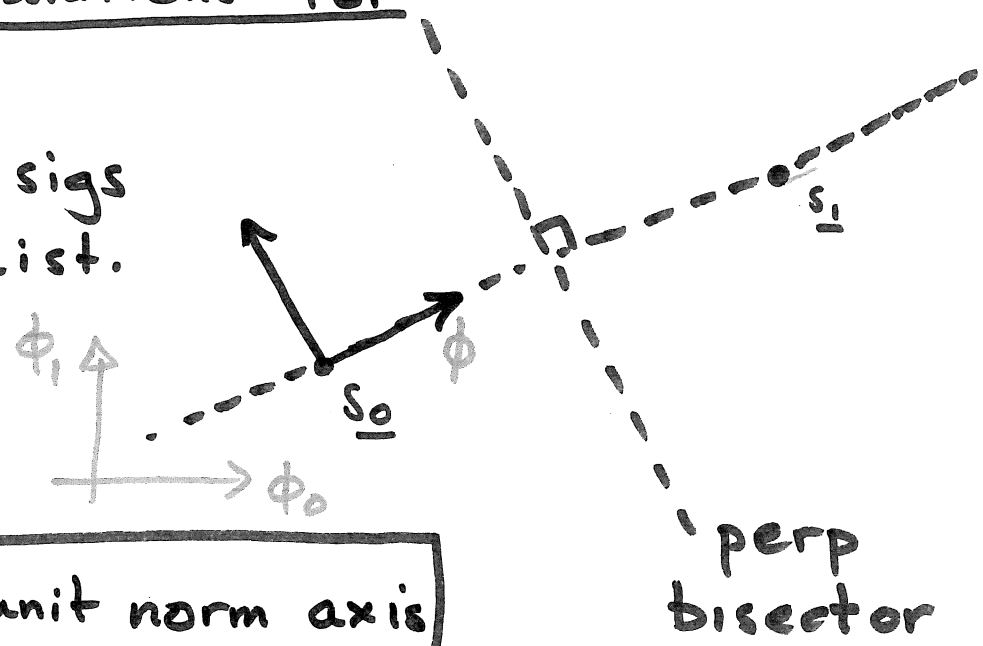
* To minimize E_s over $\underline{t}$ choose

$$\underline{t} = \frac{1}{M} \sum_{i=0}^{M-1} \underline{s}_i \qquad \text{(centroid)}$$

Prob. of Symbol Error Calculations for ML Receivers

Consider case of two sigs $\underline{s_0}$ and $\underline{s_1}$ separated by dist.

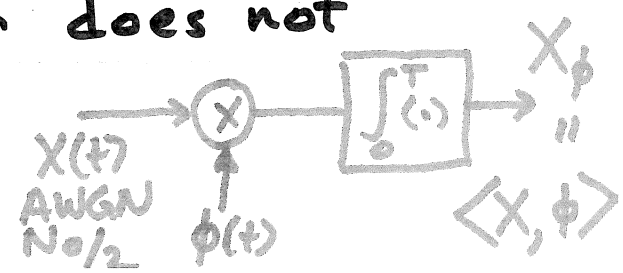
$$D = \| \underline{s_0} - \underline{s_1} \|$$

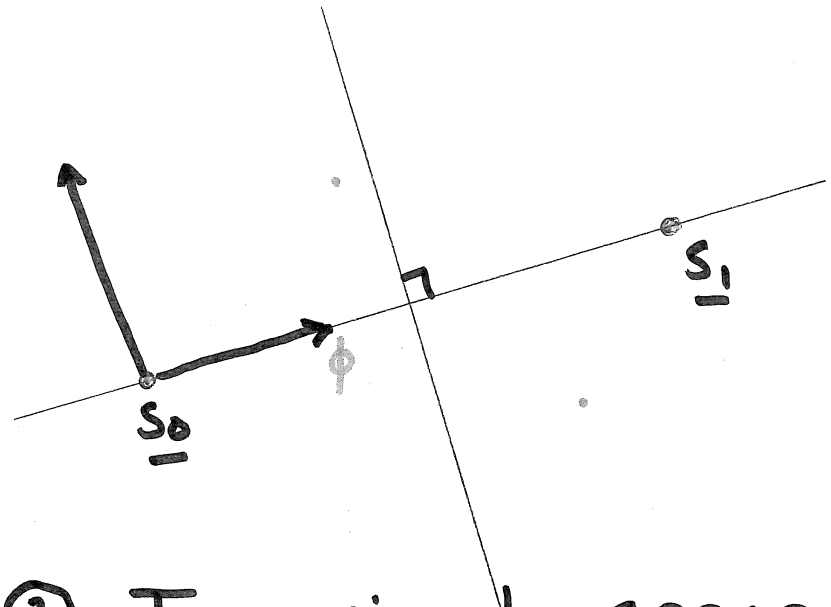


Property of AWGN

Its projection onto any unit norm axis is a Gaussian $N(0, \frac{N_0}{2})$ rv.

① Rotate axes until one axis points in direction of line connecting $\underline{s_0}$ and $\underline{s_1}$; axes rotation does not change signal energies.





② Let $N = \langle x, \phi \rangle$ be the Gaussian r.v. resulting from projection of AWGN along direction of line joining $\underline{s_0}$ and $\underline{s_1}$

③ Two signal error probability:

$P_2(1|0) \triangleq$ prob. that ML receiver chooses $\underline{s_1}$ given $\underline{s_0}$

$$= P(N > D/2) = P\left(\frac{N}{\sqrt{N_0/2}} > \frac{D/2}{\sqrt{N_0/2}}\right)$$

$$= Q\left(D/\sqrt{2N_0}\right)$$

Notes

① By symmetry $p_2(110) = p_2(011)$

② If constellation has only two signals then these pairwise conditional error probabilities tell the whole story.

③ Fact useful for bounds:

$$Q(x) \leq \frac{1}{2} \exp\left(-\frac{1}{2}x^2\right)$$

$\therefore$ as $x \to \infty$ the Gaussian tail prob. function tends to zero as

$$e^{-x^2/2}$$

Many-Signal Error Probabilities/Bounds

$\underline{s}_0 \quad \underline{s}_1 \quad \cdots \quad \underline{s}_{M-1}$ AWGN, ML receiver, observation $\underline{y}$

$$P_e = \sum_{k=0}^{M-1} P(\text{error} \mid \underline{s}_k \text{ sent})\, \pi_k = \frac{1}{M} \sum_{k=0}^{M-1} P(\underline{y} \notin \Gamma_k \mid \underline{s}_k \text{ sent})$$

Look at terms

$$P(\underline{y} \notin \Gamma_k \mid \underline{s}_k \text{ sent}) = P\Big(\underline{y} \in \bigcup_{\substack{i=0 \\ i \neq k}}^{M-1} \Gamma_i \;\Big|\; \underline{s}_k \text{ sent}\Big)$$

$$= \sum_{\substack{i=0 \\ i \neq k}}^{M-1} P(\underline{y} \in \Gamma_i \mid \underline{s}_k \text{ sent})$$

Let $p_2(i|k)$ = prob. that obs. closer to $\underline{s}_i$ than to $\underline{s}_k$ given $\underline{s}_k$ sent.

In general $P(\underline{y} \in \Gamma_i \mid \underline{s}_k \text{ sent}) \le p_2(i|k)$

$$P(\underline{y} \notin \Gamma_k \mid \underline{s}_k \text{ sent}) \le \sum_{\substack{i=0 \\ i \neq k}}^{M-1} p_2(i|k) = \sum_{\substack{i=0 \\ i \neq k}}^{M-1} Q\left(\frac{\|\underline{s}_i - \underline{s}_k\|}{\sqrt{2N_0}}\right)$$

$$\Rightarrow P_e \le \frac{1}{M} \sum_{k=0}^{M-1} \sum_{\substack{i=0 \\ i \neq k}}^{M-1} Q\left(\frac{\|\underline{s}_i - \underline{s}_k\|}{\sqrt{2N_0}}\right)$$

sum has $M(M-1)$ terms
each term decays like
$\frac{1}{2} \exp\left(-\frac{1}{2} \|\underline{s}_i - \underline{s}_k\|^2 / 2N_0\right)$
for large $\|\underline{s}_i - \underline{s}_k\|$.

Therefore, the terms with least distance will dominate the sum.

Let $K \triangleq$ # of signal pairs whose distance is equal to the min. distance.

The double sum counts these pairs twice so

$$P_e \lesssim \frac{2K}{M} Q\left(\frac{D_{min}}{\sqrt{2N_0}}\right)$$

now only an "approx." upper bound

Often write in terms of a "normalized" min. dist.

$$d_{min} \triangleq \frac{D_{min}}{\sqrt{2E_b}}$$

$$\Rightarrow \quad P_e \lesssim \frac{2K}{M} Q\left(\sqrt{\frac{d_{min}^2 E_b}{N_0}}\right)$$

Writing this ways to separate effect of bit energy E_b from constellation geometry parameters K, M, d_{min}.