EE 544 Dig. Comm. (Session 10)

MF Filtering for Colored Noise.
Approach: Whitening Filter

\[ X(t) \xrightarrow{H_w(\omega)} \text{Desire a white r.p. } W(t) \]

- \( X(t) \) mean Gaussian
- \( S_x(\omega) \) not necc. flat

Standard on LTI Filters:

\[ EX(t) = 0 \implies EW(t) = 0 \]
\[ X(t) \text{ Gauss. } \implies W(t) \text{ is Gauss.} \]
\[ S_w(\omega) = |H_w(\omega)|^2 S_x(\omega) \]
I seek a filter $H_w$ s.t.

$$1 = |H_w(\omega)|^2 S_x(\omega) \Rightarrow |H_w(\omega)|^2 = \frac{1}{S_x(\omega)}$$

Suppose there is such a filter $h_w(t) \leftrightarrow H_w(\omega)$

Then consider orig. prob but with $h_w$ inserted:

$$u_o(t) = \begin{cases} S_o * h_w(t) \\ S_1 * h_w(t) \end{cases} + W(t) \rightarrow \text{white with } S_w(\omega) = 1$$
This is already solved. Know that HF is
\[ h(t) = c \tilde{u}_{T_0}(t) = c \left[ u_0(T_0-t) - u_1(T_0-t) \right] \]

\[ H(\omega) = ce^{-j\omega T_0} \left[ \tilde{u}_0^*(\omega) - \tilde{u}_1^*(\omega) \right] \]
\[ = ce^{-j\omega T_0} H_W^*(\omega) \left[ S_0^*(\omega) - S_1^*(\omega) \right] \]

The overall filter
\[ H_W(\omega) H(\omega) = ce^{-j\omega T_0} \frac{S_0^*(\omega) - S_1^*(\omega)}{S_x(\omega)} \]

Questions
1. Existence of whitening filter
2. Optimality of approach.
Let $Z$ be a r.v. with two possible statistical descriptions ("states" of nature)

- $H_0: Z \sim f_0(z)$
- $H_1: Z \sim f_1(z)$

- Either $H_0$ or $H_1$ must occur but not both.
- May have (or may not) "side" or "prior" information about $H_0, H_1$,
  
  $\pi_0 = P(H_0 \text{ is true}) \quad \pi_1 = P(H_1 \text{ is true})$

  $\pi_0 + \pi_1 = 1$
A general decision rule is a two-valued function of the observations

\[ Z = z \]

\[ T(z) = \begin{cases} 0 \rightarrow \text{dec. rule chooses } H_0 \\
1 \rightarrow " " \quad " \quad H_1 \end{cases} \]

Let \( \Pi \) be the space of all possible values taken by \( Z \). Then specifying \( T \) is the same as specifying a partition

\[ \Pi = \Pi_0 \cup \Pi_1 \quad \text{st} \quad \Pi_0 \cap \Pi_1 = \emptyset \]

\[ z \in \Pi_0 \iff T(z) = 0 \quad \Rightarrow \quad T(z) = "\text{indicator}" \text{ of set } \Pi \]

\[ z \in \Pi_1 \iff T(z) = 1 \quad \Rightarrow \quad = 1_{\Pi}(z) \]
Every possible partition gives a decision rule. Our prev. threshold tests

\[
    z > Y \quad \text{decide } 0 \\
    z \leq Y 
    \quad \text{decide } 1
\]

corresponds to \( \Pi_0 = (\varphi, +\infty) \) and \( \Pi_1 = (-\infty, \varphi] \)

Optimizing Decision Rules

\[
    \Pi = \Pi_0 \cup \Pi_1 \quad \Pi_0 \cap \Pi_1 = \phi
\]

\[
    P_{e,0} = P(\text{say } H_1 | H_0 \text{ true}) \\
    P_{e,1} = P(\text{say } H_0 | H_1 \text{ true})
\]

\[
= P(Z \in \Pi_1 | H_0 \text{ true}) \\
= \int_{\Pi_1} f_0(z) \, dz
\]

\[
= P(Z \in \Pi_0 | H_1 \text{ true}) \\
= \int_{\Pi_0} f_1(z) \, dz
\]
If we have priors then can use Bayes criterion

\[ P_c = \pi_0 P_{c,0} + \pi_1 P_{c,1} \]

\[ = \pi_0 \int_{\Pi_0} f_0(z) \, dz + \pi_1 \int_{\Pi_1} f_1(z) \, dz \]

\[ = \pi_0 \left\{ 1 - \int_{\Pi_0} f_0(z) \, dz \right\} + \pi_1 \int_{\Pi_1} f_1(z) \, dz \]

\[ = \pi_0 + \int_{\Pi_0} \left[ \pi_1 f_1(z) - \pi_0 f_0(z) \right] \, dz \]

\[ 1 = \int_{\Gamma = \Gamma_0 \cup \Gamma_1} f_0(z) \, dz = \int_{\Pi_0 (z)} f_0(z) \, dz + \int_{\Pi_1 (z)} f_0(z) \, dz \]