ECE 544 Digital Commications (Session 17)

Recall $H_0: Y(t) = N(t)$ ← Gaussian, zero mean C_N with $\phi_k(\cdot), \lambda_k$

$0 \le t \le T$

vs.

$H_1: Y(t) = S(t) + N(t)$

$\Updownarrow$

$H_0: Y_k = N_k$

$k = 1, 2 \ldots$

vs.

$H_1: Y_k = S_k + N_k$

$$N_k = \int_0^T N(t)\phi_k(t)\,dt \sim \mathcal{N}(0, \lambda_k)$$

$$S_k = \int_0^T S(t)\phi_k(t)\,dt$$

Truncate to $k = 1, 2 \ldots n$ and then take limit $n \to \infty$

Summary of Truncated Problem

$$L_n(Y) = \exp\left\{ \boxed{\sum_{k=1}^{n} \frac{s_k Y_k}{\lambda_k}} - \frac{1}{2} \sum_{k=1}^{n} \frac{s_k^2}{\lambda_k} \right\}$$

$$G_n(Y) = \sum_{k=1}^{n} s_k Y_k / \lambda_k$$

$$G_n \sim \begin{cases} N(0, \sigma_n^2) & \text{under } H_0 \\ N(\sigma_n^2, \sigma_n^2) & \text{under } H_1 \end{cases} \qquad \sigma_n^2 = \sum_{k=1}^{n} s_k^2 / \lambda_k$$

$$G_n \begin{matrix} \geq \\ < \end{matrix} \frac{1}{2}\sigma_n^2 \quad \begin{matrix} \text{say } H_1 \\ \text{say } H_0 \end{matrix} \implies \begin{aligned} P_{e,0} &= Q(\sigma_n/2) \\ &= P_{e,1} \end{aligned}$$

$\sigma_n^2 \nearrow$ as $n \to \infty$ $\Longrightarrow$ Has limit though possibly $+\infty$

$$P_{e,0} = P_{e,1} = Q(\sigma_n/2) \downarrow \quad \text{as } n \to \infty$$

Define:

$$G = \lim_{n \to \infty} G_n = \sum_{k=1}^{\infty} s_k Y_k / \lambda_k$$

$$\mathrm{Var}\{G\} \triangleq d^2 = \lim_{n \to \infty} \sigma_n^2 = \sum_{k=1}^{\infty} s_k^2 / \lambda_k$$

Statistic G really only makes sense if $d^2 < \infty \Leftarrow$ SNR parameter.

A Simpler Way to Generate G

Say all limits exist for simplicity.

$$g_k \triangleq s_k/\lambda_k \qquad k = 1, 2 \ldots$$

Define $\quad g(t) = \sum_{k=1}^{\infty} g_k \, \phi_k(t) \quad \Rightarrow G = \sum_{k=1}^{\infty} g_k Y_k$

$$\Rightarrow \quad G = \sum_{k=1}^{\infty} g_k \underbrace{\int_0^T Y(t)\, \phi_k(t)\, dt}_{Y_k} = \int_0^T Y(t) \sum_{k=1}^{\infty} g_k \, \phi_k(t)\, dt$$

$$= \int_0^T Y(t)\, g(t)\, dt$$

i.e., the inf. bank of ϕ_k correlators reduced to a single correlator.

Can also show $g(t)$ is solution to

$$s(t) = \int_0^T C_N(u,t)\, g(u)\, du \qquad 0 \le t \le T$$

See: Van Trees, Vol I or Davenport + Root

Also

$$d^2 = \sum_{k=1}^{\infty} g_k s_k = \int_0^T s(t)\, g(t)\, dt$$

Another way to write L.R.

$$L(Y(t): 0 \le t \le T) =$$

$$\exp\left\{ \int_0^T y(t)\, g(t)\, dt - \frac{1}{2} \int_0^T s(t)\, g(t)\, dt \right\}$$

Historical Notes

$$G = \int_0^T g(t)\, Y(t)\, dt$$

was shown to be opt. for H_0 vs. H_1 under assumpt. that $g(t)$ is finite energy:

Grenander 1950 "Stoch. Proc. for Stat. Inference" Arkvs. for Matematic. pp 195-277

Often need to include deltas + their derivs. in $g(t)$ to solve integral eqn:

Kadota 1967 Diff. of KLT + Apps. to Opt. Det. IT-13 April pp 255-260.

Should also look at white noise:

$$g(t) = \frac{2}{N} s(t)$$

$$d^2 = \frac{2}{N_0} \int s^2(t)\, dt = \frac{2E}{N_0}$$

<u>Re: More on Sing. Detection</u> $\Longleftrightarrow$ <u>When would it Happen?</u>

Singular detection said to occur $\quad \sigma_n^2 \longrightarrow d^2 = +\infty$

Then $P_{e,0} = P_{e,1} \longrightarrow 0$ as $n \to \infty$

In order investigate this, need to look @ I. E. soln.; which is hard.

* Let that observ interval $[0,T] \longrightarrow (-\infty, \infty)$

* Let the noise be WSS.

The integral eq. becomes

$$s(t) = \int_{-\infty}^{\infty} C_{NN}(t-u)\, q_\infty(u)\, du$$

$$\updownarrow$$

$$S(f) = S_{NN}(f)\, Q_\infty(f) \implies Q_\infty(f) = \frac{S(f)}{S_{NN}(f)}$$

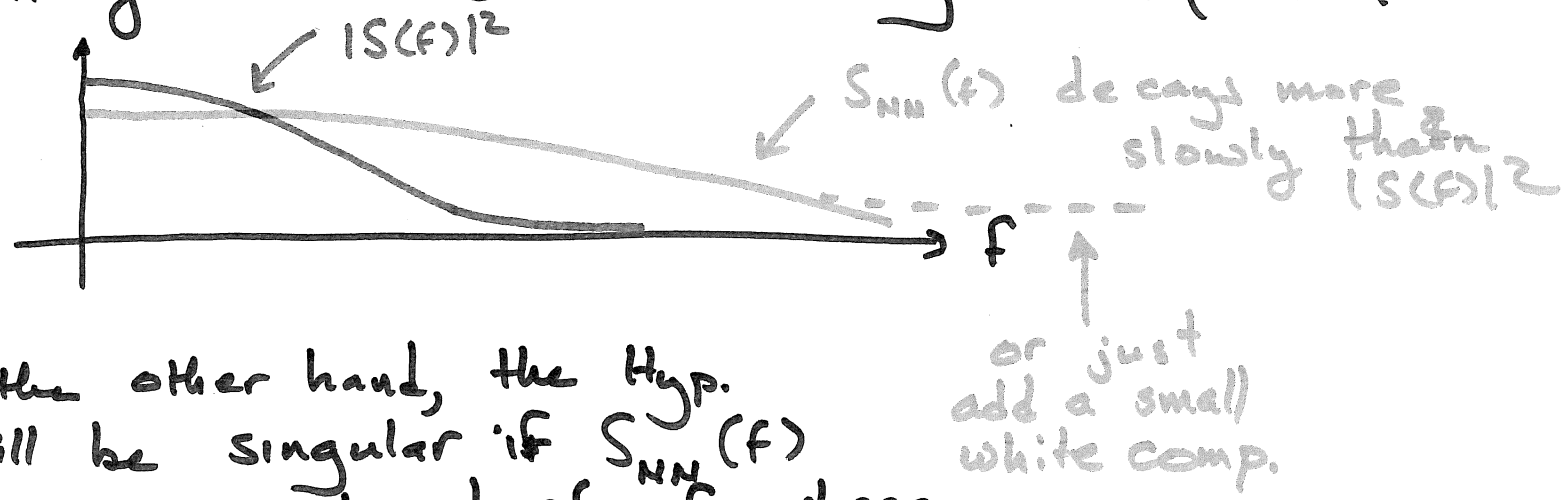
Also can solve

$$d_\infty^2 = \int_{-\infty}^{\infty} s(t)\, q_\infty(t)\, dt = \int_{-\infty}^{\infty} S^*(f)\, Q_\infty(f)\, df$$

$$d_\infty^2 = \int_{-\infty}^{\infty} |S(f)|^2 / S_{NN}(f)\, df$$

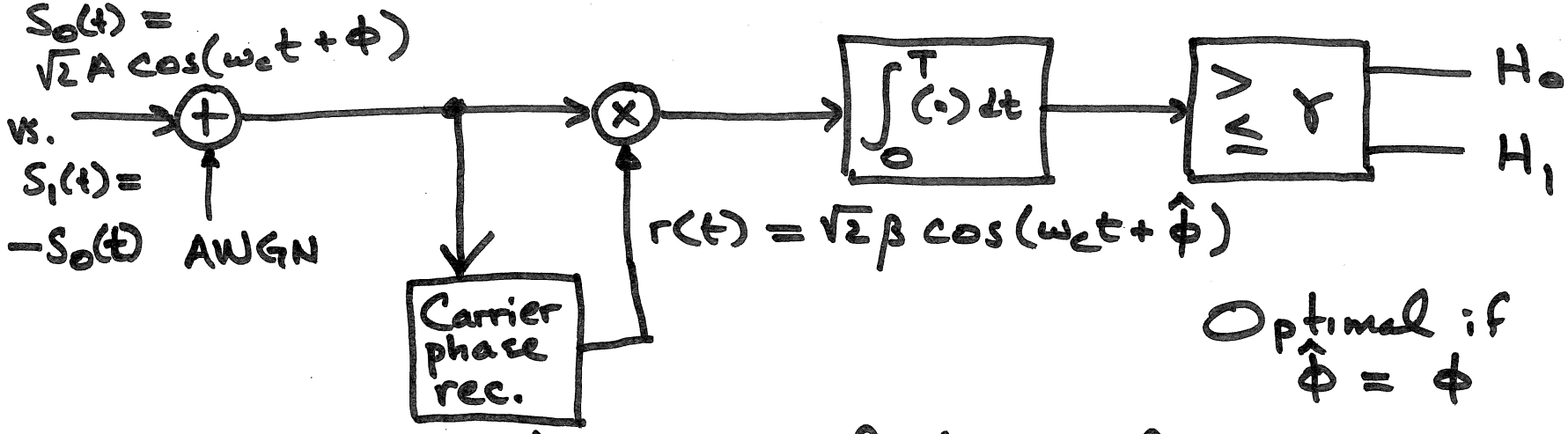
First: when will $d_\infty^2 < \infty \implies d^2(T) < \infty$ for finite T

$\implies$ need $S_{NN}(f)$ to be nonzero $\forall f$ and that it go to zero more slowly than $|S(f)|^2$



$\implies$ On the other hand, the Hyp. Test will be singular if $S_{NN}(f)$ vanishes on any interval of f where $S(f)$ does not vanish.

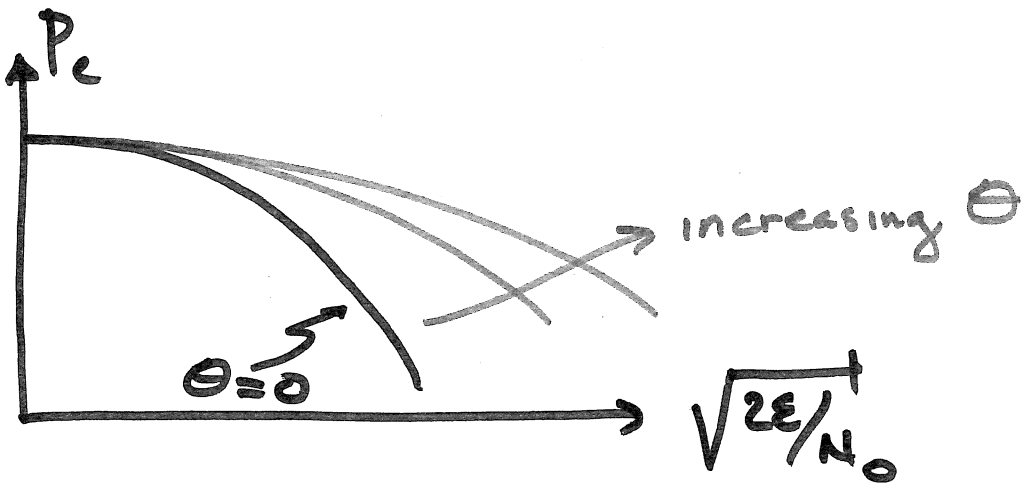
MBP Chap. 6 BPSK with Imperfect Phase Ref.



Optimal if $\hat{\phi} = \phi$

Define $\theta = \phi - \hat{\phi}$. MBP finds perf.

$$P_e(\theta) = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\cos\theta\right)$$



This gives determ. model.

More common is the phase error is unknown, and model it as a rv. Say Θ has a pdf: $f_\Theta(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma_\Theta} e^{-\theta^2/2\sigma_\Theta^2}$

Then could calc. avg. P_e:

$$P_e = \int f_\Theta(\theta)\, Q\left(\sqrt{2\mathcal{E}/N_0}\cos\theta\right) d\theta$$