5.6 The received signal in a binary baseband data communications system is given by $Y(t) = s_i(t) + X(t)$, where $X(t)$ is a stationary Gaussian random process with mean 1 and autocovariance function given by

$$C_X(u) = 4\exp(-|u|).$$

The signals $s_i(t)$ are such that $s_0(t) = 3$ for $0 < t < T$ and $s_1(t) = 1$ for $0 < t < T$; for t not in the interval $(0, T)$, $s_0(t) = s_1(t) = 0$. The receiver samples the received signal at times $T/4$, $T/2$, and $3T/4$ and sums the sample values; that is, it forms the statistic Z, which is the random variable given by

$$Z = Y(T/4) + Y(T/2) + Y(3T/4).$$

(a) First suppose the threshold is 6. That is, if $Z > 6$, the receiver decides that 0 was sent; however, if $Z < 6$, it decides that 1 was sent. Find the probability of error when 0 is transmitted and the probability of error when 1 is transmitted. Express these two probabilities in terms of the function Q. For one of these, you should be able to obtain a numerical value without the aid of a table or calculator.

(b) Is 6 the optimum threshold in the minimax sense? If not, what is the optimum minimax threshold? (Be careful—the noise has a nonzero mean.)

(c) Check your answer to part **(b)** by comparing the two error probabilities. For your value of the threshold, is the probability of error when 0 is sent equal to the probability of error when 1 is sent? Should these two probabilities be equal? Explain why or why not.

MBP Problem 5.6

$$Y(t) = s_i(t) + X(t)$$

$X(t)$ → Gaussian
$EX(t) = 1$
$C_X(\tau) = 4e^{-|\tau|}$

$s_0(t) = 3\,p_T(t)$
$s_1(t) = p_T(t)$

Receiver samples $Y(t)$ to form decision statistic

$$Z = Y(T/4) + Y(T/2) + Y(3T/4)$$

(a) Suppose the test is

$$Z \underset{<}{\overset{>}{}} 6 \quad \begin{array}{l} \text{decide } s_0 \\ \text{decide } s_1 \end{array}$$

Find $P_{e,0}$ and $P_{e,1}$.

First must find distribution of Z under the two hypotheses.

Under H_0

Signal part of Z is $s_0(T/4) + s_0(T/2) + s_0(3T/4)$
$= 9$

Under H_1

Signal part of Z is $s_1(T/4) + s_1(T/2) + s_1(3T/4)$
$= 3$

Noise part of Z is same under H_0 and H_1. It is

$$X(T/4) + X(T/2) + X(3T/4) \triangleq \tilde{X}$$

$\tilde{X}$ is Gaussian.

$$E\tilde{X} = 3$$

$$E\tilde{X}^2 = E\left\{\left(X(T/4) + X(T/2) + X(3T/4)\right)^2\right\}$$

$$= E\{X^2(T/4)\} + E\{X^2(T/2)\} + E\{X^2(3T/4)\}$$

$$+ 2E\{X(T/4)X(T/2)\} + 2E\{X(T/4)X(3T/4)\}$$

$$+ 2E\{X(T/2)X(3T/4)\}$$

$$= 3R_x(0) + 2R_x(T/4) + 2R_x(T/2) + 2R_x(T/4)$$

$$= 3R_x(0) + 4R_x(T/4) + 2R_x(T/2)$$

$$C_x(\tau) = R_x(\tau) - \left(EX(t)\right)^2 \Longrightarrow R_x(\tau) = 1 + C_x(\tau)$$

$$= R_x(\tau) - 1$$

$$R_x(0) = 1 + C_x(0) = 5$$

$$R_x(T/4) = 1 + 4e^{-T/4}$$

$$R_x(T/2) = 1 + 4e^{-T/2}$$

$$Var(\tilde{X}) = E(\tilde{X}^2) - (E\tilde{X})^2$$

$$= 15 + 4 + 16e^{-T/4} + 2 + 8e^{-T/2} - 9$$

$$= 12 + 16e^{-T/4} + 8e^{-T/2}$$

<u>Under H_0</u>

$$Z = 9 + \tilde{X} \sim N\left(12, 12 + 16e^{-T/4} + 8e^{-T/2}\right)$$

<u>Under H_1</u>

$$Z = 3 + \tilde{X} \sim N\left(6, 12 + 16e^{-T/4} + 8e^{-T/2}\right).$$

$$P_{e,0} = P(Z < 6 \mid H_0)$$

$$= P\left(\frac{Z-12}{\sigma} < \frac{6-12}{\sigma} \,\middle|\, H_0\right) \qquad \sigma = 2\sqrt{3 + 4e^{-T/4} + 2e^{-T/2}}$$

$$= \Phi(-6/\sigma) = Q(6/\sigma)$$

$$P_{e,1} = P(Z > 6 \mid H_1)$$

$$= P\left(\frac{Z-6}{\sigma} > \frac{6-6}{\sigma} \,\middle|\, H_1\right)$$

$$= Q(0) = 1/2$$

(b) No because $P_{e,0} \neq P_{e,1}$ for threshold equal to 6. If the noise had zero mean, 6 would have been minimax threshold.

To find minimax threshold we need to find average of means of Z under the two hypotheses

$$\gamma_m = \frac{E(Z \mid H_0) + E(Z \mid H_1)}{2} = \frac{12+6}{2} = 9$$

(c) with $\gamma = 9$

$$P_{e,0} = P\left(\frac{Z-12}{\sigma} < \frac{9-12}{\sigma} \,\middle|\, H_0\right) = Q(3/\sigma)$$

$$P_{e,1} = P\left(\frac{Z-6}{\sigma} > \frac{9-6}{\sigma} \,\middle|\, H_1\right) = Q(3/\sigma)$$

$\Longrightarrow$ $P_{e,0} = P_{e,1}$ as the must for the minimax soln.

5.12 Find the decision regions that give the minimum average probability of error for the probabilities π_0 and π_1 if f_0 and f_1 are zero-mean Gaussian densities with *unequal* variances σ_0^2 and σ_1^2. Express Γ_0 and Γ_1 as unions of intervals.

MBP Problem 5.12

$$f_0(z) = \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-z^2/2\sigma_0^2} \qquad f_1(z) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-z^2/2\sigma_1^2}$$

$$\pi_0 \qquad\qquad \pi_1$$

$$\Gamma_1 = \{ z : \pi_1 f_1(z) \geq \pi_0 f_0(z) \}$$

$$\frac{\pi_1 e^{-z^2/2\sigma_1^2}}{\sigma_1} \geq \frac{\pi_0 e^{-z^2/2\sigma_0^2}}{\sigma_0} \iff \frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1} \geq \exp\left\{ -\frac{1}{2}\left(\frac{z^2}{\sigma_0^2} - \frac{z^2}{\sigma_1^2} \right) \right\}$$

$$\iff \ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1} \right) \geq -\frac{1}{2} z^2 \left(\frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2 \sigma_1^2} \right) \qquad (*)$$

$$\iff -2\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1} \right) \leq z^2 \left(\frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2 \sigma_1^2} \right)$$

<u>Case</u> $\sigma_1 > \sigma_0$ $(*)$ becomes

$$z^2 \geq 2 \frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 - \sigma_0^2} \ln\left(\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0} \right)$$

<u>Subcase</u> $\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0} \leq 1 \implies \ln\left(\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0} \right) \leq 0$

and condition $(*)$ defining Γ_1 is equiv. to $z^2 \geq 0$

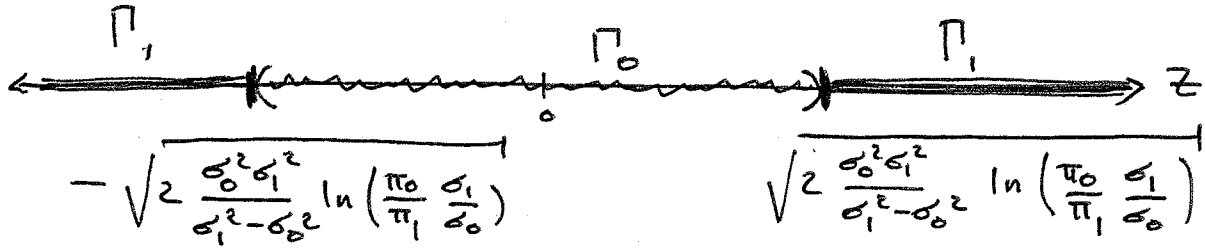
$$\implies \Gamma_1 = (-\infty, \infty), \quad \Gamma_0 = \phi$$

<u>Subcase</u> $\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0} > 1$

$$(*) \iff z \geq \sqrt{2 \frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 - \sigma_0^2} \ln\left(\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0} \right)}$$

or

$$z \leq -\sqrt{2 \frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 - \sigma_0^2} \ln\left(\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0} \right)}$$



<u>Case $\sigma_0 > \sigma_1$</u> (*) becomes

$$-2\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right)\left(\frac{\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\right) \geq z^2$$

$$\Longleftrightarrow$$

$$z^2 \leq 2\frac{\sigma_0^2\sigma_1^2}{\sigma_0^2-\sigma_1^2}\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right)$$

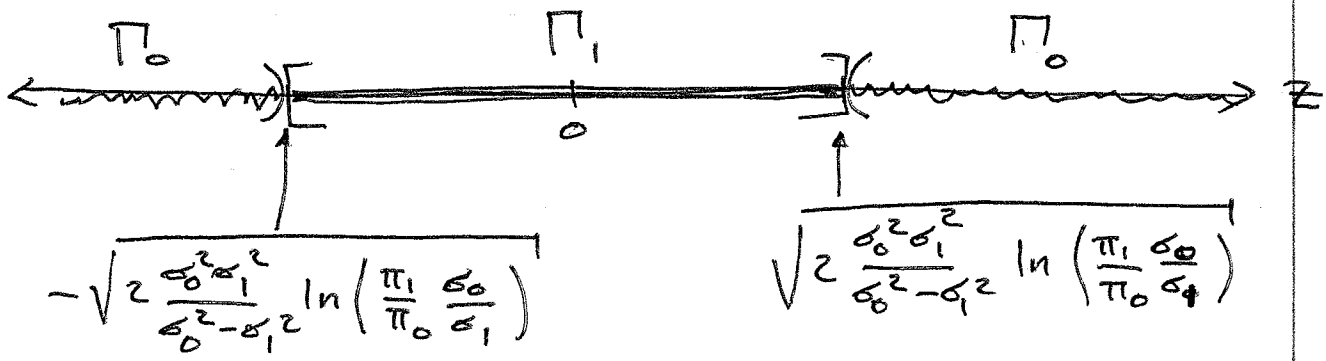
<u>Subcase</u> $\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1} \leq 1 \Longleftrightarrow \ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right) < 0$

and condition (*) defining Γ_1 is equiv. to $z^2 \leq$ a neg. number

$$\Longrightarrow \Gamma_1 = \phi, \quad \Gamma_0 = (-\infty, \infty)$$

<u>Subcase</u> $\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1} \geq 1 \Longleftrightarrow \ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right) \geq 0$

$$(*) \Longleftrightarrow -\sqrt{2\frac{\sigma_0^2\sigma_1^2}{\sigma_0^2-\sigma_1^2}\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right)} \leq z \leq \sqrt{2\frac{\sigma_0^2\sigma_1^2}{\sigma_0^2-\sigma_1^2}\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right)}$$



5.14 **(a)** Using Theorem 5-1 stated in Section 5.4, prove that the following randomized decision rule is a minimax decision rule for the densities f_0 and f_1 of Example 5-9. The decision rule is to observe z and

$$\begin{aligned} \text{if } f_1(z) &> 3f_0(z), \text{ decide 1 was sent;} \\ \text{if } f_1(z) &< 3f_0(z), \text{ decide 0 was sent;} \end{aligned}$$

and

$$\text{if } f_1(z) = 3f_0(z), \text{ make a random choice.}$$

The random choice is as follows: With probability 1/4, decide 0 was sent, and with probability 3/4, decide 1 was sent. In order to apply the theorem you must first show the decision rule is a Bayes decision rule. (What are the values of π_0 and π_1?) You must also prove that the decision rule has the property that $P_{e,0} = P_{e,1}$.

(b) Show that the following choice of Γ_0 and Γ_1 gives a nonrandomized minimax rule for the densities of Example 5-9:

$$\begin{aligned} \Gamma_0 &= [0, 5/4] \cup [2, 3]; \\ \Gamma_1 &= (5/4, 2). \end{aligned}$$

Is this decision rule a likelihood ratio test?

MBP Problem 5.14

Thm 5-1: If a decision rule is a Bayes rule and also satisfies $P_{e,0} = P_{e,1}$ then it is a minimax decision rule.

From Example 5-9: f_0 unif on $(0,3)$

f_1 unif on $(1,2)$

We are to consider the decision rule

$$f_1(z) \begin{cases} > & \text{decide 1 was sent} \\ < \; 3 f_0(z) & \text{decide 0 was sent} \\ = & \text{decide 0 with prob. } 1/4 \text{ and 1 with prob. } 3/4 \end{cases}$$

(a)

Bayes rules are of the form

$$\Gamma_1 = \{ z \in \Gamma : \pi_1 f_1(z) > \pi_0 f_0(z) \}$$

$$\Gamma_0 = \{ z \in \Gamma : \pi_1 f_1(z) < \pi_0 f_0(z) \}$$

$$\Gamma_{\text{don't care}} = \{ z \in \Gamma : \pi_1 f_1(z) = \pi_0 f_0(z) \}$$

Thus must show a solution to

$$\frac{\pi_0}{\pi_1} = 3 \qquad \pi_0 + \pi_1 = 1 \qquad \pi_0, \pi_1 \geq 0$$

$$\Longrightarrow \pi_0 = 3\pi_1 \Longrightarrow 4\pi_1 = 1 \Longrightarrow \pi_1 = \frac{1}{4}, \; \pi_0 = \frac{3}{4}$$

So this is a Bayes rule for these priors.

Can we find the regions Γ_1, Γ_0 and $\Gamma_{\text{don't care}}$?

If instead define $\Gamma_0' = \Gamma_0 \cup \Gamma_{\text{don't care}}$

$$P_{e,0} = P(Z \in \Gamma_1 \mid H_0) = 0 \qquad (\Gamma_1 = \phi)$$

$$P_{e,1} = P(Z \in \Gamma_0' \mid H_1)$$
$$= P(0 < Z < 3 \mid H_1) = 1$$

$$\Longrightarrow \overline{P}_e^* = \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{1}{4}$$

Note that we get same Bayes error in either of these two extreme cases, while the conditional error probs. behave very differently.

Let $\theta = 1/4$ be the prob. with which we decide H_0 when $Z \in \Gamma_{\text{don't care}}$.

Then

$$P_{e,0} = P(Z \in \Gamma_1 \mid H_0) + (1-\theta) P(Z \in \Gamma_{\text{don't care}} \mid H_0)$$
$$= P(Z \in \phi \mid H_0) + (1-\theta) P(1 \le Z \le 2 \mid H_0)$$
$$= 0 + (1-\theta)\frac{1}{3} = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$$

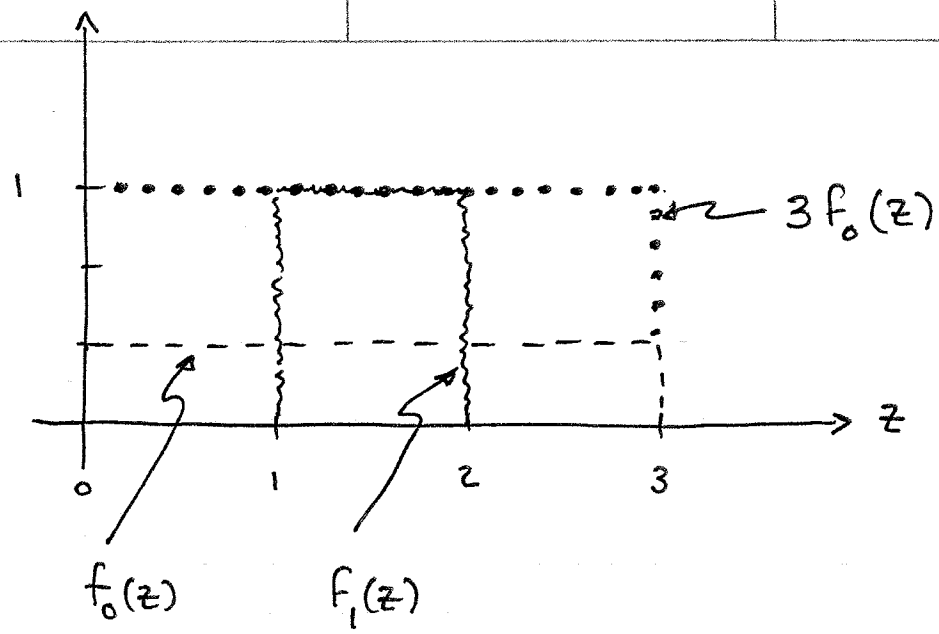
$$P_{e,1} = P(Z \in \Gamma_0 \mid H_1) + \theta P(Z \in \Gamma_{\text{don't care}} \mid H_1)$$

$$= P(0 < Z < 1 \mid H_1) + P(2 < Z < 3 \mid H_1)$$
$$+ \theta P(1 \le Z \le 2 \mid H_1)$$

$$= 0 + 0 + \theta \cdot 1 = \theta = \frac{1}{4}$$

$\therefore P_{e,0} = P_{e,1} = \frac{1}{4}$ hence minimax

Note that get the same $\overline{P}_e^*$ for this randomized rule.



Can restrict consideration to points z s.t $0<z<3$ since the interval encompasses the support of both pdfs.

Clearly

$$\Gamma_1 = \phi \qquad \Gamma_0 = (0,1) \cup (2,3) \qquad \Gamma_{\text{don't care}} = [1,2]$$

The usual Bayes error for the priors $\pi_0 = 3/4$, $\pi_1 = 1/4$ is

$$\overline{P}_e^* = \pi_0 P_{e,0} + \pi_1 P_{e,1}$$

but we do have to decide where we put $\Gamma_{\text{dont care}}$.

If define $\Gamma_1' = \Gamma_1 \cup \Gamma_{\text{dont care}}$

$$P_{e,0} = P(Z \in \Gamma_1' \mid H_0) = P(1 \le Z \le 2 \mid H_0) = 1/3$$

$$P_{e,1} = P(Z \in \Gamma_0 \mid H_1)$$

$$= P(0 < Z < 1 \mid H_1) + P(2 < Z < 3 \mid H_1) = 0$$

$$\Rightarrow \overline{P}_e^* = \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot 0 = \frac{1}{4}$$

(b) Consider the nonrandomized decision rule

$$\Gamma_0 = [0, 5/4] \cup [2,3]$$

$$\Gamma_1 = (\tfrac{5}{4}, 2)$$

$$P_{e,0} = P(Z \in \Gamma_1 \mid H_0) = P(\tfrac{5}{4} < Z < 2 \mid H_0)$$

$$= \left(2 - \tfrac{5}{4}\right)\tfrac{1}{3} = \tfrac{3}{4} \cdot \tfrac{1}{3} = \tfrac{1}{4}$$

$$P_{e,1} = P(Z \in \Gamma_0 \mid H_1) = P(0 \le Z \le \tfrac{5}{4} \mid H_1) + P(2 \le Z \le 3 \mid H_1)$$

$$= \left(\tfrac{5}{4} - 1\right) \cdot 1 + 0 = \tfrac{1}{4}$$

Clearly an equalizer rule. Also

$$\overline{P_e^*} = \pi_0 P_{e,0} + \pi_1 P_{e,1} = \tfrac{1}{4}$$

So also a Bayes rule. Hence is minimax.

This is not an LRT because we <u>cannot</u> find an η st

$$\Gamma_0 = \left\{ z : \frac{f_1(z)}{f_0(z)} < \eta \right\} \text{ and } \Gamma_1 = \left\{ z : \frac{f_1(z)}{f_0(z)} > \eta \right\}.$$

Note:

$$\frac{f_1(z)}{f_0(z)} = \begin{cases} 0 & 0 < z < 1 \\ 1/3 & 1 < z < 2 \\ 0 & 2 < z < 3 \end{cases}$$

5.15 A binary baseband data transmission system uses the antipodal signal set defined by

$$s_0(t) = \begin{cases} 2At/T, & 0 \le t < T/2, \\ A(2t - T)/T, & T/2 \le t < T, \\ 0, & \text{otherwise.} \end{cases}$$

The channel is an additive white Gaussian noise channel with noise spectral density $N_0/2$. The *minimax* criterion is to be used.

(a) What is the minimum probability of error for this system? Give your answer in terms of A, T, N_0, and the function Q.

(b) Give the impulse response of the filter that achieves the minimum error probability (i.e., the matched filter). Simplify your answer as much as possible.

(c) Give the optimum sampling time and optimum (minimax) threshold for the receiver that uses the filter of part **(b)**.

(d) Find the variance σ^2 of the output process when the input to the matched filter is a white-noise process with spectral density $N_0/2$.

(e) Suppose that the filter in the receiver is *not* the filter of part **(b)**, but is instead a filter with impulse response

$$h(t) = p_T(t).$$

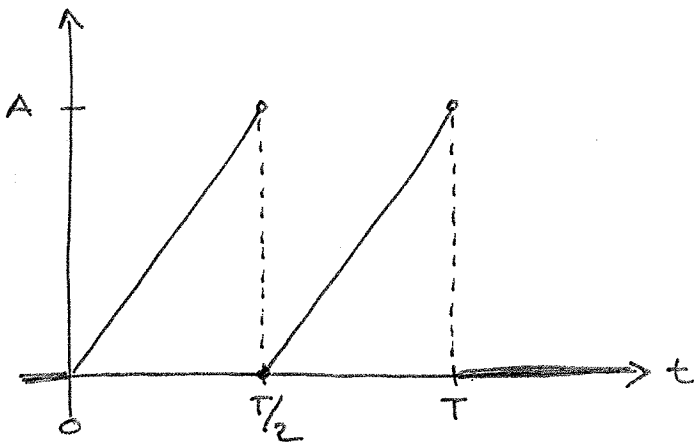
Give an expression for the output $\hat{s}_0(t)$ of this filter when the input is $s_0(t)$. Give the value of $\hat{s}_0(t)$ for *all* t in the range $-\infty < t < \infty$.

(f) For the filter of part **(e)**, find the maximum value of $\hat{s}(t)$, where the maximization is over all t in the range $-\infty < t < \infty$. (Note: This part can be solved independently of the solution to part **(e)**.)

MBP 5.15

Antipodal signalling $s_1(t) = -s_0(t)$ where

$$s_0(t) = \begin{cases} 2At/T & 0 \le t < T/2 \\ A(2t-T)/T & T/2 \le t < T \\ 0 & \text{else} \end{cases}$$



AWGN $N_0/2$

Use minimax criterion

For antipodal signals and zero mean additive Gaussian noise the minimax threshold is always

$$\gamma_m = 0$$

The minimum or best minimax error prob. occurs when the LTI filter is a matched filter

$$h(t) = c\left[s_0(T_0-t) - s_1(T_0-t)\right]$$

(b) $$= 2c\, s_0(T_0-t)$$

any sampling time T_0 will work and the choice $T_0 = T$ is the smallest which results in a causal M.F.

From class or MBP formulas the max. SNR is

$$SNR_{max} = \frac{2\,\|s_0\|}{\sqrt{2N_0}}$$

$$\|s_0\|^2 = \int_0^T s_0^2(t)\,dt = 2\int_0^{T/2} s_0^2(t)\,dt = 2\cdot\frac{4A^2}{T^2}\int_0^{T/2} t^2\,dt$$

$$\|s_0\|^2 = \frac{8A^2}{T^2}\,\frac{1}{3}\left(\frac{T}{2}\right)^3 = \frac{A^2T}{3} \Longrightarrow \|s_0\| = \frac{A\sqrt{T}}{\sqrt{3}}$$

$$SNR_{max} = \frac{2\cdot A\sqrt{T}}{\sqrt{3}}\,\frac{1}{\sqrt{2N_0}} = 2A\sqrt{\frac{T}{6N_0}}$$

$$P^*_{e,m} = Q(SNR_{max}) = Q\left(2A\sqrt{\frac{T}{6N_0}}\right).$$

(a) $$= Q\left(\sqrt{\frac{2A^2T}{3N_0}}\right) \longrightarrow = \min_h P^*_{e,m}$$

(c) As mentioned above $\gamma_m = 0$ and $T_0 = T$.

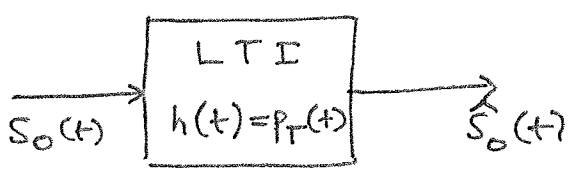
(d) Find variance of output process when input to filter is white Gaussian noise with $N_0/2$.

As shown in class and in MBP

$$\sigma^2 = \frac{N_0}{2}\|h\|^2 = \frac{N_0}{2}\,4c^2\|s_0\|^2$$

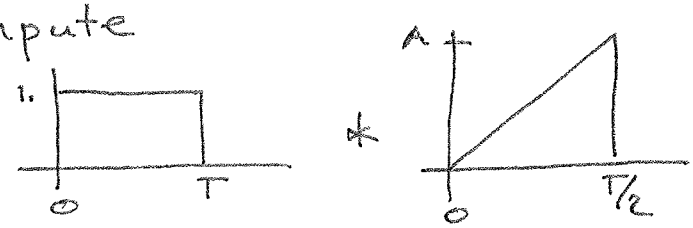
$$= 2N_0c^2A^2T/3$$

(e) Instead of using a MF let $h(t) = p_T(t)$.

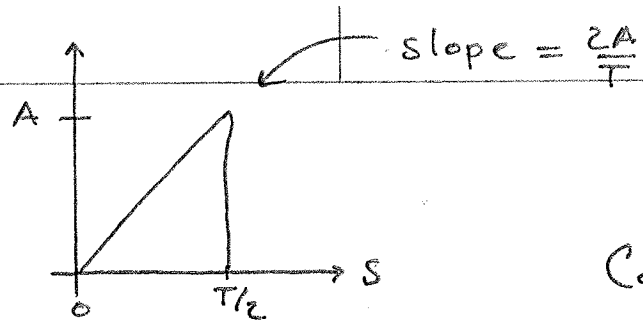


So want to compute the convolution $\hat{s}_0(t) = p_T * s_0(t)$.

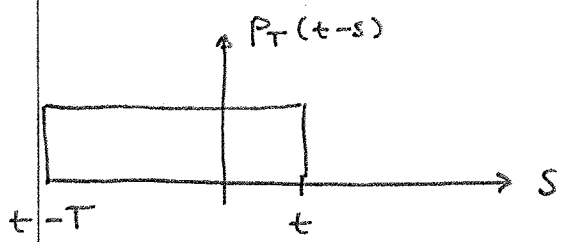
Using linearity and time invariance we first compute



Then can put pieces together to get final answer.



Call this partial result $\hat{\hat{S}}_0(t)$



From picture have cases.

Case: $t < 0 \Longrightarrow \hat{\hat{S}}_0(t) = 0$

Case: $0 < t < T/2 \Longrightarrow \hat{\hat{S}}_0(t) = \frac{1}{2} t \cdot \frac{2A}{T} t = \frac{A}{T} t^2$

Case: $T/2 < t < T \Longrightarrow \hat{\hat{S}}_0(t) = \frac{1}{2} \frac{T}{2} \cdot A = \frac{AT}{4}$

Case: $T < t < \frac{3T}{2} \Longrightarrow \hat{\hat{S}}_0(t) = \frac{AT}{4} - \frac{1}{2}(t-T) \frac{2A}{T} (t-T)$

$$= \frac{AT}{4} - \frac{A}{T}(t-T)^2$$

Case: $\frac{3T}{2} < t \Longrightarrow \hat{\hat{S}}_0(t) = 0.$

Then from linearity and time invariance

$$\hat{S}_0(t) = \hat{\hat{S}}_0(t) + \hat{\hat{S}}_0(t - T/2).$$

$$\hat{S}_o(t) = \begin{cases} 0 & t < 0 \\ \frac{A}{T} t^2 & 0 < t < T/2 \\ \frac{AT}{4} + \frac{A}{T}(t - T/2)^2 & \frac{T}{2} < t < T \\ \frac{AT}{2} - \frac{A}{T}(t-T)^2 & T < t < \frac{3T}{2} \\ \frac{AT}{4} - \frac{A}{T}\left(t - \frac{3T}{2}\right)^2 & \frac{3T}{2} < t < 2T \\ 0 & t > 2T \end{cases}$$

(f) Find max value of $\hat{S}_o(t)$.

From graphical picture of convolution and the positivity of s_o and p_T the max must occur at $t = T$

$$\max_t \hat{S}_o(t) = \frac{AT}{2}$$

5.18 Consider the communications system shown in Figure 5-4. The noise process $X(t)$ is a white Gaussian random process with spectral density $N_0/2$, and the signals $s_0(t)$ and $s_1(t)$ are given by

$$s_i(t) = (-1)^i A\, p_T(t)$$

for $i = 0$ and $i = 1$. The threshold is $\gamma = 0$ and the sampling time is $T_0 = \alpha T$ for $0 < \alpha < 2$. Investigate the effects of the sampling time by finding the error probabilities $P_{e,0}$ and $P_{e,1}$ in the following two cases:

(a) The filter is a linear time-invariant filter that is matched to the signals; that is, the impulse response is

$$h(\lambda) = s_0(T - \lambda) - s_1(T - \lambda).$$

Give $P_{e,0}$ and $P_{e,1}$ in terms of α, A, T, and N_0.

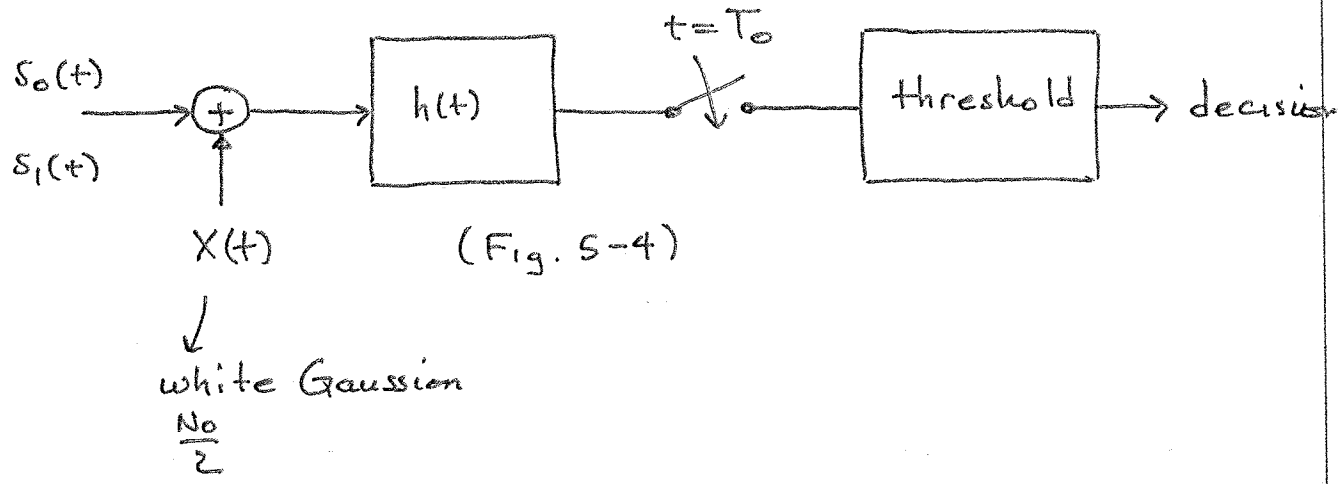
(b) The filter is an integrate-and-dump filter with output given by

$$Z(T_0) = \int_0^{T_0} Y(t)\, \mathrm{d}t.$$

Give expressions for $P_{e,0}$ and $P_{e,1}$ in terms of α, A, T, and N_0.

(c) Express your answers to **(a)** and **(b)** in terms of α, $\mathcal{E}$, and N_0 (where $\mathcal{E}$ is the energy per pulse), and compare them.

MBP Problem 5.18



(Fig. 5-4)

$$s_0(t) = A p_T(t) \qquad s_1(t) = -A p_T(t) \qquad \text{antipodal}$$

$$\gamma = 0$$

$$T_0 = \alpha T \qquad 0 < \alpha < 2$$

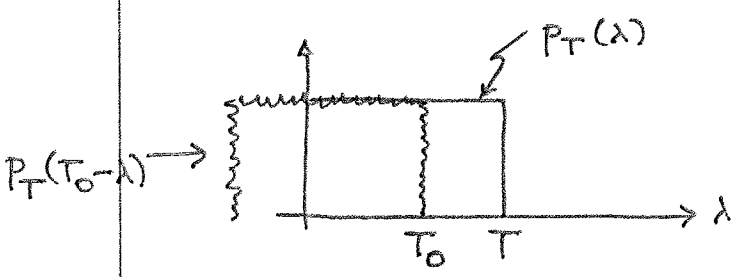
Find $P_{e,0}$ and $P_{e,1}$ assuming:

(a) $h(t) = s_0(T-t) - s_1(T-t)$

$$= 2A p_T(T-t) = 2A p_T(t)$$

Following approach taken in class

$$\mu_0 = s_0 * h(T_0) = \int s_0(\lambda)\, h(T_0-\lambda)\, d\lambda = 2A^2 \int p_T(\lambda)\, p_T(T_0-\lambda)\, d\lambda$$



$$\mu_0 = 2A^2 \int_0^{T_0} d\lambda \qquad 0 < T_0 < T$$

$$= 2A^2 T_0 = 2A^2 \alpha T \qquad 0 < \alpha \le 1$$

$$\mu_0 = 2A^2 \int_{T_0-T}^{T} d\lambda = 2A^2 (2T - T_0) \qquad T < T_0 < 2T$$

$$= 2A^2 (2-\alpha) T \qquad 1 \le \alpha < 2$$

$$\therefore \; -\mu_1 = \mu_0 = \begin{cases} 2A^2 \alpha T & 0 < \alpha \le 1 \\ 2A^2 (2-\alpha) T & 1 \le \alpha < 2 \end{cases}$$

Note that the minimax threshold $= 0$, which agrees with choice made in this problem.

For AWGN

$$\sigma^2 = \frac{N_0}{2}\int h^2(t)\,dt = \frac{N_0}{2}4A^2\int_0^T dt = 2A^2N_0T$$

$$\therefore\ SNR = \frac{\mu_0-\mu_1}{2\sigma} = \frac{4A^2\alpha T}{2\sqrt{2A^2N_0T}} \qquad 0<\alpha\le 1$$

$$= \frac{4A^2(2-\alpha)T}{2\sqrt{2A^2N_0T}} \qquad 1\le\alpha<2$$

Simplifying

$$SNR = \begin{cases} A\sqrt{\frac{2T}{N_0}}\,\alpha & 0<\alpha\le 1 \\ A\sqrt{\frac{2T}{N_0}}\,(2-\alpha) & 1\le\alpha<2 \end{cases}$$

Since minimax $P_{e,0} = P_{e,1} = Q(SNR)$.

(b) Integrate and dump with

$$Z(T_0) = \int_0^{T_0} Y(t)\,dt = \int_0^{T_0} s_i(t)\,dt + \int_0^{T_0} X(t)\,dt$$

For the signal part consider $i=0$

$$\int_0^{T_0} s_0(t)\,dt = A\int_0^{T_0} p_T(t)\,dt = \begin{cases} AT_0 & 0<T_0\le T \\ AT & T<T_0<2T \end{cases}$$

$$= \begin{cases} A\alpha T & 0<\alpha\le 1 \\ AT & 1\le\alpha<2. \end{cases}$$

Also $\int_0^{T_0} s_1(t)\,dt = -\int_0^{T_0} s_0(t)\,dt.$

For the noise part note that $\int_0^{T_0} X(t)\,dt$ is a Gaussian r.v. with zero mean and variance

$$\sigma^2 = E\left\{\left(\int_0^{T_0} X(t)\,dt\right)^2\right\} = \int_0^{T_0}\int_0^{T_0} E\{X(t)X(\lambda)\}\,dt\,d\lambda$$

$$= \frac{N_0}{2}\int_0^{T_0}\int_0^{T_0} \delta(t-\lambda)\,dt\,d\lambda = \frac{N_0}{2}\int_0^{T_0} d\lambda = \frac{N_0 T_0}{2}$$

$$= \frac{N_0}{2} T\alpha$$

So decision problem for this statistic can be written

$$Z(T_0) \sim \begin{cases} N\left(\mu(\alpha), \frac{N_0 T}{2}\alpha\right) & \text{under } H_0 \\ N\left(-\mu(\alpha), \frac{N_0 T}{2}\alpha\right) & \text{" } H_1 \end{cases}$$

where

$$\mu(\alpha) = \begin{cases} AT\alpha & 0 < \alpha \le 1 \\ AT & 1 < \alpha < 2 \end{cases}$$

$\gamma = 0$ is minimax threshold as before. The SNR depends on α

$$SNR = \frac{\mu_0 - \mu_1}{2\sigma} = \frac{\mu_0}{\sigma}$$

$$= \begin{cases} \dfrac{AT\alpha}{\sqrt{N_0 T\alpha/2}} & 0 < \alpha \le 1 \\ \dfrac{AT}{\sqrt{N_0 T\alpha/2}} & 1 < \alpha < 2 \end{cases} = \begin{cases} A\sqrt{\dfrac{2T}{N_0}\alpha} & 0 < \alpha \le 1 \\ A\sqrt{\dfrac{2T}{N_0\alpha}} & 1 < \alpha < 2 \end{cases}$$

$P_{e,0} = P_{e,1} = Q(SNR)$.

(c) Express prev. answers in terms of $\mathcal{E}$ (energy per pulse).

$$\mathcal{E} = \int_0^T A^2\,dt = A^2 T$$

$$SNR_a = \begin{cases} \sqrt{\frac{2\mathcal{E}}{N_0}}\ \alpha & 0 < \alpha \le 1 \\ \sqrt{\frac{2\mathcal{E}}{N_0}}\ (2-\alpha) & 1 < \alpha < 2 \end{cases}$$

$$SNR_b = \begin{cases} \sqrt{\frac{2\mathcal{E}}{N_0}}\ \sqrt{\alpha} & 0 < \alpha \le 1 \\ \sqrt{\frac{2\mathcal{E}}{N_0}}\ \frac{1}{\sqrt{\alpha}} & 1 < \alpha < 2 \end{cases}$$