6.20 Consider an interior point of a regular M-QASK signal constellation for a very large value of M. Let this interior point correspond to a particular signal s_n. Suppose that the nearest neighbors of the point are also interior points. In parts **(b)–(d)**, express your answers in terms of the function Q and the parameters d and N_0.

(a) If the distance between nearest neighbors is d, show that the distance between the point that corresponds to s_n and the closest exterior point is at least ~~3d~~ 2d

(b) Suppose the point that corresponds to signal s_m is a nearest neighbor of the point that corresponds to signal s_n. Use the idea conveyed in Problem 6.19 to find the conditional probability that the maximum-likelihood receiver decides that signal s_m was transmitted, given that signal s_n was actually transmitted.

(c) Continue this approach to find the probability that the receiver decides in favor of one of the nearest neighbors, and compare this with the expression for the symbol error probability for the interior point.

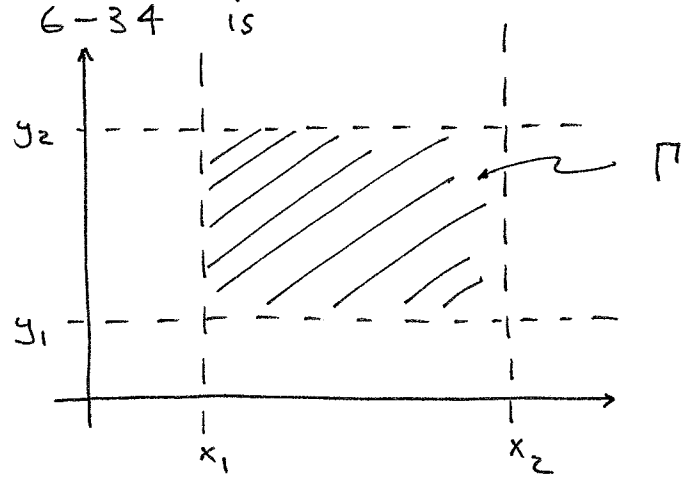
(d) Show that for large signal-to-noise ratio, the most likely error event is that the receiver decides in favor of a nearest neighbor rather than one of the other points in the signal constellation.

MBP 6.19 (mentioned in MBP 6.20)

Reconsider Fig. 6-34 and suppose $(\mu, \nu) \notin \Gamma_n$.

- Is Eq. (6.54) still valid?
- Explain how this observation can be used to compute conditional prob. the received decides $\underline{s_n}$ given $\underline{s_m}$ was actually transmitted, $\underline{s_m} \neq \underline{s_n}$.

Fig. 6-34 is



Assumption leading to Eq. (6.54) is that Z is a two-dimensional Gaussian

$$Z = (X, Y)$$

where

$$f(x,y) = \frac{1}{2\pi\sigma^2} \exp\left\{-\left[(x-\mu)^2 + (y-\nu)^2\right]/2\sigma^2\right\}$$

which is equiv. to saying $X \perp\!\!\!\perp Y$; $X \sim N(\mu, \sigma^2)$; $Y \sim N(\nu, \sigma^2)$.

$$P(Z \in \Gamma) = P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2)$$

$$= P(x_1 < X \leq x_2)\, P(y_1 < Y \leq y_2)$$

$$P(Z \in \Gamma) = P\left(\frac{x_1-\mu}{\sigma} < \frac{X-\mu}{\sigma} \le \frac{x_2-\mu}{\sigma}\right) \cdot$$

$$P\left(\frac{y_1-\nu}{\sigma} < \frac{Y-\nu}{\sigma} \le \frac{y_2-\nu}{\sigma}\right)$$

$$= \left[\Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right)\right]\left[\Phi\left(\frac{y_2-\nu}{\sigma}\right) - \Phi\left(\frac{y_1-\nu}{\sigma}\right)\right]$$

Nowhere does this depend on $(\mu, \nu) \in \Gamma$.

Conclude: Eq. (6.54) still valid if $(\mu, \nu) \notin \Gamma$.

As to the last question: So long as the decision region for $\underline{s_n}$ is rectangular the prob.

$P(\text{decide } \underline{s_n} \mid \underline{s_m} \text{ is transmitted})$

$= P(Z \in \Gamma_n \mid \underline{s_m} \text{ is transmitted})$

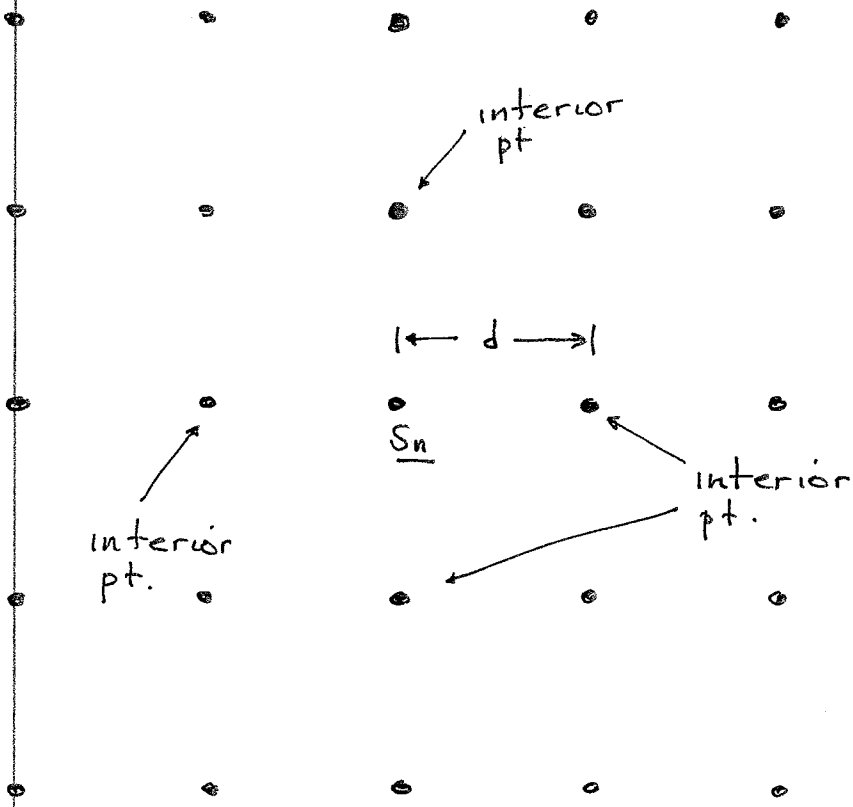
$=$ an expression of form of Eq. (6.54) where x_1 x_2 y_1 y_2 define the boundaries of Γ_n and

$$\underline{s_m} = (\mu, \nu)$$

MBP 6.20

regular M-QASK, M large. Constellation pt. $\underline{s_n}$ is interior and all of its nearest neighbors are also interior. The minimum distance is d.

Situation is like



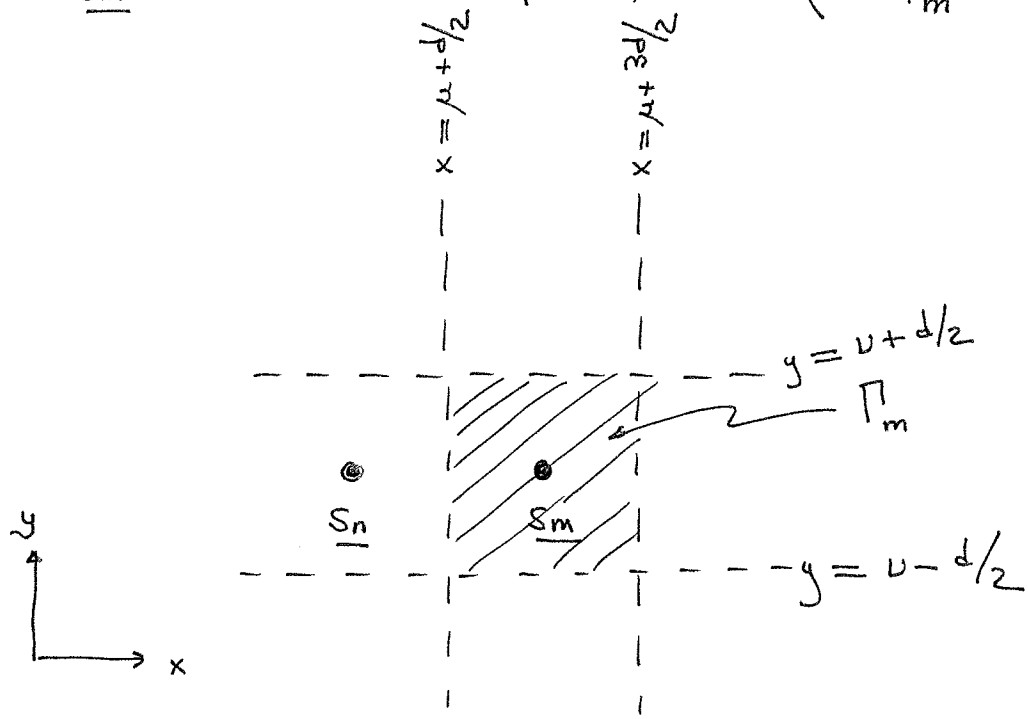
(If regular the immediate vicinity of $\underline{s_n}$ must look square)

(a) If start at $\underline{s_n}$ and move horizontally until we come to an exterior pt we must pass by at least one other interior pt. Thus horizontal distance to exterior $\geq 2d$.

Same argument for moving vertically from $\underline{s_n}$ to exterior.

(b) Say $\underline{s_m}$ is a nearest neigh. to $\underline{s_n}$. By assumpt. $\underline{s_m}$ must be an interior pt. Consider ML receiver which implies that decision region for $\underline{s_m}$ is square $d \times d$ and centered at $\underline{s_m}$.

$$P(\underset{\underline{s_m}}{\text{ML receiver chooses}} \mid \underline{s_n}) = P(\text{observation falls in } \Gamma_m \mid \underline{s_n})$$



Let $\underline{s_n} = (\mu, \nu)$. Since Prob. 6.19 holds can use the formula with

$$x_1 = \mu + d/2 \qquad y_1 = \nu - d/2$$
$$x_2 = \mu + 3d/2 \qquad y_2 = \nu + d/2$$

$$P(Z \in \Gamma_m \mid \underline{s_n}) = \left[\Phi\left(\frac{3d/2}{\sigma}\right) - \Phi\left(\frac{d/2}{\sigma}\right)\right]\left[\Phi\left(\frac{d/2}{\sigma}\right) - \Phi\left(-\frac{d/2}{\sigma}\right)\right]$$

$$= \left[\Phi\left(\frac{3d/2}{\sigma}\right) - \Phi\left(\frac{d/2}{\sigma}\right)\right]\left[-1 + 2\Phi\left(\frac{d/2}{\sigma}\right)\right]$$

Here σ^2 = variance of AWGN @ $N_0/2$ projected onto one of orthonormal axes

$= N_0/2$

$$\Longrightarrow \sigma = \sqrt{N_0/2} \Longrightarrow \frac{3d/2}{\sigma} = \frac{3d}{2\sqrt{N_0/2}} = \frac{3d}{\sqrt{2N_0}}$$

$$\frac{d/2}{\sigma} = \frac{d}{\sqrt{2N_0}}$$

$$Q(\alpha) = 1 - \Phi(\alpha)$$

$$\therefore P(Z \in \Gamma_m \mid \underline{s_n}) = \left[Q\left(\frac{d}{\sqrt{2N_0}}\right) - Q\left(\frac{3d}{\sqrt{2N_0}}\right)\right]\left[1 - 2Q\left(\frac{d}{\sqrt{2N_0}}\right)\right]$$

(c) There are 4 of these "interior" nearest neighbors.

$$P\left(\begin{matrix}\text{ML receiver chooses} \\ \text{one of 4 nearest neigh.}\end{matrix} \,\middle|\, \underline{s_n}\right)$$

$$= 4\left[Q\left(\frac{d}{\sqrt{2N_0}}\right) - Q\left(\frac{3d}{\sqrt{2N_0}}\right)\right]\left[1 - 2Q\left(\frac{d}{\sqrt{2N_0}}\right)\right]$$

The symbol error prob. for $\underline{s_n}$ is

$$1 - P\left(\begin{matrix}\text{ML receiver} \\ \text{chooses } \underline{s_n}\end{matrix} \,\middle|\, \underline{s_n}\right)$$

$$= 1 - \left[\Phi\left(\frac{d/2}{\sigma}\right) - \Phi\left(-\frac{d/2}{\sigma}\right)\right]^2$$

$$= 1 - \left[1 - 2Q\left(\frac{d}{\sqrt{2N_0}}\right)\right]^2$$

$$= 1 - 1 + 4Q\left(\frac{d}{\sqrt{2N_0}}\right) - 4Q^2\left(\frac{d}{\sqrt{2N_0}}\right)$$

$$= 4Q\left(\frac{d}{\sqrt{2N_0}}\right)\left[1 - Q\left(\frac{d}{\sqrt{2N_0}}\right)\right]$$

(d).

The leading terms for both

$$P\left(\begin{matrix}\text{ML receiver does} \\ \text{not choose } \underline{s_n}\end{matrix} \,\middle|\, \underline{s_n}\right) \qquad (*)$$

and

$$P\left(\begin{matrix}\text{ML receiver chooses one} \\ \text{of 4 nearest neigh.}\end{matrix} \,\middle|\, s_n\right) \qquad (**)$$

are same and equal to

$$4Q\left(\frac{d}{\sqrt{2N_0}}\right).$$

This implies that for large SNR the probs. in $(*)$ and $(**)$ are approx. equal.

6.21 Let $A(\omega)$ be the Fourier transform of $\alpha(t)$. Assume that the bandwidth of $A(\omega)$ is small compared with ω_c, so that $A(\omega - \omega_c)\,A(\omega + \omega_c) = 0$ for all ω. Use the modulation theorem of Fourier transforms to show that the bandwidth of the signal $w_1(t) = u\,\alpha(t)\cos(\omega_c t) - v\,\alpha(t)\sin(\omega_c t)$ depends on $A(\omega)$, but not on the parameters u, v, and ω_c. Let $w_2(t) = \alpha(t)\cos(\omega_c t)$ and show that $w_1(t)$ and $w_2(t)$ have the same bandwidth for any choice of u and v, and explain why this conclusion does not depend on how the bandwidth is defined (e.g., half-power bandwidth, rms bandwidth, equivalent noise bandwidth, etc.).

MBP 6.21

$\alpha(t) \longleftrightarrow A(\omega)$ Assume $A(\omega-\omega_c)A(\omega+\omega_c) = 0$

Find FTs of

$$w_1(t) = u\,\alpha(t)\cos(\omega_c t) - v\,\alpha(t)\sin(\omega_c t)$$

$$w_2(t) = \alpha(t)\cos(\omega_c t)$$

and compare BWs.

$$\cos\omega_c t = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$\alpha(t)\,e^{j\omega_c t} \longleftrightarrow A(\omega-\omega_c)$$

$$\alpha(t)\,e^{-j\omega_c t} \longleftrightarrow A(\omega+\omega_c)$$

$$\therefore\ W_2(\omega) = \frac{1}{2}\left[A(\omega-\omega_c) + A(\omega+\omega_c)\right]$$

$$\sin\omega_c t = \frac{e^{j\omega_c t} - e^{-j\omega_c t}}{j2}$$

$$\therefore\ \alpha(t)\sin\omega_c t \longleftrightarrow \frac{1}{j2}\left[A(\omega-\omega_c) - A(\omega+\omega_c)\right]$$

$$\therefore\ W_1(f) = \frac{u}{2}\left[A(\omega-\omega_c) + A(\omega+\omega_c)\right] - \frac{v}{j2}\left[A(\omega-\omega_c) - A(\omega+\omega_c)\right]$$

$$= \frac{1}{2}(u+jv)A(\omega-\omega_c) + \frac{1}{2}(u-jv)A(\omega+\omega_c)$$

$$A(\omega-\omega_c)\,A(\omega+\omega_c)=0 \implies |A(\omega-\omega_c)\,A(\omega+\omega_c)|=0$$

$$\implies A^*(\omega-\omega_c)\,A(\omega+\omega_c)=0,\quad A(\omega-\omega_c)\,A^*(\omega+\omega_c)=0.$$

Therefore, when finding the energy density spec. of these pulses the cross terms are zero:

$$|W_2(\omega)|^2 = \frac{1}{4}\left[|A(\omega-\omega_c)|^2 + |A(\omega+\omega_c)|^2\right]$$

$$|W_1(\omega)|^2 = \frac{u^2+v^2}{4}\left[|A(\omega-\omega_c)|^2 + |A(\omega+\omega_c)|^2\right]$$

The two are related by a scale factor u^2+v^2. Thus any def. of BW (all insensitive to scale) says the two pulses have same BW.

6.23 This is a classical problem that arises in coherent frequency-shift-key (FSK) communication systems. The binary signals are given by

$$s_0(t) = \sqrt{2}\,A\cos(\omega_0 t + \varphi), \quad 0 \le t \le T,$$

and

$$s_1(t) = \sqrt{2}\,A\cos(\omega_1 t + \varphi), \quad 0 \le t \le T,$$

and the receiver employs maximum-likelihood coherent demodulation. Assume that $\omega_1 > \omega_0$, and ignore the effects of any high-frequency signal components (e.g., ignore the effects of any components at frequency $\omega_1 + \omega_0$). What is the optimum value for the frequency separation $\omega_1 - \omega_0$? *Hint*: For some choices of $\omega_1 - \omega_0$, the two signals are orthogonal (i.e., $r = 0$), but it is possible to obtain a better correlation coefficient for other choices of $\omega_1 - \omega_0$.

MBP 6.23

$$s_0(t) = \sqrt{2}\,A\cos(\omega_0 t + \phi)$$

vs. $\qquad 0 \le t \le T$

$$s_1(t) = \sqrt{2}\,A\cos(\omega_1 t + \phi) \qquad \omega_1 > \omega_0$$

Let $\omega_d = \omega_1 - \omega_0$

$$\langle s_0, s_1 \rangle = 2A^2 \int_0^T \cos(\omega_0 t + \phi)\cos(\omega_1 t + \phi)\,dt$$

$$= A^2 \int_0^T \cos[(\omega_1 - \omega_0)t]\,dt + \underbrace{A^2 \int_0^T \cos[(\omega_0 + \omega_1)t + 2\phi]\,dt}_{\approx 0}$$

$$= A^2 \int_0^T \cos\omega_d t\,dt = \frac{A^2}{\omega_d}\sin\omega_d t\Big|_0^T$$

$$\Longrightarrow \quad \langle s_0, s_1 \rangle = \frac{A^2}{\omega_d}\sin\omega_d T$$

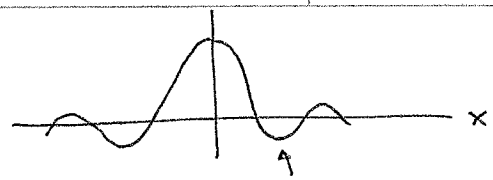
These signals have the same energy

$$\|s_0\|^2 = \|s_1\|^2 = A^2 T$$

From Chap 5 results the prob. of error is minimized by minimizing the correlation coeff

$$\frac{\langle s_0, s_1 \rangle}{\|s_0\| \cdot \|s_1\|} = \frac{\frac{A^2}{\omega_d}\sin\omega_d T}{A^2 T} = \frac{\sin\omega_d T}{\omega_d T}$$

Minimum of $\frac{\sin x}{x}$



$$\frac{d}{dx}\frac{\sin x}{x} = \frac{x\cos x - \sin x}{x^2} \overset{\text{Set}}{=} 0$$

A soln. is $x\cos x - \sin x = 0 \iff x = \tan x$

Numerical soln is

$x \approx 4.493$

$$\omega_d T \approx 4.493$$

$$\omega_d \approx \frac{4.493}{T} = 2\pi f_d$$

$$\Longrightarrow f_d \approx 0.715/T$$

The value of the corr. coeff. is then $r \approx -0.217$

$$\Longrightarrow P_e = Q\left(\sqrt{\frac{1.217E}{N_0}}\right)$$

This is about 0.85 dB better than for orthog. signals.

6.27 Consider seven signals constructed from the following sequences:

1110010
1100101
1001011
0010111
0101110
1011100
0111001

These sequences are generated by the shift register of Figure 6-40. The continuous-time signals are obtained from the sequences according to (6.65) of Section 6.6.1. Refer to these seven signals as $s_0, s_1, \ldots, s_6$, where s_0 is the signal corresponding to the first sequence in the list, s_1 to the second sequence, etc. Each signal has duration T, and each consists of a sequence of chips of duration T_c. The signals form a 7-ary signal set for communication over an AWGN channel with spectral density $N_0/2$. The receiver consists of a filter with impulse response $h(t)$, a sampler which samples the filter output at seven different times, and a decision device that bases decisions on the largest sample.

Construct the impulse response for a continuous-time filter that is matched to the sequence 1110010111001. First obtain the continuous-time signal $s(t)$ from the given sequence as described in Section 6.6.1, and then let the impulse response of the filter be $h(t) = s(T_0 - t)$, $-\infty < t < +\infty$, where T_0 is an arbitrary time reference. Each of the chips for this filter has duration T_c, so $h(t)$ has duration $13T_c = 13T/7$. The value of T_0 is not critical: $T_0 = 0$ or $T_0 = 13T_c$ are acceptable choices. ($T_0 = 13T_c$ provides a causal filter.)

(a) Find the output of the filter for each of the seven signals $s_0, s_1, \ldots, s_6$.

(b) Find the optimum sampling time for each of the seven signals.

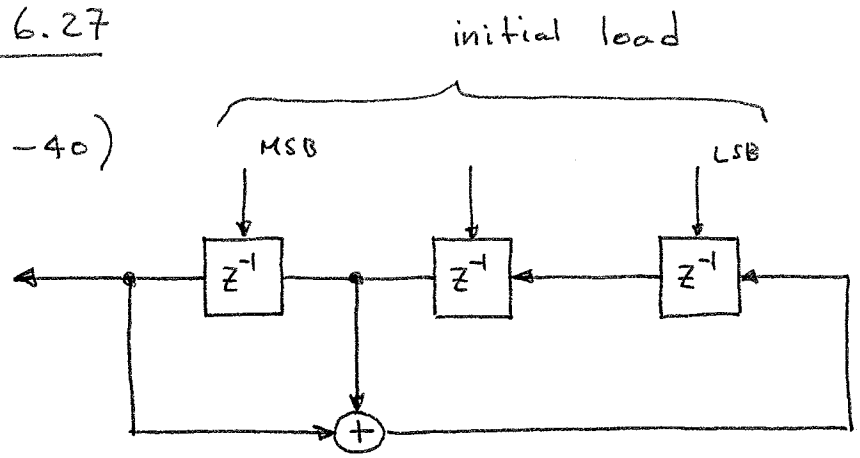
(c) How might you use the features demonstrated in part **(b)** to make symbol decisions?

(d) Consider binary signaling using s_0 and s_1 only. The receiver samples the filter output at the optimum times for signals s_0 and s_1 (from part **(b)**). Find $P_{e,i}$ for $i = 0, 1$, and compare with the error probabilities for the optimum receiver.

(e) Consider binary signaling using s_0 and s_2 only. Find $P_{e,0}$ and $P_{e,2}$ and compare your answers with the error probabilities for the optimum receiver.

MBP 6.27

(Fig. 6-40)



Mapping to signals $s_0\ s_1 \ldots s_6$

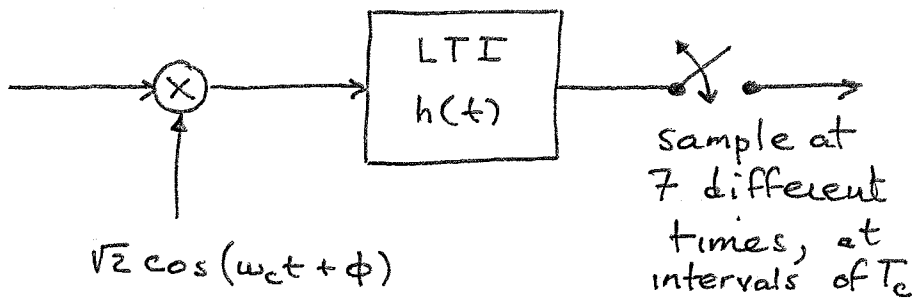
	Signal	Bit Stream $b_{i,n}$	Symbol Stream $\alpha_{i,n}$ ($n=0$ … $n=6$)
$i=0$	s_0	1 1 1 0 0 1 0	-1 -1 -1 1 1 -1 1
	s_1	1 1 0 0 1 0 1	-1 -1 1 1 -1 1 -1
	s_2	1 0 0 1 0 1 1	-1 1 1 -1 1 -1 -1
	s_3	0 0 1 0 1 1 1	1 1 -1 1 -1 -1 -1
	s_4	0 1 0 1 1 1 0	1 -1 1 -1 -1 -1 1
	s_5	1 0 1 1 1 0 0	-1 1 -1 -1 -1 1 1
$i=6$	s_6	0 1 1 1 0 0 1	+1 -1 -1 -1 1 1 -1

$$a_i(t) = \sum_{n=0}^{6} \alpha_{i,n}\, p_{T_c}(t - nT_c) \qquad i = 0, 1, 2, \ldots 6$$

$$s_i(t) = \sqrt{2}\, A\, a_i(t) \cos(\omega_c t + \phi).$$

In this problem we don't really need to worry about the carrier $\cos(\omega_c t + \phi)$ as the problem really concerns properties of the baseband waveforms.

Hence consider a receiver of the following form



Problem says that $h(t)$ should be matched to a signal $s(t)$ obtained from

1 1 1 0 0 1 0 1 1 1 0 0 1 ← bit stream

-1 -1 -1 1 1 -1 1 -1 -1 -1 1 1 -1 ← symbol stream (α_n)

$$s(t) = \sum_{n=0}^{12} \alpha_n \, p_{T_c}(t - nT_c)$$

$h(t) = s(T_0 - t)$ where choice of T_0 is arbitrary ... $T_0 = 0$ is a purely anticausal $h(t)$; $T_0 = 13T_c$ gives a causal $h(t)$.

Problem also says that signals $s_0\ s_1 \ldots s_6$ have duration T, which implies that

$$T = 7T_c$$

(a) Find filter outputs for every signal s_i

First look at mixer output

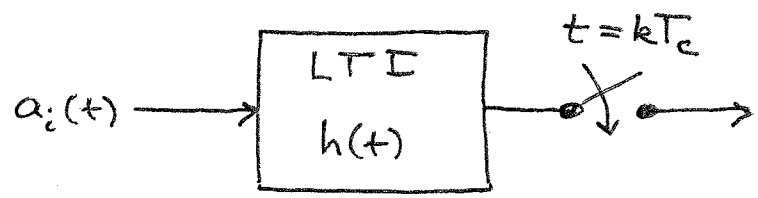
$$s_i(t)\sqrt{2}\cos(\omega_c t+\phi) = 2A\,a_i(t)\cos^2(\omega_c t+\phi)$$

$$= A\,a_i(t) + A\,a_i(t)\cos(2\omega_c t+2\phi)$$

$$= \text{input to LTI filter.}$$

Since the filter is LTI can consider inputs $A\,a_i(t)$ and $A\,a_i(t)$ & $A\,a_i(t)\cos(2\omega_c t+2\phi)$ separately. Also, <u>claim</u> that only $A\,a_i(t)$ will influence output. ~~We will come back to this point later.~~

Consider:



$h(t) = s(T_0 - t)$ and we will take $T_0 = 0$ for now because that makes it slightly simpler to line things up.

Define

$$\hat{s}_i(t) = a_i * h(t)$$

$$= \int a_i(\lambda)\,h(t-\lambda)\,d\lambda = \int a_i(\lambda)\,s(\lambda-t)\,d\lambda$$

Then sample @ $t = kT_c$

$$\hat{s}_i(kT_c) = \int a_i(\lambda)\,s(\lambda-kT_c)\,d\lambda$$

Plugging in for definitions of $a_i(\lambda)$ and $s(\lambda - kT_c)$ have

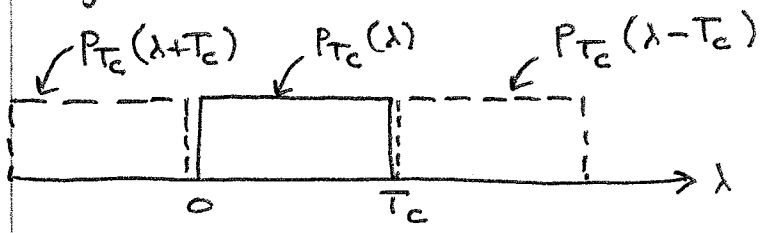
$$(*) \quad \hat{S}_i(kT_c) = \sum_{n=0}^{6} \sum_{m=0}^{12} \alpha_{i,n}\, \alpha_m \underbrace{\int p_{T_c}(\lambda - nT_c)\, p_{T_c}(\lambda - kT_c - mT_c)\, d\lambda}$$

$$\lambda' = \lambda - nT_c$$

$$\int p_{T_c}(\lambda')\, p_{T_c}(\lambda' + nT_c - kT_c - mT_c)\, d\lambda'$$

Thus consider integrals of the form

$$\int p_{T_c}(\lambda)\, p_{T_c}(\lambda - \ell T_c)\, d\lambda = \begin{cases} T_c & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$



$\therefore$ no overlap unless $\ell = 0$

Back to $\hat{S}_i(kT_c)$ calculation.

In the sum over m in Eq. $(*)$ only term $m = n-k$ is nonzero. Hence

$$\hat{S}_i(kT_c) = T_c \sum_{n=0}^{6} \alpha_{i,n}\, \alpha_{n-k}$$

If we extend the sequences $\alpha_{i,n}$ and α_n by prepending and appending with zero we can rewrite as

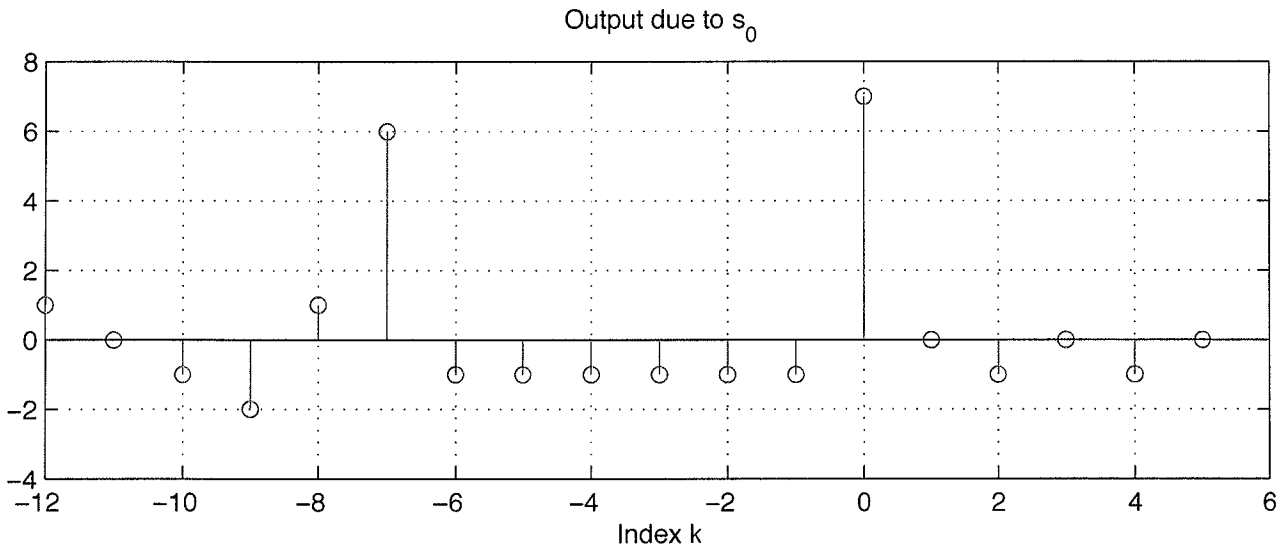
$$\hat{S}_i(kT_c) = T_c \sum_n \alpha_{i,n}\, \alpha_{n-k} = T_c \sum_m \alpha_{i,m+k}\, \alpha_m$$

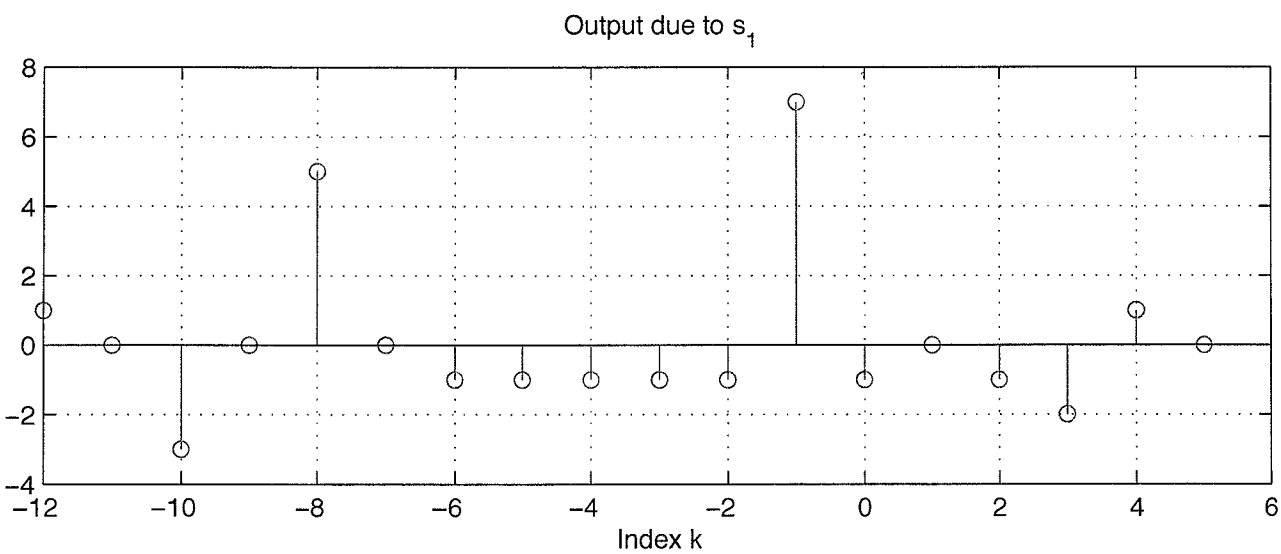
which is the cross-correlation of the two sequences.

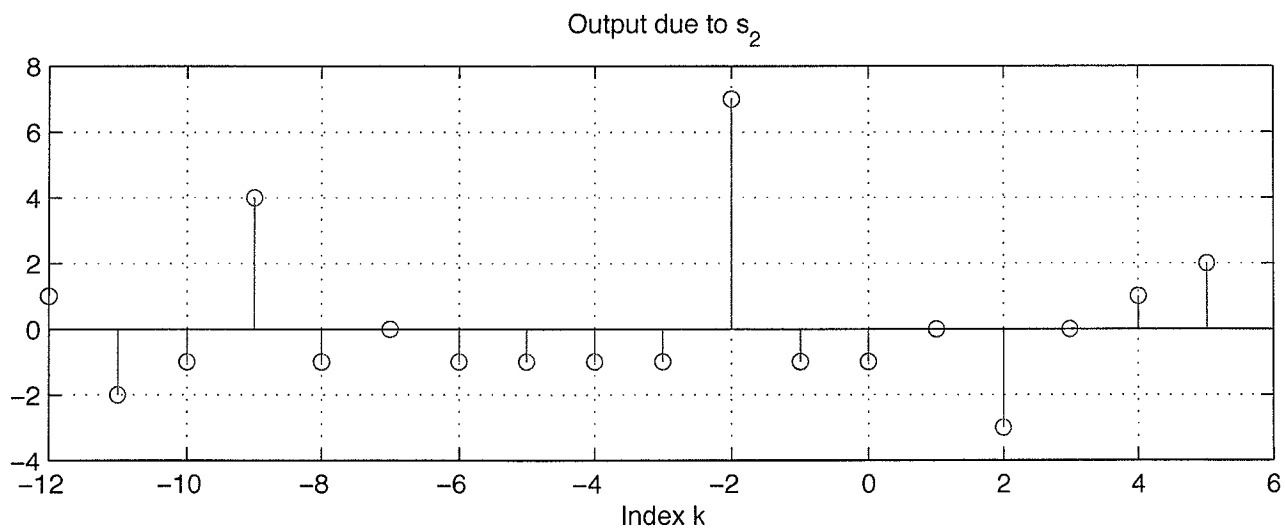
Implementing these cross-correllations (one for each signal $i = 0, 1, 2 \ldots 6$) is possible to do manually, but much easier via Matlab. This was done and results are plotted. See following pages.

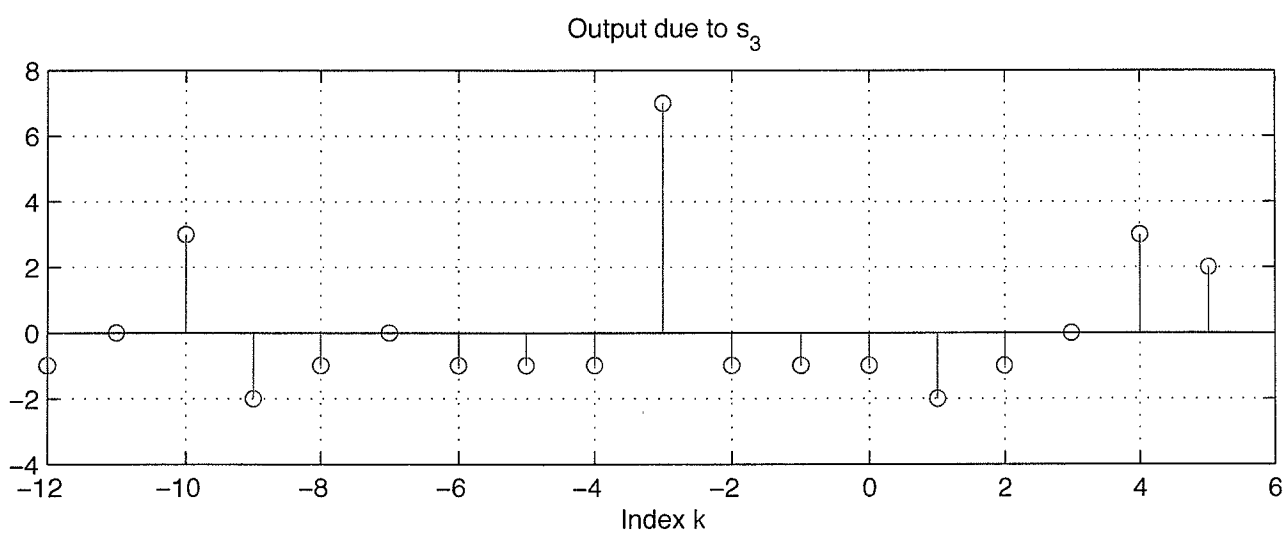
If instead of the non-causal MF we had chosen $T_0 = 13T_c$ this would only shift the outputs relative to index k. In this case the plots would start with first point at $k = 1$ (rather than $k = -12$ as plotted).

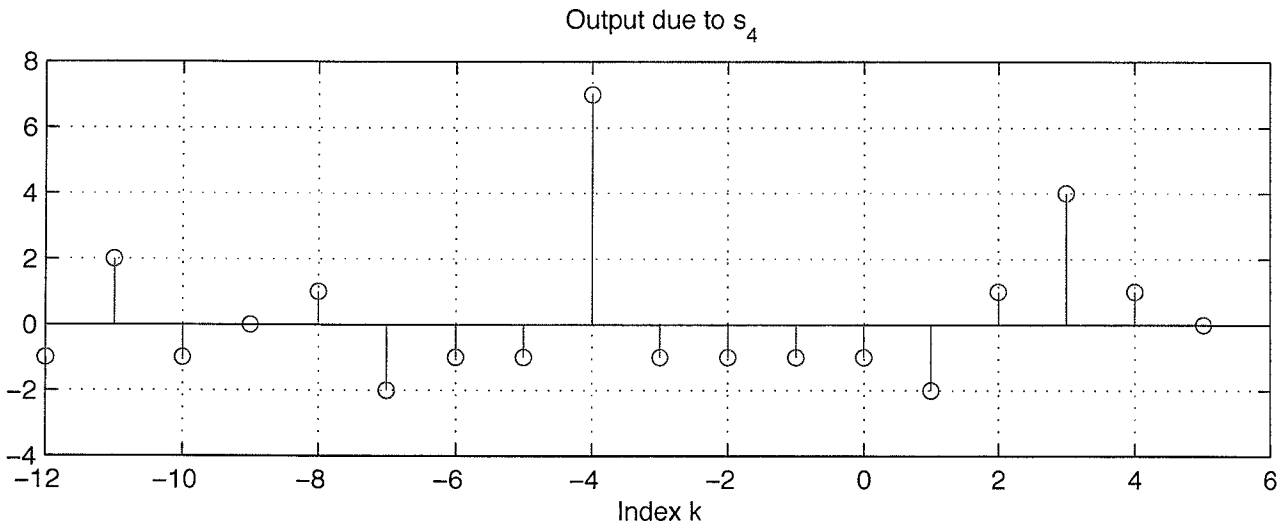
The actual outputs of the LTI filter would be the integers shown in the plots multiplied by AT_c.

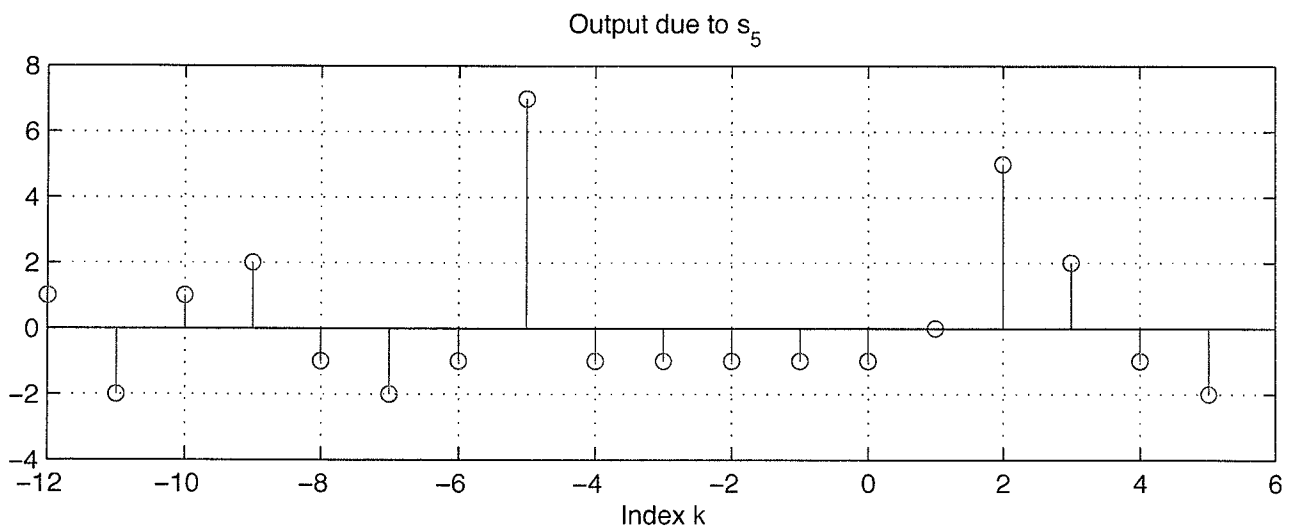

Output due to s_0
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k

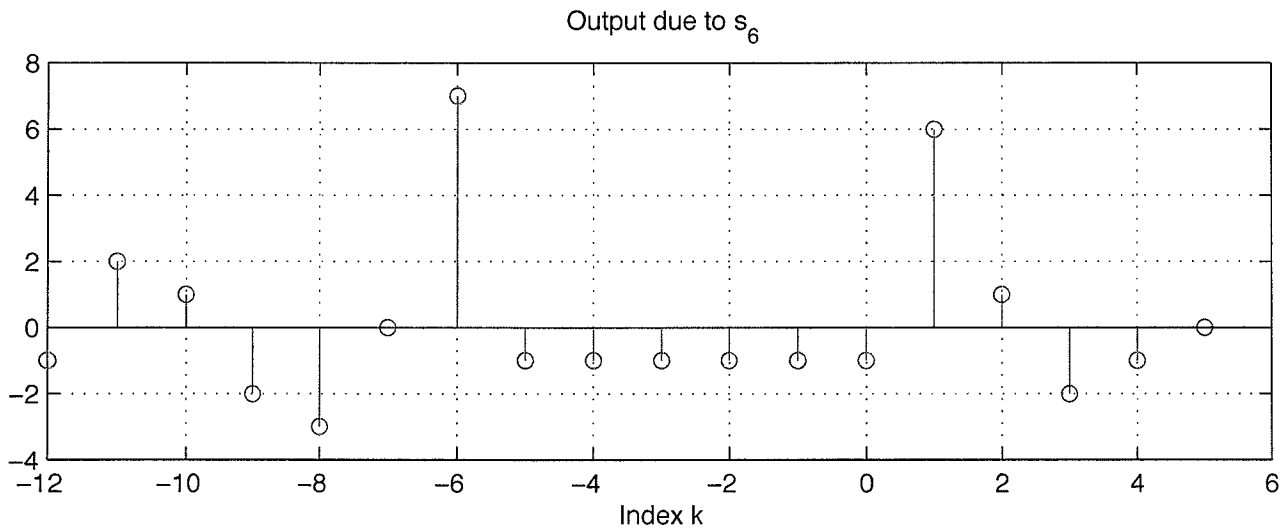

Output due to s_1
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_2
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_3
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_4
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_5
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_6
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k

(b) Opt. Sampling Times for each of seven signals s_0 s_1 ... s_6

These are the values that maximize the noiseless filter outputs:

Signal	k_{opt} for $T_0 = 0$	k_{opt} for $T_0 = 13T_c$
s_0	0	13
s_1	-1	12
s_2	-2	11
s_3	-3	10
s_4	-4	9
s_5	-5	8
s_6	-6	7

Notice that the continuous time output consists of straight line segments joining the samples prev. plotted.

(c) The receiver would sample at the seven times indicated. Then it would decide which of the seven samples was largest and choose as its estimate of the transmitted signal the one corresponding to the index of the largest sample.

(d) Let $T_0 = 0$. The opt. sampling times for s_0 and s_1 are

$$s_0 \longrightarrow k = 0$$

$$s_1 \longrightarrow k = -1$$

So we build the binary receiver on these two samples only (since the receiver front end is already picked for us).

If s_0 is sent:

$$A\hat{s}_0(0 \cdot T_c) = 7AT_c$$
$$A\hat{s}_0(-1 \cdot T_c) = -AT_c$$

If s_1 is sent:

$$A\hat{s}_1(0 \cdot T_c) = -AT_c$$
$$A\hat{s}_1(-1 \cdot T_c) = 7AT_c$$

Above are the signal components at the output of the sampler at the two times indicated. There will also be noise samples.

The noise input is AWGN $X(t)$ with $N_0/2$. The output noise is

$$\hat{X}(t) = \int X(\lambda) h(t-\lambda)\, d\lambda$$

Clearly $\hat{X}$ is Gaussian and of mean zero. Its autocorrelation function is

$$R_{\hat{X}\hat{X}}(\tau) = \frac{N_0}{2}\int h(\lambda) h(\lambda-\tau)\, d\lambda$$

$$= \frac{N_0}{2}\int s(\lambda) s(\lambda-\tau)\, d\lambda$$

We really only need the samples of this autocorr.

$$R_{\hat{X}\hat{X}}(kT_c) = \frac{N_0}{2}\int s(\lambda) s(\lambda-kT_c)\, d\lambda$$

$$R_{\hat{X}\hat{X}}(kT_c) = \sum_{n=0}^{12} \sum_{m=0}^{12} \alpha_n \alpha_m \underbrace{\int p_{T_c}(\lambda - nT_c)\, p_{T_c}(\lambda - kT_c - mT_c)\, d\lambda}_{= \begin{cases} T_c & m = n-k \\ 0 & \text{else} \end{cases}}$$

$$= \frac{N_0}{2} T_c \sum_{n=0}^{12} \alpha_n \alpha_{n-k}$$

We will need the samples for $k = 0, \pm 1$

$$R_{\hat{X}\hat{X}}(0) = \frac{N_0}{2} T_c \cdot 13$$

$$R_{\hat{X}\hat{X}}(1) = \frac{N_0}{2} T_c \left[0 \right] = 0 = R_{\hat{X}\hat{X}}(-1)$$

-1	-1	-1	1	1	-1	1	-1	-1	-1	1	1	-1	
	-1	-1	-1	1	1	-1	1	-1	-1	-1	1	1	-1
0	1	1	-1	1	-1	-1	-1	1	1	-1	1	-1	0

sum to zero

For $P_{e,0}$

The receiver compares

$$\hat{Y}(0 \cdot T_c) = A\hat{S}_0(0 \cdot T_c) + \hat{X}(0 \cdot T_c)$$

to

$$\hat{Y}(-1 \cdot T_c) = A\hat{S}_0(-1 \cdot T_c) + \hat{X}(-1 \cdot T_c)$$

and selects s_1 if latter is largest.

$$\therefore P_{e,0} = P\left(A\hat{s}_0(-T_c) + \hat{X}(-T_c) > A\hat{s}_0(0) + \hat{X}(0)\right)$$

$$= P\left(\hat{X}(-T_c) - \hat{X}(0) > A\hat{s}_0(0) - A\hat{s}_0(-T_c)\right)$$

$$= P\left(\underbrace{\hat{X}(-T_c) - \hat{X}(0)}_{\text{rv is Gaussian of mean zero and var}} > 8AT_c\right)$$

$$E\left\{\left(\hat{X}(-T_c) - \hat{X}(0)\right)^2\right\}$$

$$= 2R_{\hat{X}\hat{X}}(0) - 2R_{\hat{X}\hat{X}}(T_c)$$

$$= 2\frac{N_0}{2}\,13T_c - 0 = N_0\,13T_c$$

$$P_{e,0} = Q\left(\frac{8AT_c}{\sqrt{N_0 13T_c}}\right) = Q\left(8A\sqrt{\frac{T_c}{13N_0}}\right)$$

<u>For $P_{e,1}$</u>

$$\hat{Y}(0 \cdot T_c) = A\hat{s}_1(0 \cdot T_c) + \hat{X}(0 \cdot T_c)$$

$$\hat{Y}(-T_c) = A\hat{s}_1(-T_c) + \hat{X}(-T_c)$$

$$P_{e,1} = P\left(A\hat{s}_1(0) + \hat{X}(0) > A\hat{s}_1(-T_c) + \hat{X}(-T_c)\right)$$

$$= P\left(\hat{X}(-T_c) - \hat{X}(0) < A\hat{s}_1(0) - A\hat{s}_1(-T_c)\right)$$

$$= P\left(\hat{X}(-T_c) - \hat{X}(0) < -8AT_c\right)$$

$$= Q\left(8A\sqrt{\frac{T_c}{13N_0}}\right) = P_{e,0}.$$

The optimum binary receiver for these two signals

The opt. receiver would be designed for the two signals s_0 and s_1. They each have energy

$$7A^2T_c$$

and their cross-correlation is $-A^2T_c$. Therefore, their correlation coeff is $r = -1/7$. We can use

$$Q\left(\sqrt{\frac{\mathcal{E}(1-r)}{N_0}}\right)$$

for the error prob.

$$P_{e,0}^{opt} = P_{e,1}^{opt} = Q\left(\sqrt{\frac{7A^2T_c(8/7)}{N_0}}\right)$$

$$= Q\left(A\sqrt{\frac{8T_c}{N_0}}\right) = Q\left(8A\sqrt{\frac{T_c}{8N_0}}\right)$$

Thus opt. is better by

$$10\log_{10}\left(13/8\right) = 2.1 \text{ dB}$$

(e) Same idea. However the samples of interest are now $k=0$ and $k=-2$ and it requires to compute $R_{\hat{x}\hat{x}}(2T_c) = -3N_0T_c$ which incr. the noise variance.

$$P_{e,0} = P_{e,1} = Q\left(2A\sqrt{T_c/N_0}\right)$$

and the loss from opt. is now 3dB.

6.32 Consider the two signal sets illustrated in Figure 6-54. Assume that a maximum-likelihood receiver is used for each signal set, and the channel is an additive white Gaussian noise channel with two-sided spectral density $N_0/2$. The distance between pairs of signals is d for each set. The signals in Set 1 are equal-energy, orthogonal, three-dimensional signals. The signals in Set 2 are equal-energy, two-dimensional signals. Let $\mathcal{E}_1$ denote the energy for each signal in Set 1, and let $\mathcal{E}_2$ denote the energy for each signal in Set 2.

Fact: Since the distance between signals is the same for the two sets, the symbol error probability for maximum-likelihood reception is also the same for the two sets.

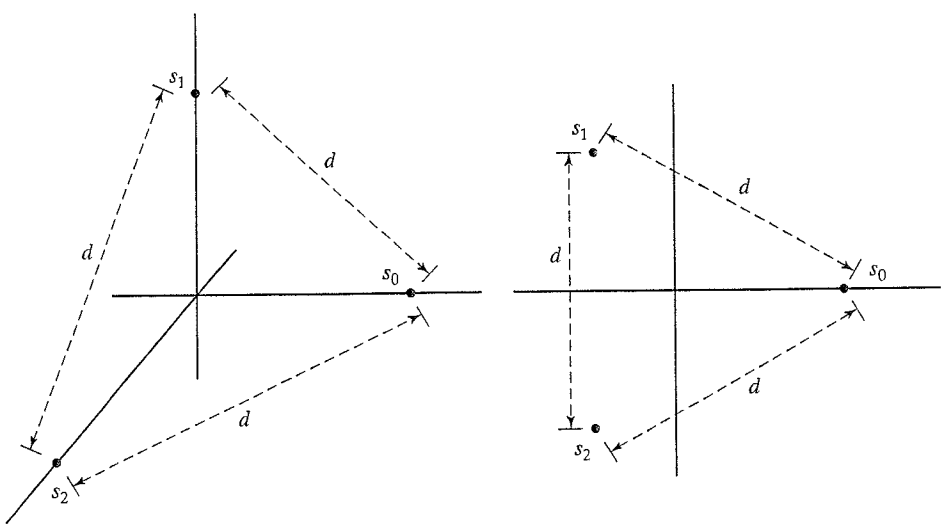
(a) Give an expression for d in terms of $\mathcal{E}_1$.

(b) For Set 1, use (6.72) in Section 6.6.3 and your result in part **(a)** to obtain an expression for the probability of symbol error in terms of d.

(c) Use simple trigonometry to find an expression for $\mathcal{E}_2$ in terms of d. *Hint*: d is the length of each side of the equilateral triangle whose vertices are the points in the constellation for Set 2. From this hint, it is easy to determine various angles that may be useful in the evaluation of $\mathcal{E}_2$.

(d) Use the fact in the problem statement and your results from parts **(b)** and **(c)** to obtain an expression for the probability of symbol error in terms of $\mathcal{E}_2$ for Set 2. For a given symbol error probability, which set requires less energy?

(e) Give a numerical value for the correlation coefficient r for pairs of signals in Set 2. What type of signal set is Set 2?



(a) Set 1 (orthogonal signals)

(b) Set 2

Figure 6-54: Signal sets for Problem 6.32.

MBP 6.32

(a) For Set 1 let $\underline{s_0} = (\alpha, 0, 0)$ and note that $\underline{s_1} = (0, \alpha, 0)$ and $\underline{s_2} = (0, 0, \alpha)$.

$$\|\underline{s_0}\|^2 = \alpha^2 = \mathcal{E}_1$$

From Mr. Pythagoras $\quad \alpha^2 + \alpha^2 = 2\alpha^2 = d^2$

$$\Longrightarrow \alpha^2 = d^2/2$$

$$\therefore \quad \frac{d^2}{2} = \mathcal{E}_1 \Longrightarrow d = \sqrt{2\mathcal{E}_1}$$

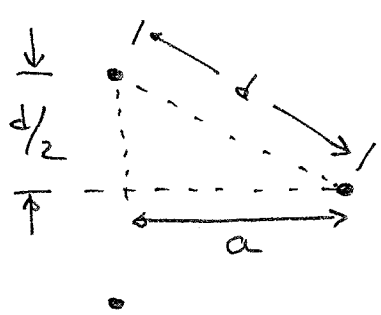
(b) Eq. (6.72) is

$$P_e = (M-1)\int_{-\infty}^{\infty} \left[\Phi(v)\right]^{M-2} \Phi\left(v - \sqrt{2\mathcal{E}/N_0}\right) e^{-v^2/2} \frac{dv}{\sqrt{2\pi}}$$

In this case $M = 3$, $\mathcal{E} = \mathcal{E}_1 = d^2/2$

$$P_e = 2\int_{-\infty}^{\infty} \Phi(v)\, \Phi\left(v - d/\sqrt{N_0}\right) e^{-v^2/2} \frac{dv}{\sqrt{2\pi}}$$

(c) For Set 2 the triangle is equilateral. All angles are 60°



$$a^2 + \frac{d^2}{4} = d^2$$

$$a^2 = \frac{3}{4} d^2$$

$$a = \frac{\sqrt{3}}{2} d$$

The points $\underline{s_0}$ $\underline{s_1}$ $\underline{s_2}$ are on a circle cent. at $(0,0)$ with radius $\sqrt{\varepsilon_2}$.

$$\underline{s_0} = (\sqrt{\varepsilon_2}, 0)$$

$$\underline{s_1} = (\beta, d/2)$$

$$\underline{s_2} = (\beta, -d/2)$$

$$\|\underline{s_1}\|^2 = \beta^2 + \frac{d^2}{4} = \varepsilon_2 \Rightarrow \beta = \sqrt{\varepsilon_2 - d^2/4}$$

Must also have

$$\beta = \sqrt{\varepsilon_2} - a = \sqrt{\varepsilon_2} - \frac{\sqrt{3}}{2} d$$

Putting these together

$$\cancel{\varepsilon_2} - \sqrt{3} d \sqrt{\varepsilon_2} + \frac{3}{4} d^2 + \frac{d^2}{4} = \cancel{\varepsilon_2}$$

$$-\sqrt{3\varepsilon_2}\, d + d^2 = 0$$

$$d(d - \sqrt{3\varepsilon_2}) = 0 \Longrightarrow d = \sqrt{3\varepsilon_2}$$

$$d^2 = 3\varepsilon_2$$

$$\varepsilon_2 = \frac{1}{3} d^2 = \frac{2}{3} \frac{d^2}{2} = \frac{2}{3} \varepsilon_1$$

(d) Since d is same for Set 1 and Set 2 and using the "Fact" and $d = \sqrt{3\mathcal{E}_2}$ we can write

$$P_e = 2\int_{-\infty}^{\infty} \Phi(v)\, \Phi\left(v - \sqrt{3\mathcal{E}_2/N_0}\right) e^{-v^2/2} \frac{dv}{\sqrt{2\pi}}$$

Set 2 requires only $\frac{2}{3}$ of the energy of Set 1 for the same symbol error prob.

(e)

$$\langle \underline{s_0}, \underline{s_1} \rangle = \mathcal{E}_2 \cos(2\pi/3) = -\mathcal{E}_2/2$$

$$\therefore\ r = \langle \underline{s_0}, \underline{s_1} \rangle / \mathcal{E}_2 = -1/2$$

This is a <u>simplex signal set.</u>

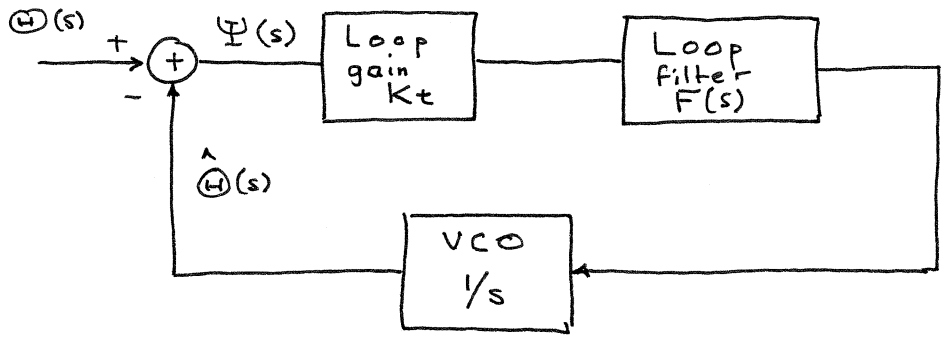
Z+T 6th Edition, Problem 3.47
p. 207.

3.47. The imperfect second-order PLL is defined as a PLL with the loop filter

$$F(s) = \frac{s+a}{s+\lambda a}$$

in which λ is the offset of the pole from the origin relative to the zero location. In practical implementations λ is small but often cannot be neglected. Use the linear model of the PLL and derive the transfer function for $\Theta(s)/\Phi(s)$. Derive expressions for ω_n and ζ in terms of K_t, a, and λ.

The linearized PLL model is



Want the transfer function mapping input phase θ to phase estimate $\hat{\theta}$.

$$\hat{\Theta} = \frac{K_t F(s)}{s}\left[\Theta - \hat{\Theta}\right]$$

$$\hat{\Theta}\left[1 + \frac{K_t F(s)}{s}\right] = \Theta \frac{K_t F(s)}{s}$$

$$\Longrightarrow \quad H(s) = \frac{\hat{\Theta}(s)}{\Theta(s)} = \frac{K_t F(s)/s}{1 + K_t F(s)/s} = \frac{K_t F(s)}{s + K_t F(s)}$$

Now $F(s) = \dfrac{s+a}{s+\lambda a}$. Substituting into the expression for $H(s)$:

$$H(s) = \frac{K_t \frac{s+a}{s+\lambda a}}{s + K_t \frac{s+a}{s+\lambda a}} = \frac{K_t (s+a)}{s(s+\lambda a) + K_t (s+a)}$$

$$= \frac{K_t (s+a)}{s^2 + [K_t + \lambda a]s + aK_t}$$

This is a 2nd order PLL. The denominator polynomial of 2nd order PLL transfer functions is always of the form

$$s^2 + 2\xi\omega_n s + \omega_n^2$$

Making the correspondence to the denominator of $H(s)$:

$$\omega_n^2 = aK_t \qquad 2\xi\omega_n = K_t + \lambda a$$

$$\omega_n = \sqrt{aK_t}$$

$$\xi = \frac{K_t + \lambda a}{2\sqrt{aK_t}}$$

Z+T 6th Edition, Problem 3.48, p. 207

3.48. Assuming the loop filter model for an imperfect second-order PLL described in the preceding problem, derive the steady-state phase errors under the three conditions of θ_0, f_Δ, and R given in Table 3.5.

Table 3.5 Steady-State Errors

PLL order	$\theta_0 \neq 0$, $f_\Delta = 0$, $R = 0$	$\theta_0 \neq 0$, $f_\Delta \neq 0$, $R = 0$	$\theta_0 \neq 0$, $f_\Delta \neq 0$, $R \neq 0$
1 ($a = 0$, $b = 0$)	0	$2\pi f_\Delta / K_t$	∞
2 ($a \neq 0$, $b = 0$)	0	0	$2\pi R / K_t$
3 ($a \neq 0$, $b \neq 0$)	0	0	0

Imperfect Loop Filter: $H_{loop}(s) = \dfrac{s+a}{s+\lambda a}$ $\quad a, \lambda > 0$

Since we are interested in steady state errors we then want the closed loop error transfer function

$$F_{phase-error}(s) = \frac{\Psi(s)}{\Theta(s)} = 1 - F_{closed-loop}(s)$$

$$= 1 - \frac{K_t H_{loop}(s)}{s + K_t H_{loop}(s)}$$

$$= \frac{s}{s + K_t H_{loop}(s)}$$

Then plug in our formula for $H_{loop}(s)$ to get

$$F_{phase-error}(s) = \frac{s}{s + K_t \frac{s+a}{s+\lambda a}}$$

$$= \frac{s(s+\lambda a)}{s(s+\lambda a) + K_t(s+a)} = \frac{s(s+\lambda a)}{s^2 + (K_t + \lambda a)s + aK_t}$$

The situation described in Table 3.5 also considers a third order PLL (row 3 in the table). We should compare to row 2, which is the perfect second order loop.

The three inputs to consider are indicated by the columns in the table:

input 1: $\theta_0 \neq 0$, $f_\Delta = 0$, $R = 0$
input 2: $\theta_0 \neq 0$, $f_\Delta \neq 0$, $R = 0$
input 3: $\theta_0 \neq 0$, $f_\Delta \neq 0$, $R \neq 0$

In general the L.T. of the PLL input is

$$\Theta(s) = \frac{2\pi R}{s^3} + \frac{2\pi f_\Delta}{s^2} + \frac{\theta_0}{s}$$

Thus the phase error LT becomes

$$\Psi(s) = \left[\frac{s(s+\lambda a)}{s^2 + (K_t + \lambda a)s + aK_t}\right]\left[\frac{2\pi R}{s^3} + \frac{2\pi f_\Delta}{s^2} + \frac{\theta_0}{s}\right]$$

and the limit we want to evaluate is

$$\lim_{s\to 0} s\,\Psi(s) = \lim_{t\to\infty} \psi(t) = \text{steady state phase error}$$

Can consider the three inputs separately for taking limits and then sum up the results.

$\boxed{\theta_0 \neq 0 \text{ term}}$

$$\cancel{s}\left[\frac{s(s+\lambda a)}{s^2 + (K_t + \lambda a)s + aK_t}\right]\frac{\theta_0}{\cancel{s}} \longrightarrow 0 \quad \text{as} \quad s \to 0$$

$\boxed{f_\Delta \neq 0 \text{ term}}$

$$\cancel{s}\left[\frac{\cancel{s}(s+\lambda a)}{s^2 + (K_t + \lambda a)s + aK_t}\right]\frac{2\pi f_\Delta}{\cancel{s^2}} \longrightarrow \frac{\lambda}{K_t}\,2\pi f_\Delta \quad \text{as} \quad s \to 0$$

$\boxed{R \neq 0 \text{ term}}$

$$s\left[\frac{s(s+\lambda a)}{s^2 + (K_t + \lambda a)s + aK_t}\right]\frac{2\pi R}{s^3} \longrightarrow \infty \quad \text{as} \quad s \to 0$$

ie no steady state exists

Therefore, Table 3.5 updated to include a row for the imperfect loop filter would look like:

	$\theta_0 \neq 0,\ f_\Delta = 0$, $R = 0$	$\theta_0 \neq 0,\ f_\Delta \neq 0$, $R = 0$	$\theta_0 \neq 0,\ f_\Delta \neq 0$, $R \neq 0$
2' $a \neq 0,\ b = 0$, $\lambda \neq 0$	0	$\frac{\lambda}{K_t} 2\pi f_\Delta$	∞.

3.8. Referring to Computer Example 3.4, draw the block diagram of the system represented by the simulation loop, and label the inputs and outputs of the various loop components with the names used in the simulation code. Using this block diagram, verify that the simulation program is correct. What are the sources of error in the simulation program?

COMPUTER EXAMPLE 3.4

A simulation program is easily developed for the PLL. Two integration routines are required; one for the loop filter and one for the VCO. The trapezoidal approximation is used for these integration routines. The trapezoidal approximation is

```
y[n] = y[n-1] + (T/2)[x[n] + x[n-1]
```

where `y[n]` represents the current output of the integrator, `y[n-1]` represents the previous integrator output, `x[n]` represents the current integrator input, `x[n-1]` represents the previous integrator input, and T represents the simulation step size, which is the reciprocal of the sampling frequency. The values of `y[n-1]` and `x[n-1]` must be initialized prior to entering the simulation loop. Initializing the integrator inputs and outputs usually result in a transient response. The parameter `settle`, which in the simulation program to follow is set equal to 10% of the simulation run length, allows any initial transients to decay to negligible values prior to applying the loop input.

The following simulation program is divided into three parts. The preprocessor defines the system parameters, the system input, and the parameters necessary for execution of the simulation, such as the sampling frequency. The simulation loop actually performs the simulation. Finally, the postprocessor allows for the data generated by the simulation to be displayed in a manner convenient for interpretation by the simulation user. Note that the postprocessor used here is interactive in that a menu is displayed and the simulation user can execute postprocessor commands without typing them.

The simulation program given here assumes a frequency step on the loop input and can therefore be used to generate Figures 3.51 and 3.52.

```
% File:  c3ce4.m
% beginning of preprocessor
clear all % be safe
fdel = input('Enter frequency step size in Hz > ');
fn = input('Enter the loop natural frequency in Hz > ');
zeta = input('Enter zeta (loop damping factor) > ');
npts = 2000; % default number of simulation points
fs = 2000; % default sampling frequency
T = 1/fs;
t = (0:(npts-1))/fs; % time vector
nsettle = fix(npts/10); % set nsettle time as 0.1*npts

Kt = 4*pi*zeta*fn; % loop gain
a = pi*fn/zeta; % loop filter parameter

filt_in_last = 0; filt_out_last=0;
vco_in_last = 0; vco_out = 0; vco_out_last=0;
% end of preprocessor

% beginning of simulation loop
for i=1:npts
    if i < nsettle
          fin(i) = 0;
          phin = 0;
    else
          fin(i) = fdel;
          phin = 2*pi*fdel*T*(i-nsettle);
    end
      s1=phin - vco_out;
      s2=sin(s1); % sinusoidal phase detector
      s3=Kt*s2;
    filt_in = a*s3;
    filt_out = filt_out_last + (T/2)*(filt_in + filt_in_last);
```

```
    filt_in_last = filt_in;
    filt_out_last = filt_out;
    vco_in = s3 + filt_out;
    vco_out = vco_out_last + (T/2)*(vco_in + vco_in_last);
    vco_in_last = vco_in;
    vco_out_last = vco_out;
    phierror(i)=s1;
    fvco(i)=vco_in/(2*pi);
    freqerror(i) = fin(i)-fvco(i);
end
% end of simulation loop

% beginning of postprocessor
kk = 0;
while kk == 0
    k = menu('Phase Lock Loop Postprocessor',...
    'Input Frequency and VCO Frequency',...
    'Phase Plane Plot',...
    'Exit Program');
    if k == 1
       plot(t,fin,t,fvco)
       title('Input Frequency and VCO Freqeuncy')
       xlabel('Time - Seconds')
       ylabel('Frequency - Hertz')
       pause
    elseif k == 2
       plot(phierror/2/pi,freqerror)
       title('Phase Plane')
       xlabel('Phase Error / pi')
       ylabel('Frequency Error - Hz')
       pause
    elseif k == 3
       kk = 1;
    end
end
% end of postprocessor
```

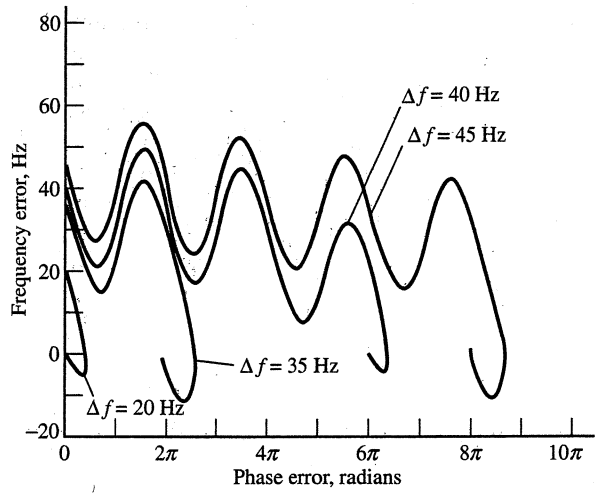


Figure 3.51
Phase-plane plot for second-order PLL.

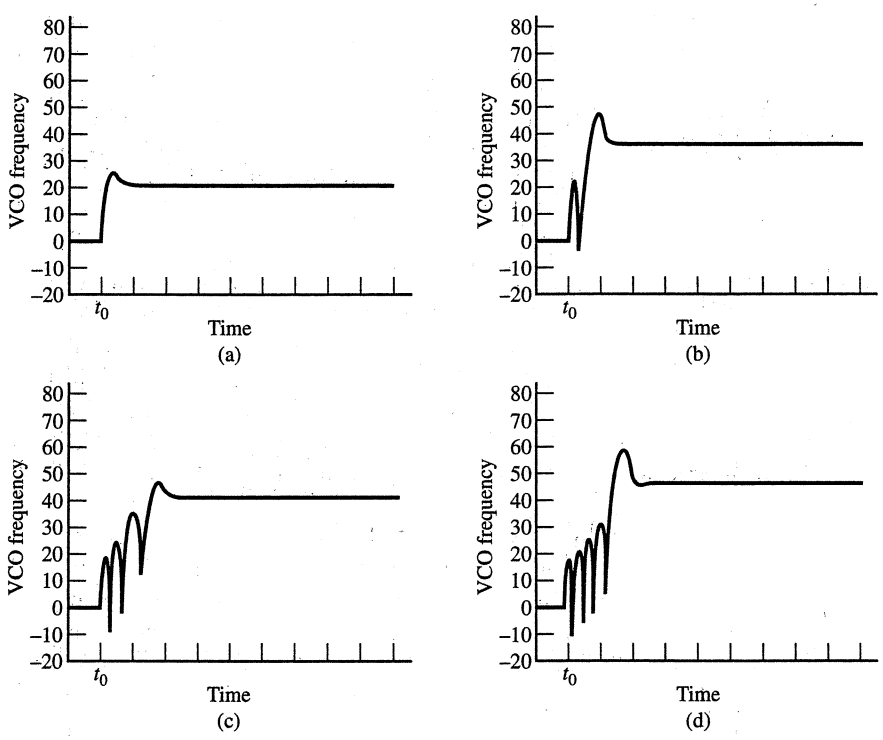


Figure 3.52
voltage-controlled oscillator frequency for four values of input frequency step. (a) VCO frequency for $\Delta f = 20$ Hz. (b) VCO frequency for $\Delta f = 35$ Hz. (c) VCO frequency for $\Delta f = 40$ Hz. (d) VCO frequency for $\Delta f = 45$ Hz.

Solution to Z&T Computer Example 3.4 page 179

File = ZT_CExample_3_4.m

Verbatim copy of code from Z&T text

Contents

Beginning of preprocessor

```
clear all;                          %Be safe

%fdel = input('Enter frequency step size in Hz > ');
%fn = input('Enter the loop natural frequency in Hz > ');
%zeta = input('Enter zeta (loop damping factor) > ');
fdel = 35;
fn = 10;
zeta = 0.707;

npts = 2000;                        %Default number of simulation points
fs = 2000;                          %Default sampling frequency
T = 1/fs;

t = (0:(npts-1))/fs;                %Time vector
nsettle = fix(npts/10);             %Set nsettle time as 0.1*npts

Kt = 4*pi*zeta*fn;                  %Loop gain
a = pi*fn/zeta;                     %Loop filter parameter

filt_in_last = 0;
filt_out_last = 0;
vco_in_last = 0;
vco_out = 0;
vco_out_last = 0;

% End of preprocessor
```

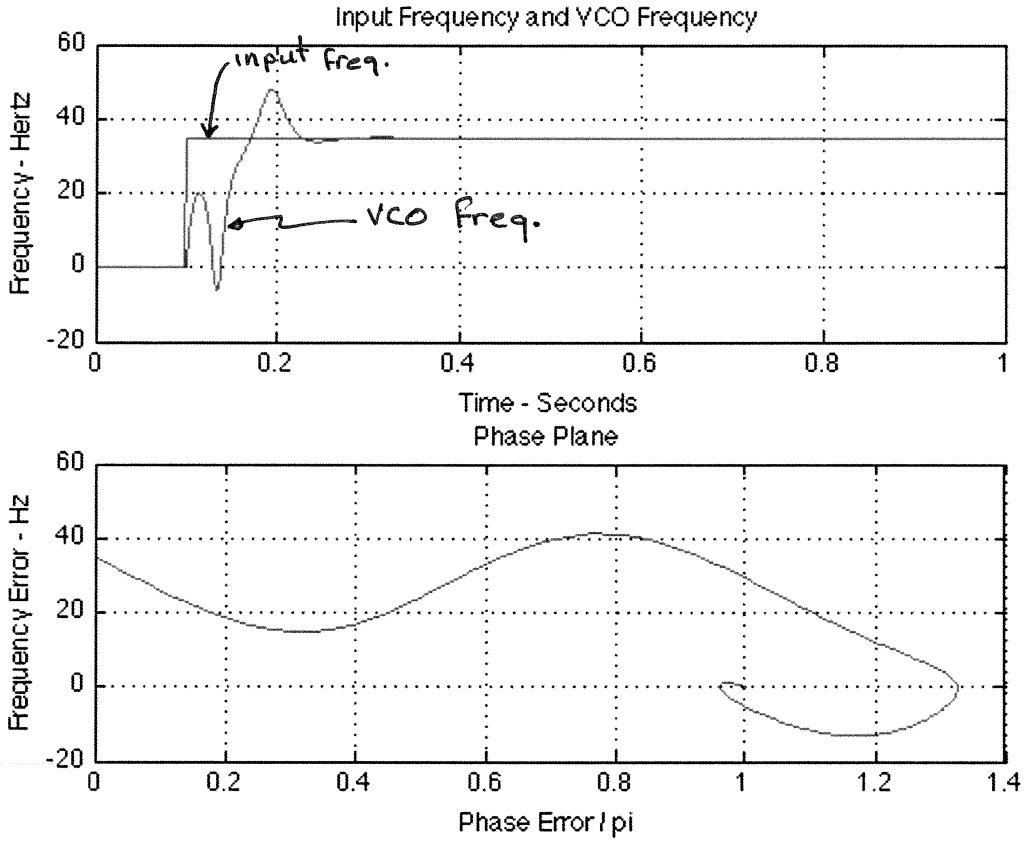
Beginning of Simulation Loop

```
for i = 1:npts

    if i < nsettle
        fin(i) = 0;
        phin = 0;
    else
        fin(i) = fdel;
        phin = 2*pi*fdel*T*(i-nsettle);
    end
```

```
    s1 = phin - vco_out;
    s2 = sin(s1);                %Sinusoidal phase detector
    s3 = Kt*s2;

    filt_in = a*s3;
    filt_out = filt_out_last + (T/2)*(filt_in + filt_in_last);
    filt_in_last = filt_in;
    filt_out_last = filt_out;
    vco_in = s3 + filt_out;
    vco_out = vco_out_last + (T/2)*(vco_in + vco_in_last);
    vco_in_last = vco_in;
    vco_out_last = vco_out;
    phierror(i) = s1;
    fvco(i) = vco_in/(2*pi);
    freqerror(i) = fin(i) - fvco(i);

end

% End of Simulation Loop
```

Beginning of Postprocessor

```
subplot(2,1,1)
plot(t,fin,t,fvco)
title('Input Frequency and VCO Frequency')
xlabel('Time - Seconds')
ylabel('Frequency - Hertz')
grid

subplot(2,1,2)
plot(phierror/2/pi,freqerror)
title('Phase Plane')
xlabel('Phase Error / pi')
ylabel('Frequency Error - Hz')
grid

% End of Postprocessor
```



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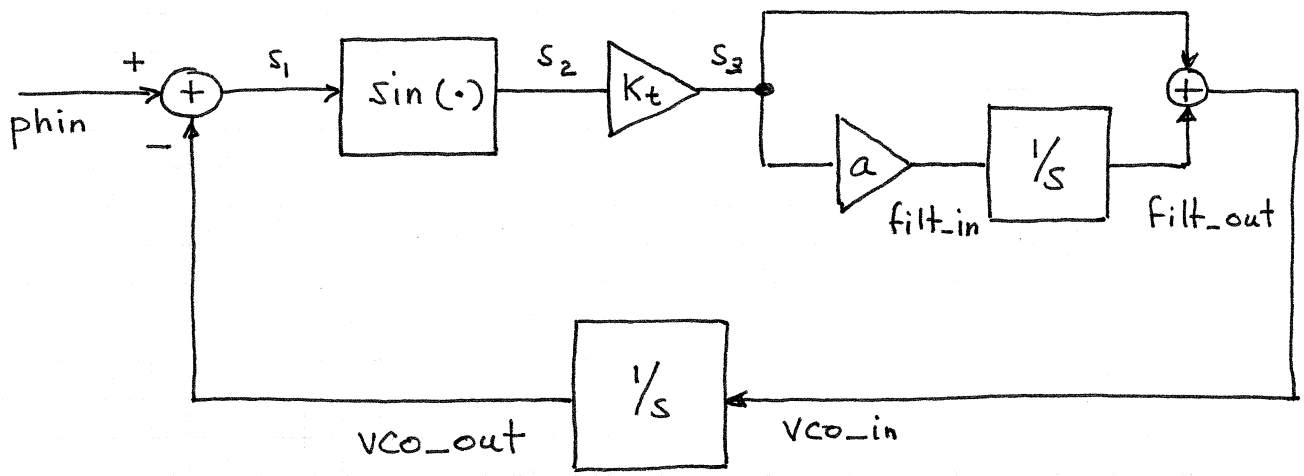
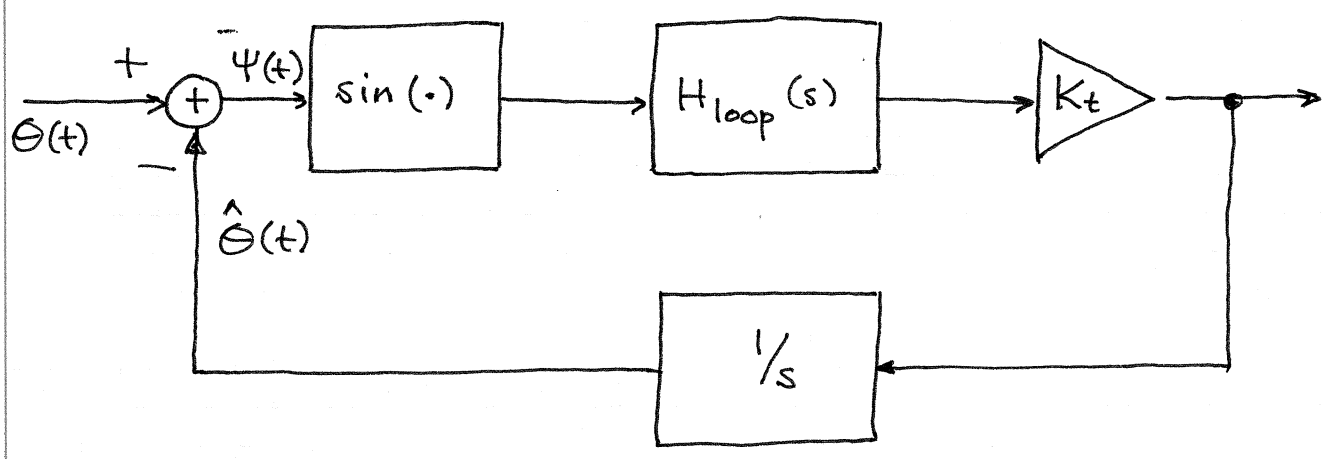
Block Diagram of System of CE 3.4



The block diagram above was created by examining the code given in CE 3.4 on p 179 of Z+T. It should be compared to the baseband block diagram from class:



$H_{loop}(s) = \dfrac{s+a}{s}$ for the so-called perfect loop filter.

Notes:

① phin $= \theta$, vco_out $= \hat{\theta}$, s1 $= \psi$

② order of K_t and H_{loop} in cascade can be reversed.

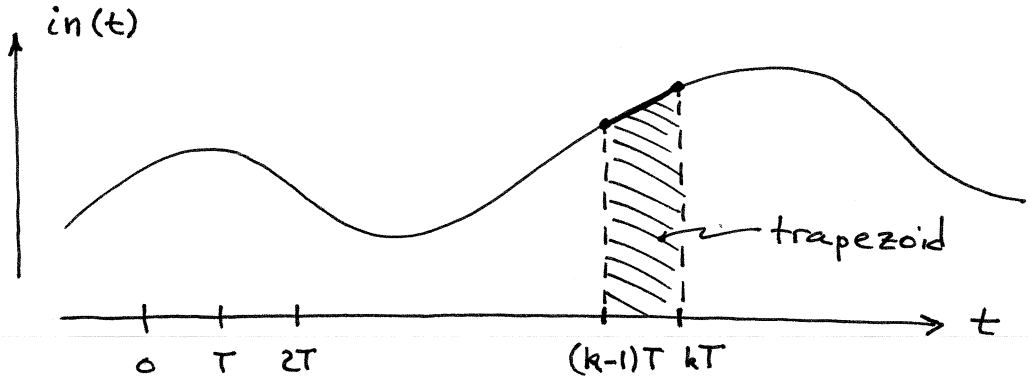
③ The implementation of H_{loop} follows from writing

$$\frac{s+a}{s} = 1 + \frac{a}{s} = 1 + a \cdot \left(\frac{1}{s}\right).$$

The CE 3.4 code computes a and K_t from the values of f_n and ξ using

$$a = \pi f_n / \xi \qquad K_t = 4\pi \xi f_n$$

The trapezoidal approximation is justified by disretizing time into short intervals of length T



The incremental area over the interval $[(k-1)T, kT)$ is approx. given by the area of a trapezoid

$$T\left[\frac{in(kT) + in((k-1)T)}{2}\right]$$

Thus can write

$$\int_0^t in(\lambda)\, d\lambda \approx \sum_{k=1}^{\lfloor t/T \rfloor} \frac{T}{2}\left[in(kT) + in((k-1)T)\right]$$

which can be implemented recursively by defining

$$out(kT) - out((k-1)T) = \frac{T}{2}\left[in(kT) + in((k-1)T)\right]$$

$$out(kT) = out((k-1)T) + \frac{T}{2}\left[in(kT) + in((k-1)T)\right]$$

and starting with

$$out(0 \cdot T) = 0$$

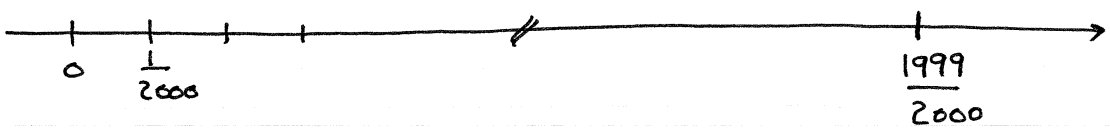
The primary error source is from discretization (since the Matlab code is floating point). Smaller T means more accurate.

The simulation code picks:

$$f_s = 2000 \qquad T = 1/f_s = 1/2000$$

$$t = (0:1999)/2000$$

$$\downarrow$$



So total simulation is approx. 1 second long. Now whether or not this sampling rate is fine or course depends on the bandwidths of the signals we expect to process.

The purpose of the system is to compute the phase estimate $\hat{\Theta}(t)$ as a function of time. The relevant transf. function is

$$\frac{\hat{\Theta}(s)}{\Theta(s)} = F_{closed-loop}(s) = \frac{K_t(s+a)}{s^2 + K_t s + aK_t}$$

$$= \frac{(4\pi\xi f_n)s + 4\pi^2 f_n^2}{s^2 + (4\pi\xi f_n)s + 4\pi^2 f_n^2}$$

which is lowpass $(s = j2\pi f)$.

In the example the simulation params were $\xi = 0.707$ and $f_n = 10\,Hz$.

$$F_{\text{closed-loop}}(j2\pi f) = \frac{4\pi\xi f_n (j2\pi f) + 4\pi^2 f_n^2}{(j2\pi f)^2 + (4\pi\xi f_n)(j2\pi f) + 4\pi^2 f_n^2}$$

$$= \frac{4\pi^2 f_n^2}{4\pi^2 f_n^2} \frac{1 + j2\xi f/f_n}{-(f/f_n)^2 + j2\xi f/f_n + 1}$$

$$= \frac{1 + j2(f/f_n)}{1 - (f/f_n)^2 + j2\xi(f/f_n)}$$

$\Longrightarrow$ Plot this mag. resp. to get an idea of potential aliasing error.