ECE 544 Digital Communications (Session 16)

Have 2 Theorems on O.N. representations for rand. processes:

$\left\{ X(t) \text{ WSS, zero mean, cov. funct } C_x(\tau) \text{ which is continuous. Let } [0,T] \text{ be an interval.} \right\}$

Mercer's Theorem

$C_x(\tau)$ has expansion as unif. + abs. converg. series

$$C_x(t-s) = \sum_{k=0}^{\infty} \lambda_k \phi_k(t) \phi_k(s); \quad 0 \le t,s \le T$$

where

$$\lambda \phi(t) = \int_0^T C_x(u-t)\phi(u)\,du$$

st $\int_0^T \phi_k(t)\phi_\ell(t)\,dt = \delta[k-\ell]$

eigenvalues λ_k

eigenfuncts $\phi_k(t)$.

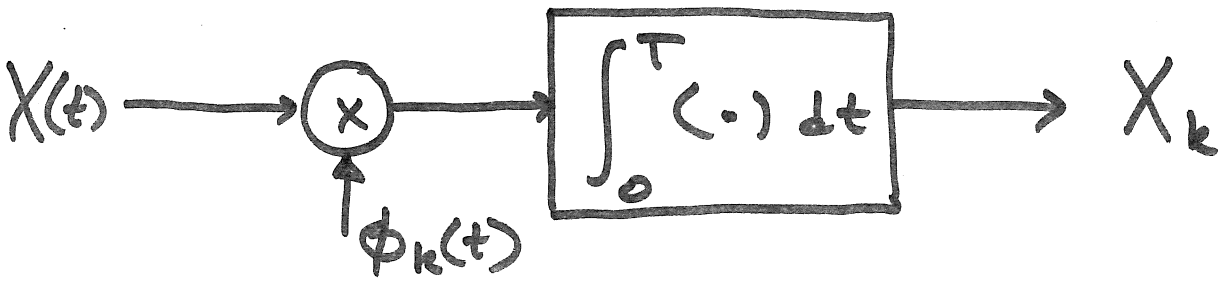
Karhunen-Loeve Expansion Thm

$X(t)$ represented as mean-square convergent series

$$X(t) = \sum_{k=0}^{\infty} X_k \, \phi_k(t) \qquad 0 \le t \le T$$

where

$$X_k = \int_0^T X(t)\, \phi_k(t)\, dt \qquad k = 0, 1, 2 \ldots$$



Comments

① Under these conditions the "Infinite Correlator Bank" architecture is optimal, i.e., informat. lossless.

② $C_x(\tau)$ strictly p.d. $\Longrightarrow$ $\phi_k(t)$ from Mercer
$\Longleftrightarrow S_x(f) > 0$ are a C.O.N.S.

③ KL coeffs are orthogonal random variables

$$E\{X_k X_\ell\} = \begin{cases} \lambda_k & k = \ell \\ 0 & k \neq \ell \end{cases}$$

④ Projection of r.p. $X(t)$ along coord. direction k is: $X_k \phi_k(t)$

$\Longrightarrow \lambda_k$ is average energy in that direction

⑤ $T C_x(0) = \sum_{k=0}^{\infty} \lambda_k$

$\boxed{\text{Example:}}$ <u>Wiener (Brownian Motion) Process</u>

$W(t)$ $t \geq 0$; Gaussian; $W(0) = 0$; $E\{W(t)\} = 0$

$E\{W^2(t)\} = \frac{N_0}{2} t$

If $s < t < u$ $\quad W(u) - W(t) \perp\!\!\!\perp W(t) - W(s)$

$C_W(t,u) = \frac{N_0}{2} \min(t,u)$

Find KL expansion

<u>Answer (HW):</u>

eigenfunctions: $\psi_k(t) = \sqrt{2/T}\,\sin\left[(k-\tfrac{1}{2})\frac{\pi}{T}t\right]$

$0 \le t \le T$

$k = 1, 2, \ldots$

eigenvalues: $\lambda_k = \dfrac{N_0 T^2}{2\pi^2 (k-\tfrac{1}{2})^2}$ $\quad k = 1, 2 \ldots$

$\boxed{\text{Example}}$ (HW) White noise process as formal derivative of Wiener.

How to KL Expansion to Detection?

$H_0: \quad Y(t) = N(t)$

vs.

$H_1: \quad Y(t) = s(t) + N(t)$

$0 \le t \le T$

$N(t)$ is Gauss, zero-mean
C_N a continuous cov.
with $\phi_k(\cdot)$, λ_k

$s(t)$ is det + finite energy
and

$$s(t) = \sum_{k=1}^{\infty} s_k \phi_k(t)$$

Now want to <u>see why KL helps us.</u>

Need to compute samples.

* Need to know that KL expansion will work for for either H_0 or H_1
* Must be able to find pdfs under H_0 / H_1.

Since $N(t)$ appears in both hypotheses, see that only the $\phi_k(\cdot)$, λ_k KL expansion is needed.

$$Y_k = \int_0^T Y(t)\,\phi_k(t)\,dt = \begin{cases} \int_0^T N(t)\,\phi_k(t)\,dt = N_k & \text{under } H_0 \\ \int_0^T \big(S(t)+N(t)\big)\,\phi_k(t)\,dt = S_k + N_k & \text{under } H_1 \end{cases}$$

The following hypoth. test are same

$$\begin{array}{l} H_0 : Y(t) = N(t) \\ \text{vs} \\ H_1 : Y(t) = S(t) + N(t) \end{array} \quad 0 \le t \le T \iff \begin{array}{l} H_0 : Y_k = N_k \\ \text{vs} \\ H_1 : Y_k = S_k + N_k \end{array} \quad k = 1, 2, \ldots \quad \begin{array}{l} N_k \sim N(0, \lambda_k) \\ \text{indep.} \end{array}$$

Remember $\rightarrow$ still need to find L.R. But still don't know what to do about pdfs.

So we truncate to n observations, solve truncated problem, maybe can take $n \rightarrow \infty$.

Truncated:

$H_0 : Y_k = N_k$ $\qquad k = 1, 2 \ldots n \qquad N_k \sim N(0, \lambda_k)$ indep.

vs

$H_1 : Y_k = s_k + N_k$

$$f_{0,n}(y) = \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi\lambda_k}} e^{-y_k^2/2\lambda_k} \quad \text{vs.} \quad f_{1,n}(y) = \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi\lambda_k}} e^{-(y_k - s_k)^2/2\lambda_k}$$

$y = (y_1, y_2 \ldots y_n)$

This fits into our procedure $\Longrightarrow$ find LR $\Rightarrow$ simplify

$$L_n(y) = \frac{f_{1,n}(y)}{f_{0,n}(y)} = \exp\left\{ \sum_{k=1}^{n} \frac{s_k y_k}{\lambda_k} - \frac{1}{2} \sum_{k=1}^{n} \frac{s_k^2}{\lambda_k} \right\}$$

$$G_n(y) = \sum_{k=1}^{n} s_k y_k / \lambda_k$$

Because L_n is a monotone inc. funct. of G_n; base test on G_n.

G_n is Gaussian under H_0 or H_1. So only need mean, var.

<u>Under H_0</u>

$$E\{G_n | H_0\} = E_0\{G_n\}$$

$$= \sum_{k=1}^{n} E_0\{Y_k\} s_k/\lambda_k = 0$$

$$\mathrm{Var}_0\{G_n\} = \mathrm{Var}_0\left\{\sum_{k=1}^{n} \frac{s_k Y_k}{\lambda_k}\right\} = \sum_{k=1}^{n} \left(\frac{s_k}{\lambda_k}\right)^2 \underbrace{\mathrm{Var}_0\{Y_k\}}_{\lambda_k}$$

$$= \sum_{k=1}^{n} s_k^2/\lambda_k \triangleq \sigma_n^2$$

<u>Under H_1</u>

$$E_1\{G_n\} = \sum_{k=1}^{n} \frac{s_k}{\lambda_k} \overset{s_k}{E_1\{Y_k\}} = \sum_{k=1}^{n} s_k^2/\lambda_k$$

$$= \sigma_n^2$$

$$Var_1\{G_n\} = Var_0\{G_n\} = \sigma_n^2$$

$\therefore$ Stat description is

$$G_n \sim \begin{cases} N(0, \sigma_n^2) & \text{under } H_0 \\ N(\sigma_n^2, \sigma_n^2) & \text{under } H_1 \end{cases}$$

Perf. of Truncated Test Bayes with $\tau = 1$

$$L_n(y) \underset{<}{\overset{\geq}{}} \tau = 1 \quad \begin{matrix} \text{say } H_1 \\ \text{say } H_0 \end{matrix} \qquad \ln L_n = G_n - \tfrac{1}{2}\sigma_n^2$$

$$\Longleftrightarrow$$

$$G_n(y) \underset{<}{\overset{\geq}{}} \underbrace{\ln \tau}_{0} + \tfrac{1}{2}\sigma_n^2 \quad \begin{matrix} \text{say } H_1 \\ \text{say } H_0 \end{matrix} \Longrightarrow \text{Need } P_{e,0} \;\&\; P_{e,1}$$

$$P_{e,0} = P_0\left(\frac{\overset{\to 0}{G_n}}{\sigma_n} \geq \frac{\frac{1}{2}\overset{\to 0}{\sigma_n^2}}{\sigma_n}\right) = P_0\left(\frac{G_n}{\sigma_n} \geq \frac{\sigma_n}{2}\right)$$

$$= Q\left(\frac{\sigma_n}{2}\right)$$

Similarly $P_{e,1} = Q(\sigma_n/2)$

Recall $\sigma_n^2 = \sum_{k=1}^{n} s_k^2/\lambda_k$ as $n \uparrow$; $\sigma_n^2 \uparrow$; $\sigma_n \uparrow$

Therefore in taking limit as $n \to \infty$ both $P_{e,0}$ and $P_{e,1}$ decrease.

Back to G_n

$$\lim_{n\to\infty} G_n = G = \sum_{k=1}^{\infty} \frac{s_k Y_k}{\lambda_k}$$

$$\sigma_n^2 \longrightarrow ?$$