**A.5.** Assuming a bandwidth of 2 MHz, find the rms noise voltage across the output terminals of the circuit shown in Figure A.9 if it is at a temperature of 400 K.

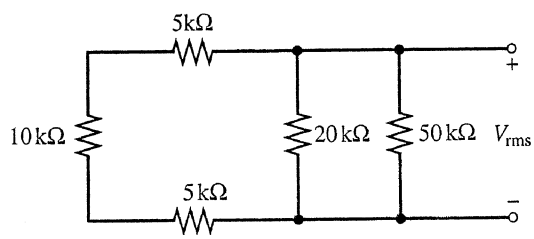
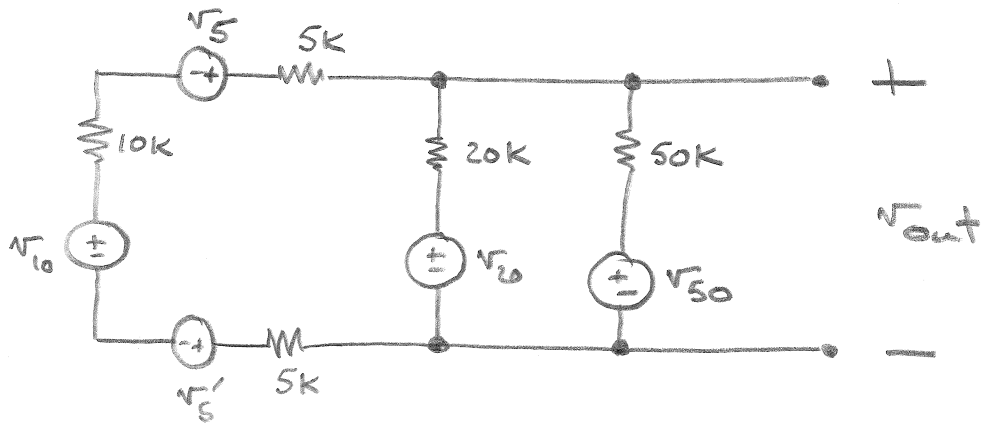



**Figure A.9**

## Z+T Problem A.5

There are two ways to attack this and they should lead to the same answer.

The Long Way Model each noisy resistor as an indep noise source and solve




$T = 400K$. The five noise sources are statistically independent, zero mean, white random processes. Their two-sided power spectral densities are

$$S_{v_5}(f) = 2kTR_5 = S_{v_5'}(f) \quad \forall f \qquad R_5 = 5k\Omega$$

$$S_{v_{10}}(f) = 2kTR_{10} \quad \forall f \qquad R_{10} = 10k\Omega$$

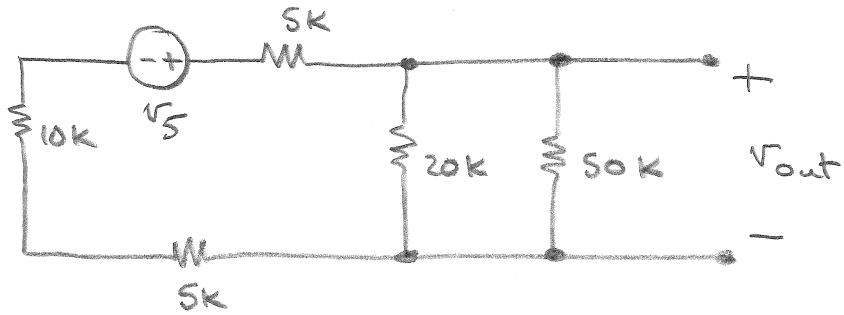
$$S_{v_{20}}(f) = 2kTR_{20} \quad '' \qquad R_{20} = 20k\Omega$$

$$S_{v_{50}}(f) = 2kTR_{50} \quad '' \qquad R_{50} = 50k\Omega$$

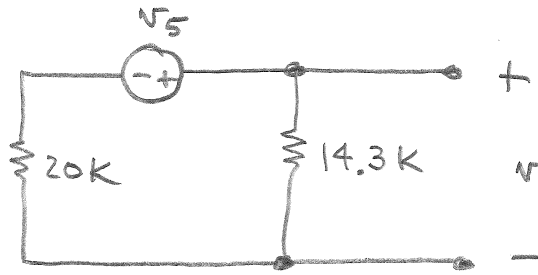
Superposition of voltages always works for LTI systems but what we want is superposition of powers and this also works here because of the statistical independence of the individual noise sources.

To use this method we need the transfer function from each input to the output. These we get from the usual 201 method.

## T.F. from 5 → output: $H_{5,out}$




Simply using circuit theory




$$v_{out} = \underbrace{\frac{14.3}{20+14.3}}_{H_{5,out} = 0.42} v_5$$

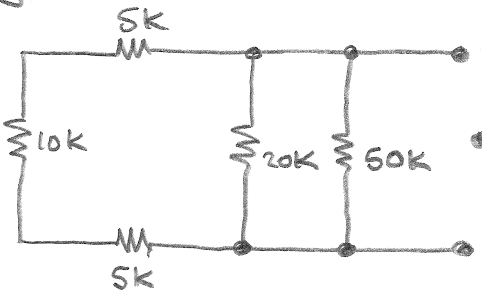
Then if find all the others

$$S_{v_{out}}(f) = H^2_{5,out} S_{v_5}(f) + H^2_{5',out} S_{v_{5'}}(f) + H^2_{10,out} S_{v_{10}}(f) + H^2_{20,out} S_{v_{20}}(f) + H^2_{50,out} S_{v_{50}}(f)$$

Then the power in a 2MHz one-sided BW would be

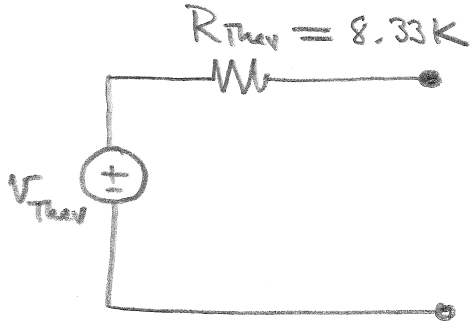
$$P_{2M} = 2\int_{\substack{2M\\BW}} S_{v_{out}}(f)\,df = 2\cdot(2MHz)\cdot S_{v_{out},height}$$

<u>The Short Way</u> Use Nyquist's Theorem to find a single resistor




$$R_{th} = 20K \,\|\, 20K \,\|\, 50K$$
$$= 10K \,\|\, 50K$$
$$= 8.33K$$

Then this noisy resistor produces the same noise voltage as the previous




$$S_{V_{Thev}}(f) = 2kTR_{Thev} \quad \forall f$$

$$[V^2/Hz]$$

Then over a 2MHz one-sided BW

$$P = 2(2\times10^6)(2)(1.38\times10^{-23})(400)(8.33\times10^3)$$

$$= (3200\times10^6)(1.38\times10^{-23})(8.33\times10^3)$$

$$= (3.2)(1.38)(8.33)\times10^{-11} \quad V^2$$

$$= 36.79\times10^{-11}\ V^2$$

$$\begin{array}{r} -23 \\ +12 \\ \hline -11 \end{array}$$

$\Longrightarrow$ RMS noise voltage $= \sqrt{P} = 19.2\ \mu V$ rms.

**A.7.** A source with equivalent noise temperature $T_s = 1000$ K is followed by a cascade of three amplifiers having the specifications shown in Table A.1. Assume a bandwidth of 50 kHz.

**Table A.1**

| Amplifier no. | $F$ | $T_e$ | Gain |
|---|---|---|---|
| 1 | | 300 K | 10 dB |
| 2 | 6 dB | | 30 dB |
| 3 | 11 dB | | 30 dB |

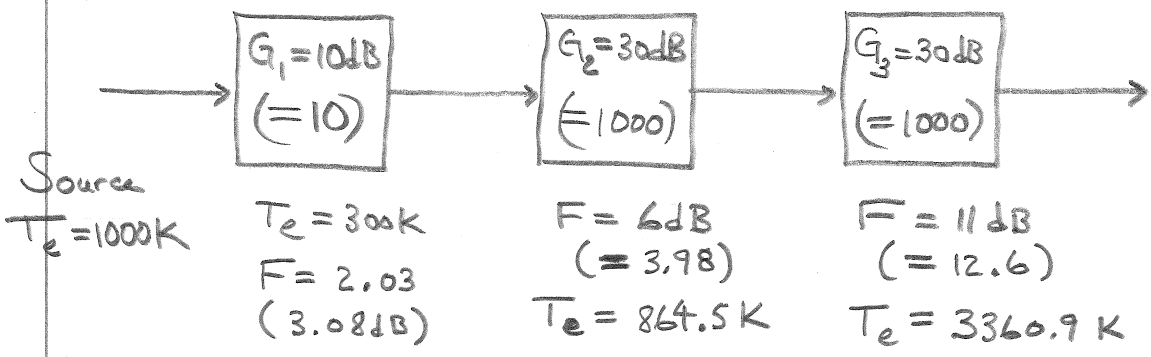
**a.** Find the noise figure of the cascade.

**b.** Suppose amplifiers 1 and 2 are interchanged. Find the noise figure of the cascade.

**c.** Find the noise temperature of the systems of parts (a) and (b).

**d.** Assuming the configuration of part (a), find the required input signal power to give an output SNR of 40 dB. Perform the same calculation for the system of part (b).

# Z+T Problem A.7




Assume that all noise figures are standardized for 290K as discussed in class notes. Then the relationship between noise figure and effective noise temp of amp is

$$T_{amp,e} = (290K)(F_{amp} - 1)$$

$$F_{amp} = \frac{T_{amp,e}}{290K} + 1$$

(a) Noise Figure of the Cascade

$$F_{cascade} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2}$$

$$= 2.03 + \frac{3.98-1}{10} + \frac{12.6-1}{10^4}$$

$$= 2.33 \quad (= 3.67\,dB)$$

(b) Interchange amps 1 and 2 and repeat (a).

$$F_{cascade} = 3.98 + \frac{2.03-1}{1000} + \frac{12.6-1}{10^4}$$

$$= 3.98 \quad (= 6.00\,dB)$$

(c) Noise temp of systems of (a), (b).

We already found the equiv. noise temps of the individual elements. To find the system equiv. temps we can use

$$T_{syst,e} = (290K)(F_{system} - 1)$$

$\therefore$

$$T_{syst\_a,e} = (290K)(2.33 - 1)$$
$$= 385.7\ K$$

$$T_{syst\_b,e} = (290K)(3.98 - 1)$$
$$= 864.2\ K$$

(d) Assuming the 1000K noise source find the noise power in the output for each of the two systems. Then for output SNR of 40 dB, find the required input signal power.

The one-sided BW is given as 50KHz.

Output noise power $= N_o = k(T_{input} + T_{syst,e})BG_1G_2G_3$

$$N_{o,a} = (1.38\times10^{-23})[1000 + 385.7](50\times10^3)(10^7)$$
$$= -50.2\ dBm$$

$$N_{o,b} = (1.38\times10^{-23})[1000 + 864.2](50\times10^3)(10^7)$$
$$= -48.9\ dBm$$

To have SNR of 40 dB need

$$S_{o,a} = -50.2 + 40 \text{ dBm}$$
$$= -10.2 \text{ dBm}$$

Since cascade gain is 70dB then need

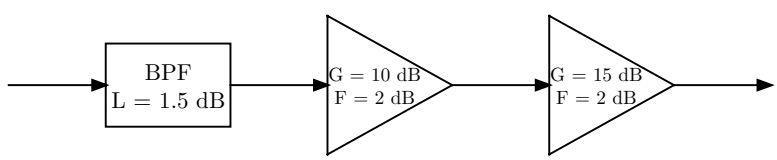
$$S_{in,a} = -10.2 \text{ dBm} - 70 \text{ dB}$$
$$= -80.2 \text{ dBm} \quad \text{input sig. power}$$

For the system of (b)

$$S_{o,b} = -48.9 \text{ dBm} + 40 \text{ dB}$$
$$= -8.9 \text{ dBm}$$

$$\Rightarrow S_{in,b} = -8.9 \text{ dBm} - 70 \text{ dBm}$$
$$= -78.9 \text{ dBm} \quad \text{input sig. power}$$

(Pozar P3.18, p. 108) Consider the wireless local area network receiver front end shown below, where the bandwidth of the bandpass filter is 150 MHz centered at 2.4 GHz. If the system is at room temperature, find the noise figure of the overall system. What is the output SNR if the input signal level is -85 dBm? Can the components be rearranged to give a better noise figure?



Pozar P3.18

No information is given about the input noise, hence we will assume that the only noise present in the system is that internally generated by the 3 component blocks.

BPF at $T_o = 290K$

$G = -1.5\ dB = 0.708$ ; $L = 1.5\ dB = 1.413$

The BPF looks like an attenuator over the band of interest.

Since it is at room temp the standard noise fig. for the BPF is equal to the loss

$F = 1.5\ dB = 1.413$

First PA

$G = 10\ dB = 10$
$F = 2\ dB = 1.585$

Second PA

$G = 15\ dB = 31.623$
$F = 2\ dB = 1.585$

The overall noise figure of the system is calculated from the cascade formula

$$F_{overall} = F_{BPF} + \frac{F_{1st\,PA} - 1}{G_{BPF}} + \frac{F_{2nd\,PA} - 1}{G_{BPF} \cdot G_{1st\,PA}}$$

$$= 1.413 + \frac{0.585}{0.708} + \frac{0.585}{(0.708)(10)}$$

$$= 2.322$$

$$= 3.66\ dB$$

The equivalent noise temp. of the entire cascade referred to the input is

$$T_{overall} = (F_{overall} - 1)(290K)$$

$$= (1.322)(290K) = 383.38K$$

Thus the equiv. noise power in a one sided bandwidth of 150 MHz at input is

$$N_{input} = k T_{overall} B$$

$$= (1.38 \times 10^{-23} J/K)(383.38K)(150 \times 10^{6} s^{-1})$$

$$= 7.94 \times 10^{-10} mW$$

$$= -91.0 dBm$$

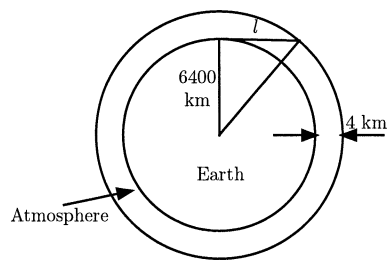
Input signal level is $-85$ dBm. Therefore, input SNR is

6dB. (in BW 150MZ)

This will also be the output SNR since we've referred signal & noise to input and modeled blocks as noiseless.

A better noise figure would be obtained by putting high gain amplifiers first (although there could be good reasons <u>not</u> to do this).

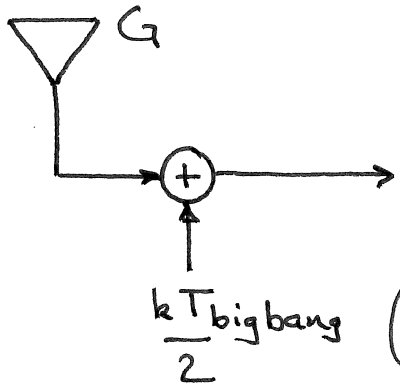
(Pozar P4.14, p. 148) The atmosphere does not have a definite thickness, since it gradually thins with altitude, with a consequent decrease in attenuation. But if we use a simplified "orange peel" model, and assume that the atmosphere can be approximated by a uniform layer of fixed thickness, we can estimate the background noise temperature seen through the atmosphere. Thus, let the thickness of the atmosphere be 4000 m and find the maximum distance $l$ to the edge of the atmosphere along the horizon, as shown in the figure below (the radius of the earth is 6400 km). Now assume an average atmospheric attenuation of 0.005 dB/km, with a background noise temperature beyond the atmosphere of 4 K, and find the noise temperature seen on earth by treating the cascade of the background noise with the attenuation of the atmosphere. Do this for an ideal antenna pointing toward the zenith, and toward the horizon.



# Pozar 4.14

The correct answer seems to depend on exactly how we choose to model this antenna. First, imagine the antenna outside of the earth's atmosphere pointing into empty space. Then

## Model for Antenna Above Atmosphere

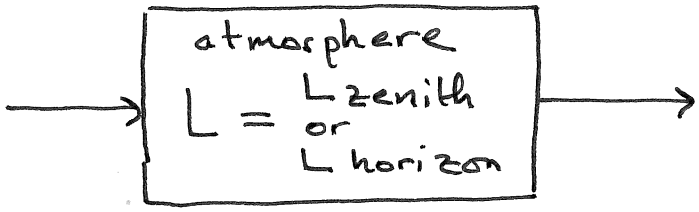



$T_{bigbang} = 4K$

$G$ = antenna gain for a signal arriving along mainlobe direction

(psd height of equiv. noise source)

Then when the antenna is placed in the atmosphere both the signal <u>and</u> the big bang noise are antenuated by some Loss factor L.




We may assume that this atmospheric attenuator is at the normal earth temperature

$$T_o = 290\,K$$

The loss depends on the direction the antenna is pointing (zenith or horizon) since that determines the loss incurred.

<u>Toward Zenith</u> 4 km of atmosphere

$$\Rightarrow (4\text{ km})(0.005\ \text{dB/km}) = 0.02\ \text{dB}$$

$\Rightarrow$ Loss factor for $L_{dB} = 0.02$ dB is

$$L = 10^{L_{dB}/10} = 1.0046$$

$\Rightarrow$ Gain

$$G = \frac{1}{L} = 0.9954$$

<u>Toward Horizon</u> $\ell = \sqrt{6404^2 - 6400^2} = 226.3$ km

$$\Rightarrow (226.3\text{ km})(0.005\ \text{dB/km}) = 1.132\ \text{dB}$$

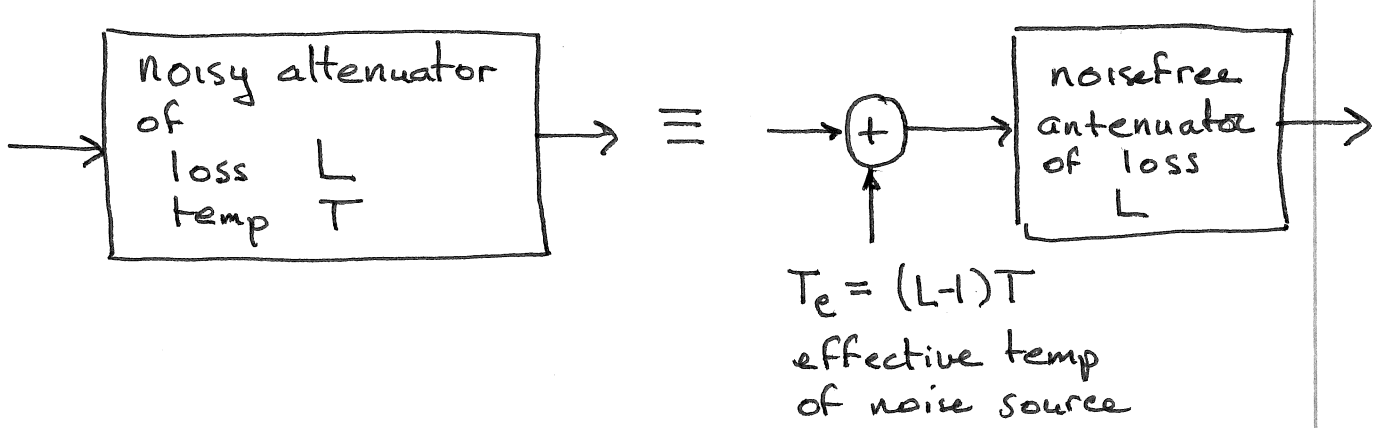
$\Rightarrow$ Loss factor for $L_{dB} = 1.132$ dB is

$$L = 10^{L_{dB}/10} = 1.2976$$

$\Rightarrow$ Gain

$$G = \frac{1}{L} = 0.7706$$

Either of these attenuators is lossy and therefore introduce noise as well. An arbitrary noisy attenuator with loss L and physical temp. T is modeled:



Thus for the two cases we have

$$L_{zenith} = 1.0046$$

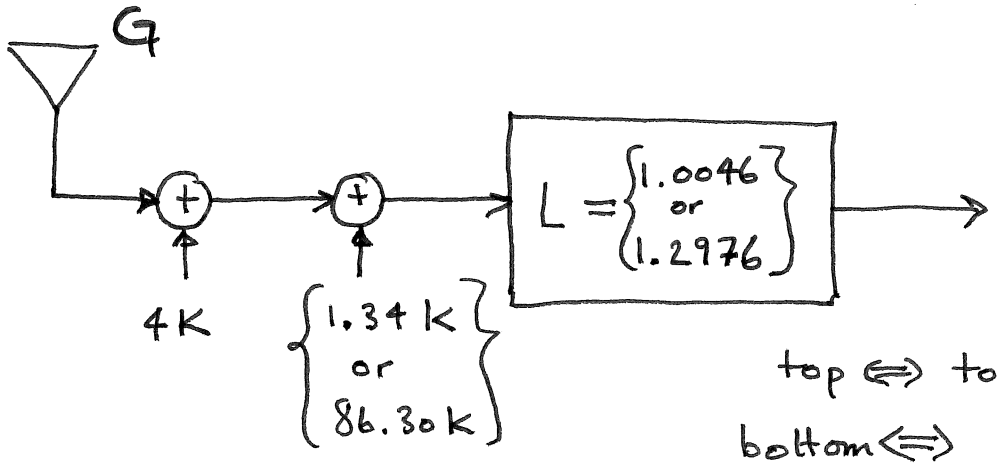
$$T_{e,zenith} = (1.0046-1)(290\,K) = 1.34\,K$$

$$L_{horizon} = 1.2976$$

$$T_{e,horizon} = (1.2976-1)(290\,K) = 86.30\,K$$

Putting these together the model so far is:

<u>Toward Zenith (top) / Horizon (bottom)</u>




top $\Leftrightarrow$ toward zenith
bottom $\Leftrightarrow$ " horizon.

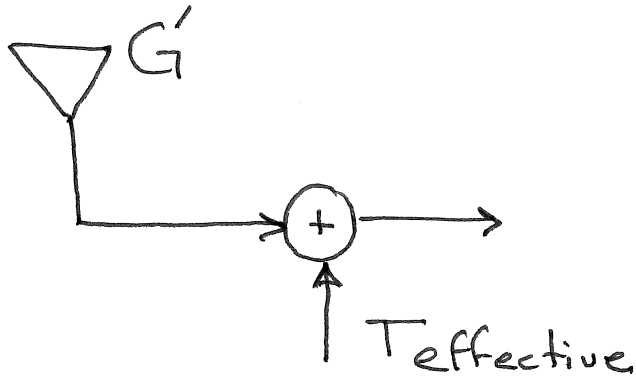
So one model simply combines the noise sources at the input of the noiseless atten. block above. This would give equiv. noise temperatures of

$4 + 1.34 = 5.34\,K$ followed by 0.02 dB loss

or

$4 + 86.30 = 90.30\,K$ " " 1.13 dB loss

But in the spirit of modeling the antenna in the usual way we want something like

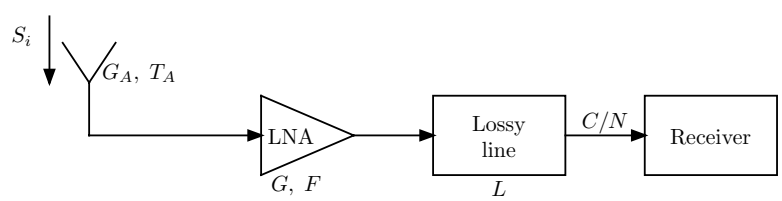



where the noise source should account for the noise at the output of the noiseless attenuator and the new gain would be

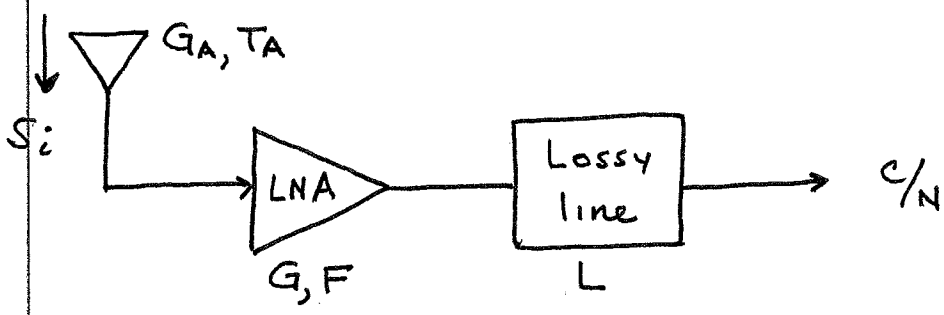
$$G' = G/L = \left\{ \begin{array}{l} G/1.0046 \\ \text{or} \\ G/1.2976 \end{array} \right\}$$

$$T_{effective} = \left\{ \begin{array}{ll} 5.34K/1.0046 & = 5.32K \\ \text{or} & \\ 90.30K/1.2976 & = 69.59K \end{array} \right\}$$

(Pozar P4.17, p. 149) Consider the GPS receiver system shown below. The guaranteed minimum L1 (1575 MHz) carrier power received by an antenna on Earth having a gain of 0 dBi is $S_i$ = -160 dBW. A GPS receiver is usually specified as requiring a minimum carrier-to-noise ratio, relative to a 1 Hz bandwidth, of $C/N$ (Hz). If the receiver antenna actually has a gain $G_A$, and a noise temperature $T_A$, derive an expression for the maximum allowable amplifier noise figure $F$, assuming an amplifier gain $G$, and a connecting line loss, $L$. Evaluate this expression for $C/N$ = 32 dB-Hz, $G_A$ = 5 dB, $T_A$ = 300 K, $G$ = 10 dB, and $L$ = 25 dB.



Pozar 4.17




Derivation of Expression for max. allowable LNA noise figure st. a minimum require C/N is met

$$S_{i,dB} = -160 \text{ dBW} = 10 \log_{10}\left(\frac{S_i}{1W}\right)$$

$$\Longrightarrow S_i = 10^{-S_{i,dB}/10} \text{ W}$$

$$= 10^{-16} \text{ W} \longrightarrow$$ this is relative to a 0 dBi antenna.

The signal power in the output of this GPS receiver will be

$$C = G_A G L^{-1} S_i$$

Next step is to find the noise power present in the output, relative to a one Hz bandwidth ... this is equivalent to finding the one-sided noise psd height at the output since

$$N_o(1 \text{ Hz}) = N$$

per the problem's definition of C/N.

The noise can be first modeled as coming from two sources:

① From the antenna described by its temp $T_A$

$\Longrightarrow$ Noise available at antenna output in a one sided BW B is $kT_AB$ W.

② Noise internal to the LNA and lossy line, which will be amplified, appearing in the output.

We follow the usual approach to ② wherein the internal noise sources are referred to the input of the LNA – Lossy Line cascade and then are amplified by noiseless versions of the LNA – Lossy Line before appearing in the output.

$$T_{LNA} = (F_{LNA} - 1)\,290\,K$$

$T_{Loss} = (F_{Loss} - 1)\,290K$ and for an attenuator at 290K we have $F_{Loss} = L$, the attenuation factor.

From the formula for a cascade:

$$T_e = T_{LNA} + \frac{T_{Loss}}{G}$$

$$= \left[F_{LNA} - 1 + \frac{L-1}{G}\right] 290\,K$$

Therefore, the effective input referenced noise of the LNA – Lossy line in a BW B is

$$kT_eB = \left[F_{LNA} - 1 + \frac{L-1}{G}\right] k\,(290k)\,B$$

and the total noise power in a BW B at input is

$$kT_AB + kT_eB$$

$$= \left[T_A + \left(F_{LNA} - 1 + \frac{L-1}{G}\right)290\right] kB$$

Setting $B = 1\,Hz$ and applying the cascade gain $GL^{-1}$ we have the noise power present in the output in a 1 Hz BW:

$$N = GL^{-1}\left[T_A + \left(F_{LNA} - 1 + \frac{L-1}{G}\right)290\right]k$$

$\therefore$

$$\frac{C}{N} = \frac{G_A S_i}{k\left[T_A + \left(F_{LNA} - 1 + \frac{L-1}{G}\right)290\right]} \geq \gamma$$

Say this is min C/N needed.

Solve above inequality for $F_{LNA}$:

$$F_{LNA} \leq \frac{G_A S_i}{k\gamma \cdot 290} - \frac{T_A}{290} + 1 - \frac{L-1}{G}$$

<u>Evaluate Expression with:</u>

$\frac{C}{N} = \gamma = 32$ dB-Hz $\longrightarrow$ 1584.89

$G_A = 5$dB $\longrightarrow$ 3.16

$T_A = 300K$

$G = 10$dB $\longrightarrow$ 10.0

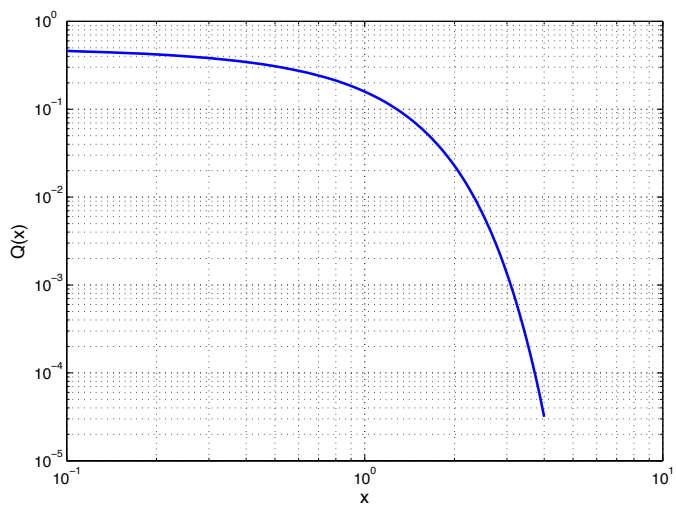
$L = 25$ dB $\longrightarrow$ 316.23

$$F_{LNA} \leq \frac{(3.16)(10^{-16})}{(1584.89)(1.38\times10^{-23})(290)} - \frac{300}{290} + 1 - \frac{316.23 - 1}{10}$$

$\Longrightarrow F_{LNA} \leq 18.26 \rightsquigarrow 12.62$ dB

(Final Exam, Fall 2009) GPS satellites transmit data to GPS receivers using BPSK modulation (in addition to spreading codes used for correlation and time delay estimation, which we ignore here). Suppose that the carrier-to-noise ratio in a 1 Hz bandwidth is $C/N = 32$ dB-Hz.

At what maximum rate $R$ (in bits per second) may we transmit using BPSK if we are to keep the bit error rate below $10^{-4}$?



For BPSK $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

If the carrier power is C W and bits are transmitted at a rate of R bits per second then $T = 1/R$ is the time interval between bits. Thus

$$E_b = CT = C/R$$

If N = noise power in a one sided BW of B Hz then

$$N = N_0 B$$

$\Rightarrow$ For $B = 1\,Hz$: $N = N_0$

$$\Rightarrow \frac{C}{N} = \frac{R E_b}{N_0} \Rightarrow \frac{E_b}{N_0} = \frac{C}{RN}$$

Therefore

$$P_e = Q\left(\sqrt{\frac{2C}{RN}}\right)$$

For $P_e \leq 10^{-4}$ need $\sqrt{\frac{2C}{RN}} \geq 3.7$ (from plot)

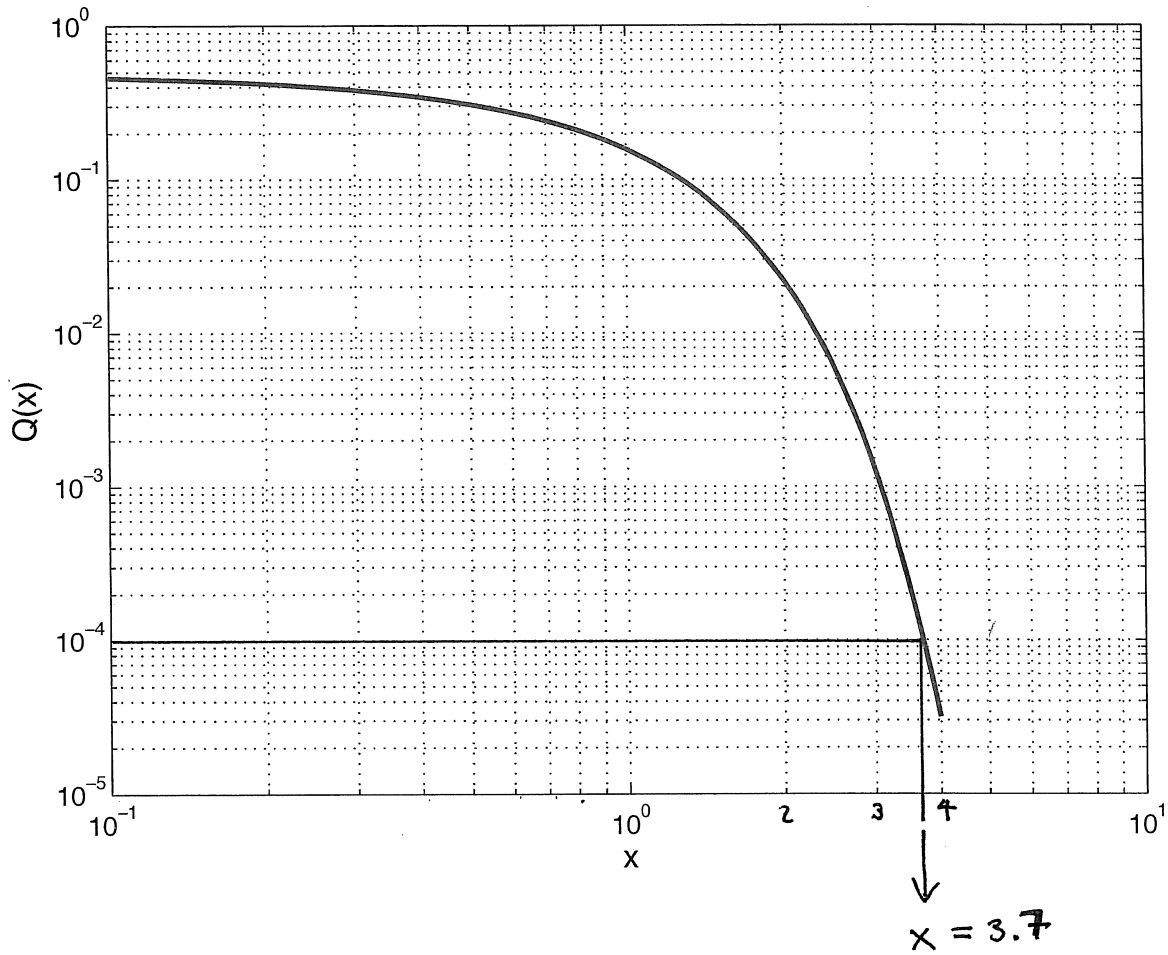
$$\frac{2C}{RN} \geq (3.7)^2 \Leftrightarrow \frac{2}{(3.7)^2}\frac{C}{N} \geq R$$

$$\frac{C}{N} = 32\ \text{dB-Hz} \longrightarrow \frac{C}{N} = 1584.9$$

$$\Rightarrow R \leq (1584.9)\frac{2}{(3.7)^2} = 231.5 \text{ bits per second.}$$

GPS satellites transmit data to GPS receivers using BPSK modulation (in addition to spreading codes used for correlation and time delay estimation). Suppose that the carrier-to-noise ratio in a 1 Hz bandwidth is $C/N = 32$ dB-Hz.

At what rate $R$ (in bits per second) may we transmit using BPSK if we are to keep the bit error rate below $10^{-4}$?



**6.10** Suppose that the receiver of Figure 6-14 is used for the $M$-ASK signal set of Exercise 6-2, and the maximum-likelihood decision regions are employed. Show that

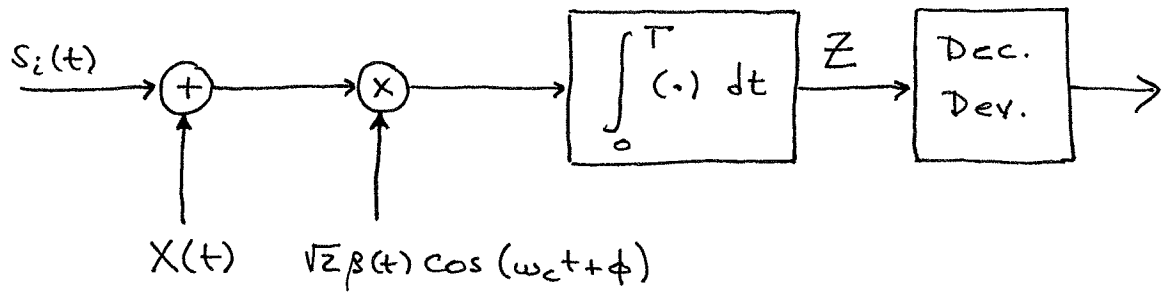
$$P_{e,0} = P_{e,M-1} = Q\left(\sqrt{2A^2\mathcal{E}_\beta/N_0}\right)$$

and that

$$P_{e,i} = 2\,Q\left(\sqrt{2A^2\mathcal{E}_\beta/N_0}\right)$$

for $1 \leq i \leq M-2$. Find the average probability of symbol error that results if $\pi_i = 1/M$ for each $i$.

# MBP 6.10




M-ASK of Exercise 6-2 (p 316) is

$$s_i(t) = \sqrt{2}\, A\, u_i\, \beta(t) \cos(\omega_c t + \phi) \qquad 0 \le t \le T$$

where

$$u_i \in \{-M+1, -M+3, \ldots, -1, +1, \ldots M-3, M-1\} \quad M \text{ even}$$

$$\in \{-M+1, -M+3, \ldots, -2, 0, +2, \ldots M-3, M-1\} \quad M \text{ odd.}$$

Use ML decision rule and compute error probabilities.

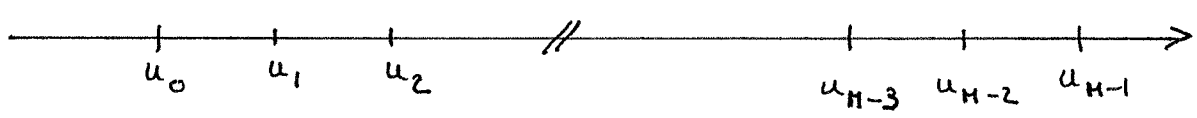
Can show that

$$Z = u_i A \|\beta\|^2 + X$$

$X \longrightarrow$ zero mean Gaussian with variance $N_0 \|\beta\|^2/2$

Lets assume for either the M odd or M even cases that the symbol amplitudes are ordered st.

$$u_0 < u_1 < \cdots < u_{M-2} < u_{M-1}$$



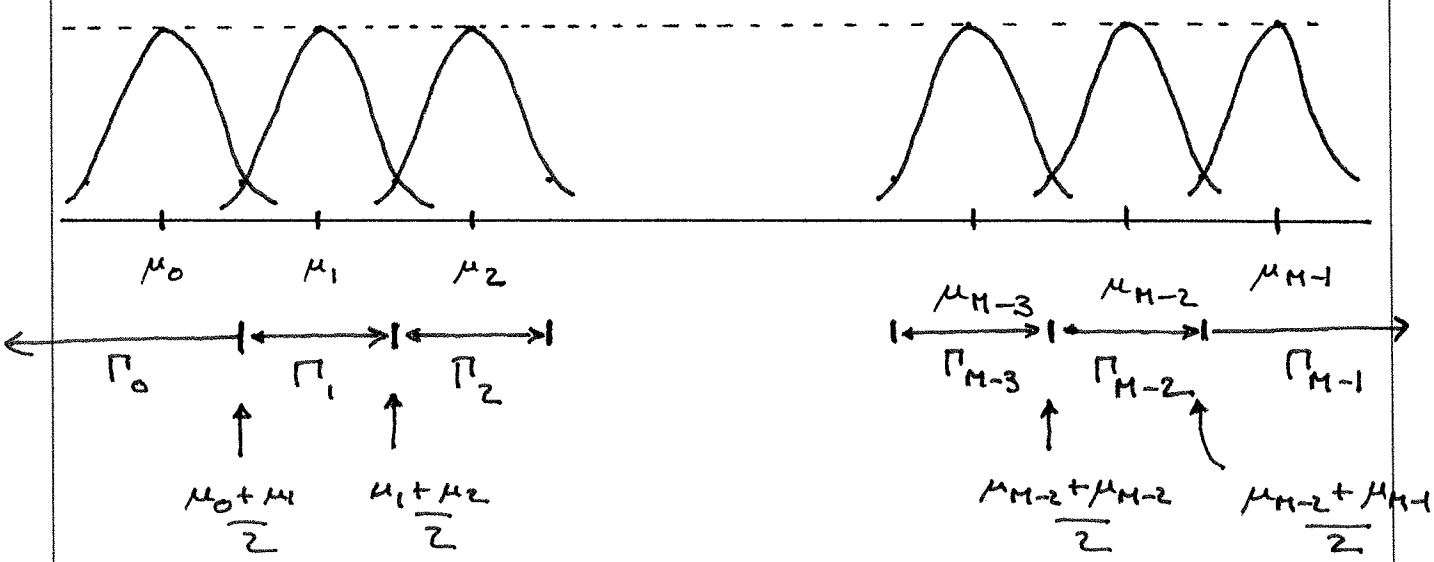
The pdfs characterizing the hypotheses on $Z$ all have same variance and just differ in mean:

$$f_i(z) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(z-\mu_i)^2/2\sigma^2} \qquad \mu_i = u_i A \|\beta\|^2 \qquad \sigma^2 = N_0 \|\beta\|^2/2$$

The ML decision rule picks the hypothesis indexed by $\hat{k}$ given observation $Z$ where

$$\hat{k} = \underset{0 \le k \le M-1}{\arg\max} \; f_k(z)$$

The sketch below illustrates the ML rule as a partition of observation space:




Symmetry gives the boundaries shown. Also clear that

$$P_{e,i} = P_{e,1} \qquad \text{for } i = 1, 2, 3 \ldots M-2$$

$$P_{e,0} = P_{e,M-1} = \frac{1}{2} P_{e,1}$$

$$P_{e,0} = P\left(Z > \frac{\mu_0+\mu_1}{2} \,\middle|\, H_0\right)$$

$$\frac{\mu_0+\mu_1}{2} = \frac{u_0+u_1}{2} A\mathcal{E}_\beta$$

$$= \frac{-M+1-M+3}{2} A\mathcal{E}_\beta$$

$$= \frac{-2M+4}{2} A\mathcal{E}_\beta$$

$$= (-M+2) A\mathcal{E}_\beta$$

Under $H_0$

$$Z \sim N(\mu_0, \sigma^2)$$

$$P_{e,0} = P\left(\frac{Z-\mu_0}{\sigma} > \frac{\mu_1-\mu_0}{2\sigma} \,\middle|\, H_0\right)$$

$$= Q\left(\frac{\mu_1-\mu_0}{2\sigma}\right) \qquad \frac{\mu_1-\mu_0}{2\sigma} = \frac{A\mathcal{E}_\beta}{\sqrt{N_0\mathcal{E}_\beta/2}} = \sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}$$

$$= Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right) = P_{e,M-1}$$

$$\Longrightarrow \quad P_{e,i} = 2P_{e,0}$$

$$= 2Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right) \qquad 1 \le i \le M-2$$

Avg. Prob. Error for equal priors $\pi_i = 1/M$

$$\overline{P_e} = \frac{1}{M}\sum_{i=0}^{M-1} P_{e,i}$$

$$= \frac{2Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right)}{M} + \frac{(M-2)}{M} 2Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right)$$

$$= \frac{2}{M}(M-1)\, Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right)$$

# HW 5 : MBP 6.12
# Corrected Solution

**6.12** A digital communication system uses QASK with ternary modulation on the inphase and quadrature components. Specifically, the transmitted signal is of the form

$$s(t) = A\{a_1(t)\cos(\omega_c t + \varphi) + a_2(t)\sin(\omega_c t + \varphi)\},$$

where $a_1(t)$ and $a_2(t)$ are sequences of rectangular pulses, each of duration $T$. The possible pulse amplitudes for each of the signals $a_1(t)$ and $a_2(t)$ are $-1$, 0, and $+1$. The parameters $A$ and $\varphi$ are fixed quantities that are known to the receiver, and the channel is an additive white Gaussian noise (AWGN) channel.

**(a)** What data rate is possible with this signaling scheme?

**(b)** Give a block diagram for an optimum coherent receiver for this signal and channel.

**(c)** Give *expressions* that specify the minimax decision regions or thresholds for the coherent receiver in part **(b)**. The expressions should be written in terms of the standard Gaussian distribution function $\Phi$ and simplified as much as possible, but it is not necessary to obtain explicit solutions.

$$X_I = \sqrt{2}\int_0^T X(t)\cos(\omega_c t + \phi)\,dt \qquad EX_I = 0$$

$$E(X_I^2) = 2\int_0^T\int_0^T \underbrace{E(X(t)X(s))}_{\frac{N_0}{2}\delta(t-s)} \cos(\omega_c t + \phi)\cos(\omega_c s + \phi)\,dt\,ds$$

$$= N_0\int_0^T \cos^2(\omega_c t + \phi)\,dt = \frac{N_0}{2}T + \underbrace{\frac{N_0}{2}\int_0^T \cos(2\omega_c t + 2\phi)\,dt}_{\approx 0.}$$

Corrected variance calc.

MBP 6.12

$$s(t) = A\left[a_1(t)\cos(\omega_c t + \phi) + a_2(t)\sin(\omega_c t + \phi)\right]$$

$$a_1(t) = \sum_n a_{1,n}\, p_T(t - nT) \qquad a_{1,n},\, a_{2,n} \in \{-1, 0, +1\}$$

$$a_2(t) = \sum_n a_{2,n}\, p_T(t - nT)$$

Coherent receiver, AWGN.

(a) What data rate is possible?

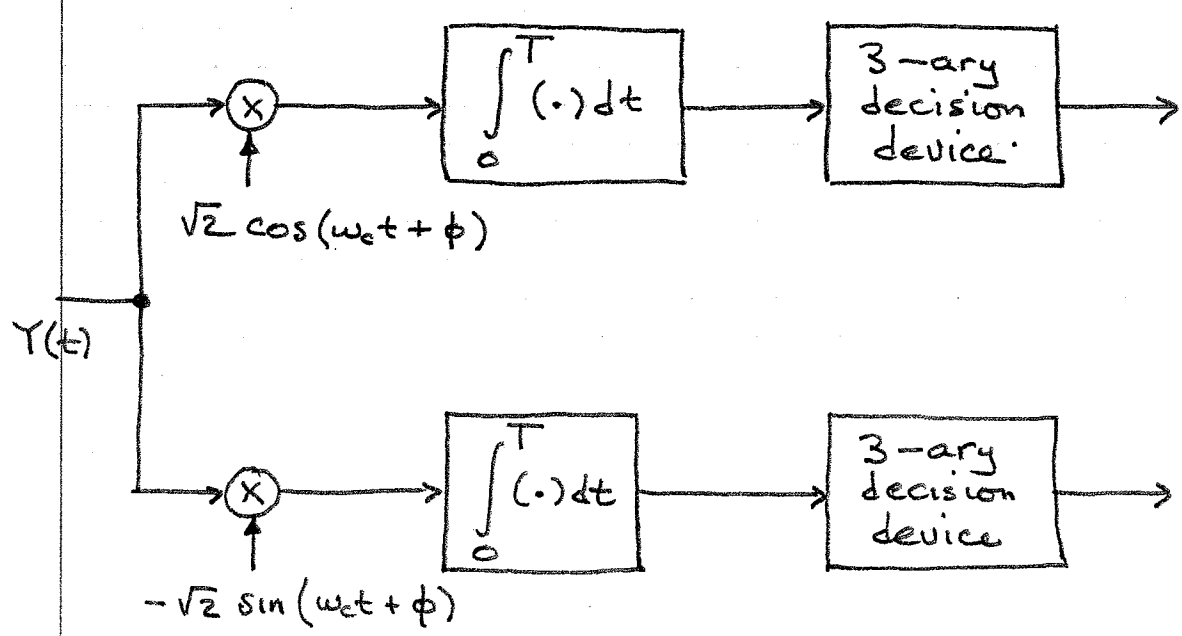
At each signalling interval can send one of $9 = 3 \times 3$ symbols representing

$$\log_2 9 \cong 3.17 \text{ bits}$$

Data rate is

$$R_b = \frac{\log_2 9}{T} \text{ bits/s.}$$

(b) Block diagram for opt. coherent receiver.



(c) Find the minimax thresholds for the 3-ary decision device.

Should see that the design of the I/Q decision devices are identical. Therefore, concentrate on I channel.

$$Z_I = \int_0^T \sqrt{2}\, Y(t) \cos(\omega_c t + \phi)\, dt$$

$$= \sqrt{2} A \int_0^T \left[ a_{1,0} \cos(\omega_c t + \phi) + a_{2,0} \sin(\omega_c t + \phi) \right] \cos(\omega_c t + \phi)\, dt$$

$$+ \sqrt{2} \int_0^T X(t) \cos(\omega_c t + \phi)\, dt$$

$$= \frac{\sqrt{2}}{2} A T a_{1,0} + X_I$$

$X_I$ → Gaussian of mean 0 and variance $N_0 T/2$ --- See front page.

$a_{1,0}$ → $\{-1, 0, \text{ or } +1\}$

By symmetry and Gaussianinity only have to specify one number $\gamma > 0$ with interp

$$Z_I > \gamma \longrightarrow \text{decide } a_{1,0} = +1$$

$$-\gamma < Z_I \leq \gamma \longrightarrow \text{decide } a_{1,0} = 0$$

$$Z_I \leq -\gamma \longrightarrow \text{decide } a_{1,0} = -1$$

$$P(Z_I > \gamma \mid a_{1,0} = +1) = P\left(X_I + \frac{\sqrt{2}}{2}AT > \gamma\right)$$

$$= P\left(\frac{X_I}{\sqrt{N_0T/2}} > \frac{\gamma - \frac{\sqrt{2}}{2}AT}{\sqrt{N_0T/2}}\right) = Q\left(\frac{\gamma - \frac{\sqrt{2}}{2}AT}{\sqrt{N_0T/2}}\right)$$

$$= P(Z_I \le -\gamma \mid a_{1,0} = -1)$$

$$P(-\gamma < Z_I \le \gamma \mid a_{1,0} = 0) = P\left(\frac{-\gamma}{\sqrt{N_0T/2}} \le \frac{X_I}{\sqrt{N_0T/2}} \le \frac{\gamma}{\sqrt{N_0T/2}}\right)$$

$$= \Phi\left(\frac{\gamma}{\sqrt{N_0T/2}}\right) - \Phi\left(\frac{-\gamma}{\sqrt{N_0T/2}}\right) = 2\Phi\left(\frac{\gamma}{\sqrt{N_0T/2}}\right) - 1$$

Above are probs of being correct. They should be equal for minimax threshold.

$$2\Phi\left(\frac{\gamma}{\sqrt{N_0T/2}}\right) - 1 = Q\left(\frac{\gamma - \sqrt{2}AT/2}{\sqrt{N_0T/2}}\right)$$

This would need a numerical solution.

6.16 Suppose the density for the Gaussian random vector $\mathbf{Z} = (Z_1, Z_2)$ is given by

$$f_{Z_1,Z_2}(x, y) = \exp\{-[(x-\mu)^2 + (y-\nu)^2]/2\sigma^2\}/2\pi\sigma^2,$$
$$-\infty < x < \infty, \quad -\infty < y < \infty.$$

Consider an infinite line in two-dimensional space that is distance $\varepsilon$ from the point $(\mu, \nu)$ as illustrated in Figure 6-52. The set $\Gamma$ is defined to be the set of all points that are on the same side of the line as the point $(\mu, \nu)$. Find the probability that the random vector $\mathbf{Z}$ is in the set $\Gamma$. Notice that the probability in question does not change if the line is rotated about the point $(\mu, \nu)$, because the given Gaussian density is invariant with respect to such a rotation. (It has circular symmetry about the point $(\mu, \nu)$.)

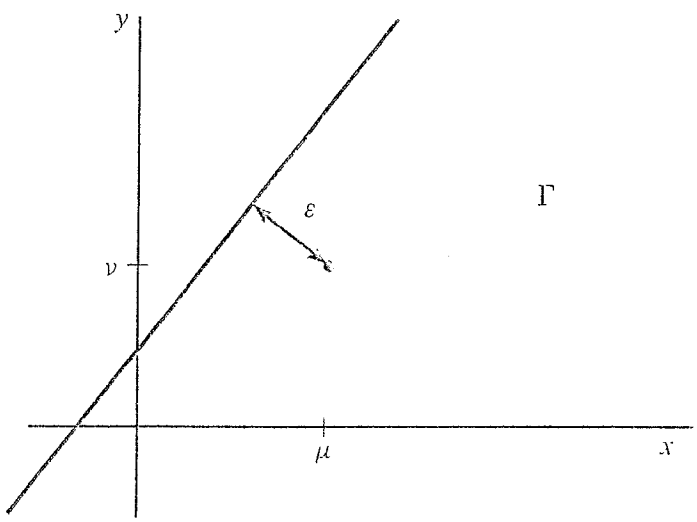



**Figure 6-52**: Illustration for Problem 6.16.

MBP 6.16

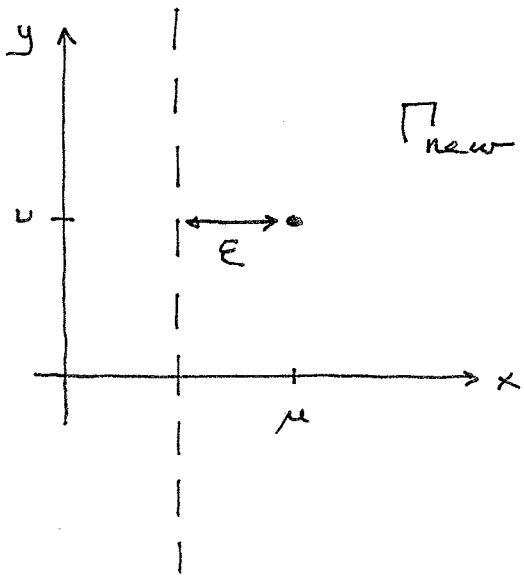
$Z = (Z_1, Z_2)$ Gaussian with pdf

$$f_{Z_1,Z_2}(x,y) = \frac{1}{2\pi\sigma^2} e^{-[(x-\mu)^2 + (y-\nu)^2]/2\sigma^2}$$

$$P(Z \in \Gamma) = \iint_{\Gamma} f_{Z_1,Z_2}(x,y)\, dx\, dy$$

The rotational symm. of $f_{Z_1,Z_2}(x,y)$ implies that we may rotate the line about the point $(\mu,\nu)$ and the prob. would be the same.

Simplest thing to do is rotate until the line is vertical or horizontal. For vertical:




$$P(Z \in \Gamma) = P(Z \in \Gamma_{new}) = P(Z_1 > \mu - \varepsilon)$$

$$= P\left(\frac{Z_1 - \mu}{\sigma} > \frac{\mu - \varepsilon - \mu}{\sigma}\right) = Q(-\varepsilon/\sigma)$$

$$= \Phi(\varepsilon/\sigma).$$