Recall \( H_0 : Y(t) = N(t) \) \( \text{Gaussian, zero mean } N(0, \Lambda_k) \quad 0 \leq t \leq T \)

vs.

\( H_1 : Y(t) = S(t) + N(t) \)

\[ N_k = \int_0^T N(t) \phi_k(t) \, dt \sim N(0, \Lambda_k) \]

\[ S_k = \int_0^T S(t) \phi_k(t) \, dt \]

\[ H_0 : Y_k = N_k \quad k = 1, 2, \ldots \]

vs.

\( H_1 : Y_k = S_k + N_k \)

\( \text{Truncate to } k = 1, 2, \ldots n \) and then take limit \( n \to \infty \)
Summary of Truncated Problem

\[ L_n(Y) = \exp \left\{ \sum_{k=1}^{n} \frac{s_k Y_k}{\lambda_k} - \frac{1}{2} \sum_{k=1}^{n} \frac{s_k^2}{\lambda_k} \right\} \]

\[ G_n(Y) = \sum_{k=1}^{n} \frac{s_k Y_k}{\lambda_k} \]

\[ N(0, \sigma_n^2) \text{ under } H_0 \quad \sigma_n^2 = \sum_{k=1}^{n} \frac{s_k^2}{\lambda_k} \]

\[ G_n \sim N(\sigma_n^2, \sigma_n^2) \text{ under } H_1 \]

\[ G_n \geq \frac{1}{2} \sigma_n^2 \text{ say } H_1 \Rightarrow P_{e,0} = O(\sigma_n^2) \]

\[ G_n < \frac{1}{2} \sigma_n^2 \text{ say } H_0 \Rightarrow P_{e,1} = P_{e,1} \]
\[ \sigma_n^2 \to 1 \text{ as } n \to \infty \implies \text{Has limit through possibly } +\infty \]

\[ P_{e,0} = P_{e,1} = O(\sigma_n^2) \downarrow \text{ as } n \to \infty \]

Define:
\[ G = \lim_{n \to \infty} G_n = \sum_{k=1}^{\infty} \frac{s_k T_k}{\lambda_k} \]
\[ \text{Var}\{G\} = d^2 = \lim_{n \to \infty} \sigma_n^2 = \sum_{k=1}^{\infty} \frac{s_k^2}{\lambda_k} \]

Statistic \( G \) really only makes sense if \( d^2 < \infty \iff \text{SNR parameter.} \)
A Simpler Way to Generate $G$

Say all limits exist for simplicity.

$$q_k = \frac{s_k}{\lambda_k} \quad k = 1, 2 \ldots$$

Define

$$q(t) = \sum_{k=1}^{\infty} q_k \phi_k(t) \quad \Rightarrow \quad G = \sum_{k=1}^{\infty} q_k T_k$$

$$\Rightarrow \quad G = \sum_{k=1}^{\infty} q_k \int_0^T Y(t) \phi_k(t) dt = \int_0^T Y(t) \sum_{k=1}^{\infty} q_k \phi_k(t) dt$$

$$= \int_0^T Y(t) g(t) dt$$

i.e., the mf. bank of $\phi_k$ correlators reduced to a single correlator.
Can also show $g(t)$ is solution to

$$s(t) = \int_0^T C_M(u,t) g(u) \, du \quad 0 \leq t \leq T$$

Also

$$d^2 = \sum_{k=1}^{\infty} g_k s_k = \int_0^T s(t) g(t) \, dt$$

Another way to write L.R.

$$L(Y(t) : 0 \leq t \leq T) = \exp \left\{ \int_0^T y(t) g(t) \, dt - \frac{1}{2} \int_0^T s(t) g(t) \, dt \right\}$$
Historical Notes

\[ G = \int_{0}^{T} g(t) Y(t) \, dt \]

was shown to be opt. for \( H_0 \) vs. \( H_1 \) under assumt. that \( g(t) \) is finite energy:


Often need to include deltas + their derivs. in \( g(t) \) to solve integral eqn:

Should also look at white noise:

\[ q(t) = \frac{2}{N} s(t) \]

\[ d^2 = \frac{2}{N_0} \int s^2(t) \, dt = \frac{2E}{N_0} \]

Re: More on Sng. Detection \( \iff \) When would it happen?

Singular detection said to occur \( \sigma_n^2 \to d^2 = +\infty \)

Then \( P_{e,0} = P_{e,1} \to 0 \) as \( n \to \infty \)
In order investigate this, need to look @ E.E. soln. j which is hard.

* Let that observ interval \([0, T] \rightarrow (-\infty, \infty)\)
* Let the noise be WSS.

The integral eq. becomes

\[
S(t) = \int_{-\infty}^{\infty} C_{NN}(t-u) g_{\infty}(u) \, du
\]

↓

\[
S(f) = S_{NN}(f) \, \Phi_{\infty}(f) \quad \Rightarrow \quad \Phi_{\infty}(f) = \frac{S(f)}{S_{NN}(f)}
\]

Also can solve

\[
d_{\infty}^2 = \int_{-\infty}^{\infty} S(t) \, g_{\infty}(t) \, dt = \int_{-\infty}^{\infty} S^*(f) \, \Phi_{\infty}(f) \, df
\]
\[ d_\infty^2 = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_{nn}(f)} \, df \]

First: When will \( d_\infty^2 < \infty \Rightarrow d^2(T) < \infty \) for finite \( T \)

\[ \Rightarrow \text{need } S_{nn}(f) \text{ to be nonzero } \forall f \text{ and that it go to zero more slowly than } (|S(f)|^2) \]

\[ S_{nn}(f) \text{ decays more slowly than } |S(f)|^2 \]

\[ \text{or just add a small white comp.} \]

\[ \Rightarrow \text{On the other hand, the Hyp. Test will be singular if } S_{nn}(f) \text{ vanishes on any interval of } f \text{ where } S(f) \text{ does not vanish.} \]
Define $\Theta = \phi - \hat{\phi}$. MBP finds perf.

$$P_e(\Theta) = \Phi\left(\frac{\sqrt{2E}}{N_0} \cos \Theta\right)$$
More common is the phase error is unknown, and model it as a rv. Say \( \Theta \) has a pdf: 

\[ f_\Theta(\theta) = e^{-\theta^2/2} \sqrt{2\pi} \]

Then could calc. avg. \( P_e \):

\[ P_e = \int f_\Theta(\theta) \Phi(\sqrt{2\pi/N_0} \cos \theta) \, d\theta \]