ECE 544  Digital Communications (Session 16)

Have 2 Theorems on O.N. representations for rand.
processes:

\{ X(t) \text{ WSS, zero mean, cov. funct } C_X(t) \text{ which is continuous. Let } [0, T] \text{ be an interval.} \}

**Mercer's Theorem**

- \( C_X(t-s) = \sum_{k=0}^{\infty} \lambda_k \phi_k(t) \phi_k(s) ; 0 \leq t, s \leq T \)

where

- \( \lambda \phi(t) = \int_{0}^{T} C_X(u-t) \phi(u) du \) eigenvalues

- \( \sum_{k=0}^{\infty} \lambda_k \phi_k(t) \phi_k(s) dt = \delta[k-l] \) eigenfunctions

\( \phi_k(t) \)
Karhunen-Loève Expansion Thm

$X(t)$ represented as mean-square convergent series

$$X(t) = \sum_{k=0}^{\infty} X_k \phi_k(t) \quad 0 \leq t \leq T$$

where

$$X_k = \int_{0}^{T} X(t) \phi_k(t) \, dt \quad k = 0, 1, 2 \ldots$$
Comments

1. Under these conditions the "Infinite Correlator Bank" architecture is optimal, i.e., informat. lossless.

2. $C_x(t)$ strictly p.d. $\Rightarrow \phi_k(t)$ from Mercer are a C.O.N.S.

3. KL coeffs are orthogonal random variables

$$E\{X_k X_\ell\} = \begin{cases} \lambda_k & k = \ell \\ 0 & k \neq \ell \end{cases}$$

4. Projection of r.p. $X(t)$ along coord. direction $k$ is: $X_k \phi_k(t)$

$\Rightarrow \lambda_k$ is average energy in that direction
3) \( T C_x(0) = \sum_{k=0}^{\infty} \lambda k \)

Example: Wiener (Brownian Motion) Process

\( W(t) \) \( t \geq 0 \); Gaussian; \( W(0) = 0 \); \( E\{W(t)^2\} = 0 \)
\( E\{W^2(t)\} = \frac{N_0 t}{2} \)

If \( s < t < u \) \( W(u) - W(t) \perp W(t) - W(s) \)

\( C_W(t,u) = \frac{N_0}{2} \min(t,u) \)

Find KL expansion
Answer (HW):

eigenfunctions: \[ \psi_k(t) = \sqrt{\frac{1}{T}} \sin \left[ (k-\frac{1}{2}) \frac{\pi}{T} t \right] \]
\[ 0 \leq t \leq T \]
\[ k = 1, 2, \ldots \]

eigenvalues: \[ \lambda_k = \frac{N_0 T^2}{2 \pi^2 (k-\frac{1}{2})^2} \]
\[ k = 1, 2, \ldots \]

Example (HW) White noise process as formal derivative of Wiener.
How to KL Expansion to Detection?

\[ H_0 : Y(t) = N(t) \quad 0 \leq t \leq T \]

vs.

\[ H_1 : Y(t) = S(t) + N(t) \]

\[ N(t) \text{ is Gauss, zero-mean} \]

\[ C_N \text{ a continuous conv.} \]

\[ s(t) \text{ is det + finite energy} \]

Need to compute samples.

Need to know that KL expansion will work for either \( H_0 \) or \( H_1 \).

Must be able to find pdfs under \( H_0 \) / \( H_1 \).
Since $N(t)$ appears in both hypotheses, see that only the $\phi_k(t)$, $\lambda_k$ expansion is needed.

$$Y_k = \int_0^T Y(t) \phi_k(t) dt = \begin{cases} \int_0^T N(t) \phi_k(t) dt = N_k & \text{under } H_0 \\ \int_0^T (S(t) + N(t)) \phi_k(t) dt = S_k + N_k & \text{under } H_1 \end{cases}$$

The following hypotheses are same.

$$H_0 : Y(t) = N(t) \quad \quad H_0 : Y_k = N_k \quad \quad N_k \sim N(0, \lambda_k)$$

$$0 \leq t \leq T \quad \quad k = 1, 2, \ldots \quad \quad \text{indep.}$$

$$H_1 : Y(t) = S(t) + N(t) \quad \quad H_1 : Y_k = S_k + N_k$$

Remember still need to find L.R. But still don't know what to do about pdfs.

So we truncate to $n$ observations, solve truncated problem, maybe can take $n \to \infty$. 
Truncated:

\[ H_0 : Y_k = N_k \]
\[ \text{vs} \]
\[ H_1 : Y_k = S_k + N_k \]

\[ f_{0,n}(y) = \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi \lambda_k}} e^{-y_k^2/2\lambda_k} \]
\[ f_{1,n}(y) = \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi \lambda_k}} e^{-(y_k - s_k)^2/2\lambda_k} \]

This fits into our procedure: find LR \Rightarrow simplify

\[ L_n(y) = \frac{f_{1,n}(y)}{f_{0,n}(y)} = \exp \left\{ \sum_{k=1}^{n} \frac{s_k y_k}{\lambda_k} - \frac{1}{2} \sum_{k=1}^{n} \frac{s_k^2}{\lambda_k} \right\} \]

\[ G_n(y) = \sum_{k=1}^{n} \frac{s_k y_k}{\lambda_k} \]

Because \( L_n \) is a monotone inc. funct. of \( G_n \); base test on \( G_n \).
$G_n$ is Gaussian under $H_0$ or $H_1$. So only need mean, var.

**Under $H_0$**

$$E\{G_n | H_0\} = E_0 \{G_n\}$$

$$= \sum_{k=1}^{n} E\{Y_k\} \frac{S_k}{\lambda_k} = 0$$

$$\text{Var}_0 \{G_n\} = \text{Var}_0 \left\{ \sum_{k=1}^{n} \frac{S_k Y_k}{\lambda_k} \right\} = \sum_{k=1}^{n} \left( \frac{S_k}{\lambda_k} \right)^2 \text{Var}_0 \{Y_k\}$$

$$= \sum_{k=1}^{n} \frac{S_k^2}{\lambda_k} \Delta = \sigma_n^2$$
Under $H_1$,

$$E_i \{G_n\} = \sum_{k=1}^{n} \frac{S_k}{\lambda_k} E_i \{Y_k\} = \sum_{k=1}^{n} \frac{S_k^2}{\lambda_k}$$

$$= \alpha_n^2$$

$$\text{Var}_i \{G_n\} = \text{Var}_o \{G_n\} = \alpha_n^2$$

$\therefore$ Stat description is

$$G_n \sim N(0, \alpha_n^2) \text{ under } H_0$$

$$G_n \sim N(\alpha_n^2, \alpha_n^2) \text{ under } H_1$$
Perf. of Truncated Test  Bayes with \( \tau = 1 \)

\[
\ln(y) \leq \tau = 1 \quad \text{Say } H_1 \quad \text{say } H_0 \\
\ln L_n = G_n - \frac{1}{2}\epsilon_n^2
\]

\[
\Leftrightarrow \\
G_n(y) \geq \ln\tau + \frac{1}{2}\epsilon_n^2 \quad \text{Say } H_1 \quad \text{say } H_0 \Rightarrow \text{Need } P_{e,0} \text{ & } P_{e,1}
\]

\[
P_{e,0} = P_0 \left( G_n \geq \frac{1}{2}\epsilon_n^2 \right) = P_0 \left( \frac{G_n}{\epsilon_n} \geq \frac{\epsilon_n}{2} \right) \\
= Q\left( \frac{\epsilon_n}{2} \right)
\]

Similarly, \( P_{e,1} = Q\left( \frac{\epsilon_n}{2} \right) \)
Recall \( \sigma_n^2 = \sum_{k=1}^{n} \frac{s_k^2}{\lambda_k} \) as \( n \rightarrow \infty \) \( \sigma_n^2 \rightarrow ? \)

Therefore in taking limit as \( n \rightarrow \infty \) both \( P_{x,0} \) and \( P_{x,1} \) decrease.

Back to \( G_n \)

\[
\lim_{n \rightarrow \infty} G_n = G = \sum_{k=1}^{\infty} \frac{s_k Y_k}{\lambda_k}
\]