5.6 The received signal in a binary baseband data communications system is given by
\( Y(t) = s_i(t) + X(t) \), where \( X(t) \) is a stationary Gaussian random process with mean
1 and autocovariance function given by
\[ C_X(u) = 4 \exp(-|u|). \]

The signals \( s_i(t) \) are such that \( s_0(t) = 3 \) for \( 0 < t < T \) and \( s_1(t) = 1 \) for \( 0 < t < T \);
for \( t \) not in the interval \((0, T)\), \( s_0(t) = s_1(t) = 0 \). The receiver samples the received
signal at times \( T/4, T/2, \) and \( 3T/4 \) and sums the sample values; that is, it forms the
statistic \( Z \), which is the random variable given by
\[ Z = Y(T/4) + Y(T/2) + Y(3T/4). \]

(a) First suppose the threshold is 6. That is, if \( Z > 6 \), the receiver decides that 0 was
sent; however, if \( Z < 6 \), it decides that 1 was sent. Find the probability of error
when 0 is transmitted and the probability of error when 1 is transmitted. Express
these two probabilities in terms of the function \( Q \). For one of these, you should
be able to obtain a numerical value without the aid of a table or calculator.

(b) Is 6 the optimum threshold in the minimax sense? If not, what is the optimum
minimax threshold? (Be careful—the noise has a nonzero mean.)

(c) Check your answer to part (b) by comparing the two error probabilities. For
your value of the threshold, is the probability of error when 0 is sent equal to
the probability of error when 1 is sent? Should these two probabilities be equal?
Explain why or why not.
MBP Problem 5.6

\[ Y(t) = \xi(t) + X(t) \]

\[ \xi \sim \text{Gaussian} \]
\[ \mathbb{E}X(t) = 1 \]
\[ \mathcal{C}_X(\tau) = 4e^{-|\tau|} \]
\[ s_0(t) = 3p(t) \]
\[ s_1(t) = p(t) \]

Receiver samples \( Y(t) \) to form decision statistic

\[ Z = Y(T/4) + Y(T/2) + Y(3T/4) \]

(a) Suppose the test is

\[ Z > 6 \quad \text{decide } s_0 \]
\[ Z < 6 \quad \text{decide } s_1 \]

Find \( P_{e_0} \) and \( P_{e_1} \).

First must find distribution of \( Z \) under the two hypotheses.

Under \( H_0 \)

Signal part of \( Z \) is

\[ s_0(T/4) + s_0(T/2) + s_0(3T/4) = 9 \]

Under \( H_1 \)

Signal part of \( Z \) is

\[ s_1(T/4) + s_1(T/2) + s_1(3T/4) = 3 \]
Noise part of $X$ is same under $H_0$ and $H_1$. It is
\[ X(T/4) + X(T/2) + X(3T/4) \equiv \tilde{X} \]
\(\tilde{X}\) is Gaussian.
\[
E\tilde{X} = 3
\]
\[ E\tilde{X}^2 = E\left\{ (X(T/4) + X(T/2) + X(3T/4))^2 \right\} \]
\[ = E\{X^2(T/4)\} + E\{X^2(T/2)\} + E\{X^2(3T/4)\} + 2E\{X(T/4)X(T/2)\} + 2E\{X(T/4)X(3T/4)\} + 2E\{X(T/2)X(3T/4)\} \]
\[ = 3R_x(0) + 2R_x(T/4) + 2R_x(T/2) + 2R_x(3T/4) \]
\[ = 3R_x(0) + 4R_x(T/4) + 2R_x(T/2) \]
\[ C_x(\tau) = R_x(\tau) - (E\tilde{X})^2 \]
\[ \Rightarrow R_x(\tau) = 1 + C_x(\tau) \]
\[ R_x(0) = 1 + C_x(0) = 5 \]
\[ R_x(T/4) = 1 + 4e^{-T/4} \]
\[ R_x(T/2) = 1 + 4e^{-T/2} \]
\[ \text{Var} (\tilde{X}) = E(\tilde{X}^2) - (E\tilde{X})^2 \]
\[ = 15 + 4 + 16e^{-T/4} + 2 + 8e^{-T/2} - 9 \]
\[ = 12 + 16e^{-T/4} + 8e^{-T/2} \]
Under $H_0$

$$Z = 9 + \tilde{X} \sim N\left(12, 12 + 16 e^{-T/4} + 8 e^{-T/2}\right)$$

Under $H_1$

$$Z = 3 + \tilde{X} \sim N\left(6, 12 + 16 e^{-T/4} + 8 e^{-T/2}\right).$$

\[ \Phi(-6/\sigma) = Q(6/\sigma) \]

\[ \Phi(-6/\sigma) = Q(6/\sigma) \]

\[ \Phi(-6/\sigma) = Q(6/\sigma) \]

\[ \Phi(-6/\sigma) = Q(6/\sigma) \]

\[ \Phi(-6/\sigma) = Q(6/\sigma) \]

(b) No because $P_{e_0} \neq P_{e_1}$ for threshold equal to 6. If the noise had zero mean, 6 would have been minimax threshold.

To find minimax threshold we need to find average of means of $Z$ under the two hypotheses

$$\gamma_m = \frac{E(Z|H_0) + E(Z|H_1)}{2} = \frac{12 + 6}{2} = 9$$
(c) With $r = 9$

\[ P_{e_{j0}} = P \left( \frac{z_{-12}}{\sigma} < \frac{9-12}{\sigma} \mid H_0 \right) = Q \left( \frac{3}{\sigma} \right) \]

\[ P_{e_{j1}} = P \left( \frac{z_{-6}}{\sigma} > \frac{9-6}{\sigma} \mid H_1 \right) = Q \left( \frac{3}{\sigma} \right) \]

\[ \Rightarrow P_{e_{j0}} = P_{e_{j1}} \text{ as the must for the minimax soln.} \]
5.12 Find the decision regions that give the minimum average probability of error for the probabilities $\pi_0$ and $\pi_1$ if $f_0$ and $f_1$ are zero-mean Gaussian densities with unequal variances $\sigma_0^2$ and $\sigma_1^2$. Express $\Gamma_0$ and $\Gamma_1$ as unions of intervals.
MHP Problem 5.12

\[ f_0(z) = \frac{1}{\sqrt{2\pi} \sigma_0} e^{-\frac{z^2}{2\sigma_0^2}} \quad f_1(z) = \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{z^2}{2\sigma_1^2}} \]

\[ \Pi_0 \quad \Pi_1 \]

\[ \Pi_1 = \left\{ z : \frac{\Pi_1}{\Pi_0} f_1(z) \geq \frac{\Pi_0}{\Pi_0} f_0(z) \right\} \]

\[ \frac{\Pi_1 e^{-\frac{z^2}{2\sigma_1^2}}}{\sigma_1} \geq \frac{\Pi_0 e^{-\frac{z^2}{2\sigma_0^2}}}{\sigma_0} \iff \frac{\Pi_1}{\Pi_0} \frac{\sigma_0}{\sigma_1} \geq \exp \left\{ -\frac{1}{2} \left( \frac{z^2}{\sigma_0^2} - \frac{z^2}{\sigma_1^2} \right) \right\} \]

\[ \iff \ln \left( \frac{\Pi_1}{\Pi_0} \frac{\sigma_0}{\sigma_1} \right) \geq -\frac{1}{2} z^2 \left( \frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2 \sigma_1^2} \right) \] \hspace{1cm} \(*\)

\[ \iff -2 \ln \left( \frac{\Pi_1}{\Pi_0} \frac{\sigma_0}{\sigma_1} \right) \leq z^2 \left( \frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2 \sigma_1^2} \right) \]

**Case** \( \sigma_1 > \sigma_0 \) \( \bullet \) becomes

\[ z^2 \geq 2 \frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 - \sigma_0^2} \ln \left( \frac{\Pi_0}{\Pi_1} \frac{\sigma_1}{\sigma_0} \right) \]

Subcase \( \frac{\Pi_0}{\Pi_1} \frac{\sigma_1}{\sigma_0} \leq 1 \) \( \iff \ln \left( \frac{\Pi_0}{\Pi_1} \frac{\sigma_1}{\sigma_0} \right) \leq 0 \)

and condition \( \bullet \) defining \( \Pi_1 \) is equiv. to \( z^2 \geq 0 \)

\[ \Rightarrow \Pi_1 = (-\infty, \infty), \quad \Pi_0 = \emptyset \]

Subcase \( \frac{\Pi_0}{\Pi_1} \frac{\sigma_1}{\sigma_0} > 1 \)

\[ \bullet \iff z \geq \sqrt{2 \frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 - \sigma_0^2} \ln \left( \frac{\Pi_0}{\Pi_1} \frac{\sigma_1}{\sigma_0} \right)} \]

\[ \text{OR} \quad z \leq -\sqrt{2 \frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 - \sigma_0^2} \ln \left( \frac{\Pi_0}{\Pi_1} \frac{\sigma_1}{\sigma_0} \right)} \]
Case $\sigma_0 > \sigma_1$ becomes

$$-2 \ln \left( \frac{\pi_1}{\pi_0} \cdot \frac{\sigma_0}{\sigma_1} \right) \left( \frac{\sigma_0^2 \sigma_1^2}{\sigma_0^2 - \sigma_1^2} \right) \geq z^2$$

$$\Rightarrow$$

$$z^2 \leq 2 \frac{\sigma_0^2 \sigma_1^2}{\sigma_0^2 - \sigma_1^2} \ln \left( \frac{\pi_1}{\pi_0} \cdot \frac{\sigma_0}{\sigma_1} \right)$$

Subcase $\frac{\pi_1}{\pi_0} \cdot \frac{\sigma_0}{\sigma_1} < 1 \iff \ln \left( \frac{\pi_1}{\pi_0} \cdot \frac{\sigma_0}{\sigma_1} \right) < 0$

and condition $\bigcirc$ defining $\Pi_1$ is equiv. to $z \leq \text{number}$

$\Rightarrow$ $\Pi_1 = \emptyset$, $\Pi_0 = (-\infty, \infty)$

Subcase $\frac{\pi_1}{\pi_0} \cdot \frac{\sigma_0}{\sigma_1} \geq 1 \iff \ln \left( \frac{\pi_1}{\pi_0} \cdot \frac{\sigma_0}{\sigma_1} \right) \geq 0$

$\bigcirc \quad \Rightarrow \quad -\sqrt{2} \frac{\sigma_0^2 \sigma_1^2}{\sigma_0^2 - \sigma_1^2} \ln \left( \frac{\pi_1}{\pi_0} \cdot \frac{\sigma_0}{\sigma_1} \right) \leq z \leq \sqrt{2} \frac{\sigma_0^2 \sigma_1^2}{\sigma_0^2 - \sigma_1^2} \ln \left( \frac{\pi_1}{\pi_0} \cdot \frac{\sigma_0}{\sigma_1} \right)$
5.14 (a) Using Theorem 5-1 stated in Section 5.4, prove that the following randomized decision rule is a minimax decision rule for the densities $f_0$ and $f_1$ of Example 5-9. The decision rule is to observe $z$ and

- if $f_1(z) > 3f_0(z)$, decide 1 was sent;
- if $f_1(z) < 3f_0(z)$, decide 0 was sent;

and

- if $f_1(z) = 3f_0(z)$, make a random choice.

The random choice is as follows: With probability 1/4, decide 0 was sent, and with probability 3/4, decide 1 was sent. In order to apply the theorem you must first show the decision rule is a Bayes decision rule. (What are the values of $\pi_0$ and $\pi_1$?) You must also prove that the decision rule has the property that $P_{e,0} = P_{e,1}$.

(b) Show that the following choice of $\Gamma_0$ and $\Gamma_1$ gives a nonrandomized minimax rule for the densities of Example 5-9:

- $\Gamma_0 = [0, 5/4] \cup [2, 3]$;
- $\Gamma_1 = (5/4, 2)$.

Is this decision rule a likelihood ratio test?
MBP Problem 5.14

Thm 5-1: If a decision rule is a Bayes rule and also satisfies $P_{x_0} = P_{x_1}$, then it is a minimax decision rule.

From Example 5-9: $f_o$ unif on $(0,3)$

$f_1$ unif on $(1,2)$

We are to consider the decision rule

\[
\begin{align*}
\frac{f_1(z)}{f_o(z)} < 3 \quad &\text{decide 1 was sent} \\
\frac{f_1(z)}{f_o(z)} = 1 \quad &\text{decide 0 with prob. } \frac{1}{4} \\
\frac{f_1(z)}{f_o(z)} > 3 \quad &\text{decide 0 with prob. } \frac{3}{4}
\end{align*}
\]

(a)

Bayes rules are of the form

\[
\begin{align*}
\Gamma_1 &= \left\{ z \in \Gamma : \frac{f_1(z)}{f_o(z)} > 3 \right\} \\
\Gamma_0 &= \left\{ z \in \Gamma : \frac{f_1(z)}{f_o(z)} < 3 \right\} \\
\Gamma_\text{don't care} &= \left\{ z \in \Gamma : \frac{f_1(z)}{f_o(z)} = 3 \right\}
\end{align*}
\]

Thus must show a solution to

\[
\frac{P_o}{P_1} = 3 \quad P_o + P_1 = 1 \quad P_o, P_1 \geq 0
\]

\[
\Rightarrow P_o = 3P_1 \quad \Rightarrow 4P_1 = 1 \quad \Rightarrow P_1 = \frac{1}{4}, P_o = \frac{3}{4}
\]

So this is a Bayes rule for these priors.

Can we find the regions $\Gamma_1$, $\Gamma_0$ and $\Gamma$ don't care?
If instead define $\Gamma'_0 = \Gamma_0 \cup \Gamma_{\text{don't care}}$

\[
P_{e_{\theta}} = P(Z \in \Gamma_1 | H_0) = 0 \quad (\Gamma_1 = \emptyset)
\]

\[
P_{e_{\theta}} = P(Z \in \Gamma'_0 | H_1)
\]

\[
= P(0 < Z < 3 | H_1) = 1
\]

\[\Rightarrow P_{e^*} = \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{1}{4}
\]

Note that we get same Bayes error in either of these two extreme cases, while the conditional error probs. behave very differently.

Let $\Theta = \frac{1}{4}$ be the prob. with which we decide $H_0$ when $Z \in \Gamma_{\text{don't care}}$.

Then

\[
P_{e_{\theta}} = P(Z \in \Gamma_1 | H_0) + (1-\Theta) P(Z \in \Gamma_{\text{don't care}} | H_0)
\]

\[
= P(Z \in \emptyset | H_0) + (1-\Theta) P(1 \leq Z \leq 2 | H_0)
\]

\[
= 0 + (1-\Theta) \frac{1}{3} = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}
\]

\[
P_{e_{\theta}} = P(Z \in \Gamma_0 | H_1) + \Theta P(Z \in \Gamma_{\text{don't care}} | H_1)
\]

\[
= P(0 < Z < 1 | H_1) + P(Z < 3 | H_1)
\]

\[
+ \Theta P(1 \leq Z \leq 2 | H_1)
\]

\[
= 0 + 0 + \Theta \cdot 1 = \Theta = \frac{1}{4}
\]

\[\therefore P_{e_{\theta}} = P_{e_{\theta}} = \frac{1}{4} \quad \text{hence minimax}
\]

Note that get the same $P_{e^*}$ for this randomized rule.
Can restrict consideration to points \( z \) s.t. \( 0 < z < 3 \) since the interval encompasses the support of both pdfs.

Clearly,
\[
\Gamma_1 = \emptyset \\
\Gamma_0 = (0, 1) \cup (2, 3) \\
\Gamma_{\text{don't care}} = [1, 2]
\]

The usual Bayes error for the priors \( \pi_0 = \frac{3}{4}, \pi_1 = \frac{1}{4} \)
is
\[
\overline{P}_e^* = \pi_0 P_{e,0} + \pi_1 P_{e,1}
\]

but we do have to decide where we put \( \Gamma_{\text{don't care}} \).

If define \( \Gamma_1' = \Gamma_1 \cup \Gamma_{\text{don't care}} \),
\[
P_{e,0} = P(Z \in \Gamma_1' | H_0) = P(1 \leq z \leq 2 | H_0) = \frac{1}{3}
\]
\[
P_{e,1} = P(Z \in \Gamma_0 | H_1)
\]
\[
= P(0 < z < 1 | H_1) + P(z \geq 3 | H_1) = 0
\]
\[
\Rightarrow \overline{P}_e^* = \frac{3}{4}, \frac{1}{3} + \frac{1}{4} = 0 = \frac{1}{4}
\]
(b) Consider the non-randomized decision rule

\[ P_0 = [0, \frac{5}{4}] \cup [2, 3] \]
\[ P_1 = (\frac{5}{4}, 2) \]

\[ P_{e_0} = P(Z \in P_1 | H_0) = P(\frac{5}{4} < Z < 2 | H_0) \]
\[ = (2 - \frac{5}{4}) \frac{1}{3} = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4} \]

\[ P_{e_1} = P(Z \in P_0 | H_1) = P(0 \leq Z \leq \frac{5}{4} | H_1) + P(2 \leq Z \leq 3 | H_1) \]
\[ = (\frac{5}{4} - 1) \cdot 1 + 0 = \frac{1}{4} \]

Clearly an equalizer rule. Also

\[ \overline{P_e} = P_0 P_{e_0} + P_1 P_{e_1} = \frac{1}{4} \]

so also a Bayes rule. Hence is minimax.

This is not an LRT because we cannot find an \( \eta \) s.t.

\[ \Pi_0 = \{ Z : \frac{f_1(Z)}{f_0(Z)} < \eta \} \] and \[ \Pi_1 = \{ Z : \frac{f_1(Z)}{f_0(Z)} > \eta \} \]

Note:

\[ \frac{f_1(Z)}{f_0(Z)} = \begin{cases} 0 & 0 < Z < 1 \\ \frac{1}{3} & 1 < Z < 2 \\ 0 & 2 < Z < 3 \end{cases} \]
A binary baseband data transmission system uses the antipodal signal set defined by

\[ s_0(t) = \begin{cases} 
2At/T, & 0 \leq t < T/2, \\
A(2t - T)/T, & T/2 \leq t < T, \\
0, & \text{otherwise.}
\end{cases} \]

The channel is an additive white Gaussian noise channel with noise spectral density \( N_0/2 \). The minimax criterion is to be used.

(a) What is the minimum probability of error for this system? Give your answer in terms of \( A, T, N_0 \), and the function \( Q \).

(b) Give the impulse response of the filter that achieves the minimum error probability (i.e., the matched filter). Simplify your answer as much as possible.

(c) Give the optimum sampling time and optimum (minimax) threshold for the receiver that uses the filter of part (b).

(d) Find the variance \( \sigma^2 \) of the output process when the input to the matched filter is a white-noise process with spectral density \( N_0/2 \).

(e) Suppose that the filter in the receiver is not the filter of part (b), but is instead a filter with impulse response

\[ h(t) = p_T(t). \]

Give an expression for the output \( \hat{s}_0(t) \) of this filter when the input is \( s_0(t) \). Give the value of \( \hat{s}_0(t) \) for all \( t \) in the range \(-\infty < t < \infty\).

(f) For the filter of part (e), find the maximum value of \( \hat{s}(t) \), where the maximization is over all \( t \) in the range \(-\infty < t < \infty\). (Note: This part can be solved independently of the solution to part (e).)
Antipodal signalling \( s_1(t) = -s_0(t) \) where

\[
\begin{align*}
    s_0(t) &= \begin{cases} 
        2A t/T & 0 \leq t < T/2 \\
        A (2t-T)/T & T/2 \leq t < T \\
        0 & \text{else}
    \end{cases} 
\end{align*}
\]

AWGN \( N_0/2 \)

Use minimax criterion

For antipodal signals and zero mean additive Gaussian noise the minimax threshold is always

\( \gamma_m = 0 \)

The minimum or best minimax error probability occurs when the LTI filter is a matched filter

\[
h(t) = c \left[ s_0(T_0-t) - s_1(T_0-t) \right] = 2c s_0(T_0-t)
\]

any sampling time \( T_0 \) will work and the choice \( T_0 = T \) is the smallest which results in a causal HIF.

From class or MBP formulas the max. SNR is

\[
\text{SNR}_{\text{max}} = \frac{2 \|s_0\|}{\sqrt{\frac{T}{2} N_0}} \quad T/2 \\
\|s_0\|^2 = \int_0^T s_0^2(t) \, dt = 2 \int_0^{T/2} s_0^2(t) \, dt = 2 \cdot \frac{4A^2}{T^2} \int_0^{T/2} t^2 \, dt
\]
\[ \|S_0\|^2 = \frac{8A^2}{T^2} \frac{1}{3} \left( \frac{T}{2} \right)^3 = \frac{A^2 T^2}{3} \Rightarrow \|S_0\| = \frac{A \sqrt{T}}{\sqrt{3}} \]

\[ \text{SNR}_{\text{max}} = \frac{2 \cdot A \sqrt{T}}{\sqrt{3}} \frac{1}{\sqrt{2} N_0} = 2A \sqrt{\frac{T}{6N_0}} \]

\[ P_{e_{\text{m}}}^* = Q(\text{SNR}_{\text{max}}) = Q \left( 2A \sqrt{\frac{T}{6N_0}} \right). \]

(a) \[ = Q \left( \sqrt{\frac{2A^2 T}{3N_0}} \right) \rightarrow \min_{h} P_{e_{\text{m}}}^* \]

(c) As mentioned above \( \gamma_{\text{m}} = 0 \) and \( T_0 = T \).

(d) Find variance of output process when input to filter is white Gaussian noise with \( N_0/T \).

As shown in class and in MBP

\[ \sigma^2 = \frac{N_0}{T} \|h\|^2 = \frac{N_0}{T} 4c^2 \|S_0\|^2 \]

\[ = \frac{2N_0 c^2 A^2 T}{3} \]

(e) Instead of using a HF let \( h(t) = p_T(t) \).

So want to compute the convolution \( \hat{S}_0(t) = p_T * S_0(t) \).

Using linearity and time invariance we first compute

\[ \begin{array}{ccc}
\text{LTI} \\
S_0(t) \quad h(t) = p_T(t) \quad 2 \hat{S}_0(t)
\end{array} \]

Then can put pieces together to get final answer.
From picture have cases.

Case: $t < 0$ \implies \hat{S}_o(t) = 0$

Case: $0 < t < \frac{T}{2}$ \implies \hat{S}_o(t) = \frac{1}{2} t \cdot \frac{2A}{T} t = \frac{A}{T} t^2$

Case: $\frac{T}{2} < t < T \implies \hat{S}_o(t) = \frac{1}{2} \frac{T}{T} \cdot A = \frac{AT}{4}$

Case: $T < t < \frac{3T}{2}$ \implies \hat{S}_o(t) = \frac{AT}{4} - \frac{1}{2} (t-T) \frac{2A}{T} (t-T)$

Case: $\frac{3T}{2} < t \implies \hat{S}_o(t) = 0$.

Then from linearity and time invariance

\[ \hat{S}_o(t) = \hat{S}_o(t) + \hat{S}_o(t-T/2). \]
\( \hat{S}_o(t) = \begin{cases} 
0 & t < 0 \\
\frac{A}{T} t^2 & 0 < t < \frac{T}{2} \\
\frac{A T}{4} + \frac{A}{T} (t - \frac{T}{2})^2 & \frac{T}{2} < t < T \\
\frac{A T}{2} - \frac{A}{T} (t - T)^2 & T < t < \frac{3T}{2} \\
\frac{A T}{4} - \frac{A}{T} (t - \frac{3T}{2})^2 & \frac{3T}{2} < t < 2T \\
0 & t > 2T 
\end{cases} \)

(f) Find max value of \( \hat{S}_o(t) \).

From graphical picture of convolution and the positivity of \( S_o \) and \( p_T \) the max must occur at \( t = T \)

\[
\max_{t} \hat{S}_o(t) = \frac{AT}{2}
\]
5.18 Consider the communications system shown in Figure 5.4. The noise process \(X(t)\) is a white Gaussian random process with spectral density \(N_0/2\), and the signals \(s_0(t)\) and \(s_1(t)\) are given by

\[
s_i(t) = (-1)^i A p_T(t)
\]

for \(i = 0\) and \(i = 1\). The threshold is \(\gamma = 0\) and the sampling time is \(T_0 = \alpha T\) for \(0 < \alpha < 2\). Investigate the effects of the sampling time by finding the error probabilities \(P_{e,0}\) and \(P_{e,1}\) in the following two cases:

(a) The filter is a linear time-invariant filter that is matched to the signals; that is, the impulse response is

\[
h(\lambda) = s_0(T - \lambda) - s_1(T - \lambda).
\]

Give \(P_{e,0}\) and \(P_{e,1}\) in terms of \(\alpha, A, T,\) and \(N_0\).

(b) The filter is an integrate-and-dump filter with output given by

\[
Z(T_0) = \int_0^{T_0} Y(t) \, dt.
\]

Give expressions for \(P_{e,0}\) and \(P_{e,1}\) in terms of \(\alpha, A, T,\) and \(N_0\).

(c) Express your answers to (a) and (b) in terms of \(\alpha, E,\) and \(N_0\) (where \(E\) is the energy per pulse), and compare them.
MBP Problem 5.18

$S_0(t)$ → $+$ → $h(t)$ → $t=T_0$ → Threshold → decision

$X(t)$ (Fig. 5-4)

\[ \frac{N_0}{2} \]

White Gaussian

$S_0(t) = A p_T(t)$  $S_1(t) = -A p_T(t)$ antipodal

$\gamma = 0$

$T_0 = \alpha T$  $0 < \alpha < 2$

Find $P_{e,0}$ and $P_{e,1}$ assuming:

(a) $h(t) = s_0(T-t) - s_1(T-t)$

$= 2A p_T(T-t) = 2 A p_T(t)$

Following approach taken in class

$\mu_0 = s_0 * h(T_0) = \int s_0(\lambda) h(T_0-\lambda) \, d\lambda = 2A \int p_T(\lambda) p_T(T_0-\lambda) \, d\lambda$

$P_T(T_0-\lambda) \rightarrow \begin{array}{c}
\vdots \\
T_0 \\
T \\
\end{array}$

$P_T(\lambda) \rightarrow \begin{array}{c}
\vdots \\
T_0 \rightarrow \\
T \\
\end{array}$

$\mu_0 = 2A^2 \int_0^{T_0} d\lambda = 2A^2 T_0 = 2A^2 \alpha T$  $0 < \alpha \leq 1$

$= 2A^2 T_0 = 2A^2 (2T-T_0)$  $T < T_0 < T$

$\mu_0 = 2A^2 \int_0^{T_0} d\lambda = 2A^2 (2T-T_0)$  $T < T_0 < T$

$T_0 - T = 2A^2 (2-\alpha)T$  $1 \leq \alpha < 2$

$-\mu_1 = \mu_0 = \left\{ \begin{array}{ll}
2A^2 \alpha T & 0 < \alpha \leq 1 \\
2A^2 (2-\alpha)T & 1 \leq \alpha < 2 \\
\end{array} \right.$
Note that the minimax threshold = 0, which agrees with choice made in this problem.

For AWGN
\[ \sigma^2 = \frac{N_0}{2} \int h^2(t) \, dt = \frac{N_0}{2} \alpha^2 \int_0^T \, dt = 2\alpha^2 N_0 T \]

\[ \therefore \text{SNR} = \frac{\mu_0 - \mu_1}{2\sigma^2} = \frac{4\alpha^2 \alpha T}{2 \sqrt{2\alpha^2 N_0 T}} \]

\[ = \frac{4\alpha^2 (2-\alpha) T}{2 \sqrt{2\alpha^2 N_0 T}} \quad 1 \leq \alpha < 2 \]

Simplifying
\[ \text{SNR} = \begin{cases} \frac{A}{\sqrt{2 T N_0}} & 0 < \alpha \leq 1 \\ \frac{A}{\sqrt{2 T N_0}} (2-\alpha) & 1 \leq \alpha < 2 \end{cases} \]

Since minimax \( P_{e,0} = P_{e,1} = Q(\text{SNR}) \).

(b) Integrate and dump with \( T_0 \)
\[ Z(T_0) = \int_0^{T_0} Y(t) \, dt = \int_0^{T_0} S_i(t) \, dt + \int_0^{T_0} X(t) \, dt \]

For the signal part consider \( i = 0 \)
\[ \int_0^{T_0} S_o(t) \, dt = A \int_0^{T_0} P_T(t) \, dt = \begin{cases} AT_0 & 0 < T_0 \leq T \\ AT & T < T_0 < 2T \\ A \alpha T & 0 < \alpha \leq 1 \\ AT & 1 \leq \alpha < 2. \end{cases} \]

Also \[ \int_0^{T_0} S_i(t) \, dt = -\int_0^{T_0} S_o(t) \, dt. \]
For the noise part note that \( \int_0^{T_0} X(t) \, dt \) is a Gaussian r.v. with zero mean and variance
\[
\sigma^2 = E \left\{ \left( \int_0^{T_0} X(t) \, dt \right)^2 \right\} = \int_0^{T_0} \int_0^{T_0} E \{ X(t) X(\lambda) \} \, dt \, d\lambda
\]
\[
= \frac{N_0}{2} \int_0^{T_0} \int_0^{T_0} \delta(t-\lambda) \, dt \, d\lambda = \frac{N_0}{2} \int_0^{T_0} d\lambda = \frac{N_0 T_0}{2}
\]
\[
= \frac{N_0 T_0}{2}
\]

So decision problem for this statistic can be written
\[
\begin{align*}
Z(T_0) & \sim N \left( \mu(\alpha), \frac{N_0 T_0}{2} \alpha \right) \quad \text{under } H_0 \\
& \sim N \left( -\mu(\alpha), \frac{N_0 T_0}{2} \right) \quad \text{under } H_1
\end{align*}
\]

where
\[
\mu(\alpha) = \begin{cases} 
AT \alpha & 0 < \alpha \leq 1 \\
AT & 1 < \alpha < 2 
\end{cases}
\]

\( \gamma = 0 \) is minimax threshold as before. The SNR depends on \( \alpha \)
\[
\text{SNR} = \frac{\mu_0 - \mu_1}{\sigma} = \frac{\mu_0}{\sigma}
\]
\[
= \begin{cases} 
\frac{AT \alpha}{\sqrt{N_0 T_0} / 2} & 0 < \alpha \leq 1 \\
\frac{AT}{\sqrt{N_0 T_0} / 2} & 1 < \alpha < 2 
\end{cases} = \begin{cases} 
A \sqrt{\frac{2T}{N_0} \alpha} & 0 < \alpha \leq 1 \\
A \sqrt{\frac{2T}{N_0} \alpha} & 1 < \alpha < 2 
\end{cases}
\]
\[
P_{e_0} = P_{e_1} = Q(\text{SNR}).
\]
(c) Express prev. answers in terms of $E$ (energy per pulse).

$$E = \int_0^T A^2 \, dt = A^2 T$$

$$\text{SNR}_a = \begin{cases} \sqrt{\frac{2E}{N_0}} & 0 < \alpha \leq 1 \\ \sqrt{\frac{2E}{N_0}} (2-\alpha) & 1 < \alpha < 2 \end{cases}$$

$$\text{SNR}_b = \begin{cases} \sqrt{\frac{2E}{N_0}} \sqrt{\alpha} & 0 < \alpha \leq 1 \\ \sqrt{\frac{2E}{N_0}} \frac{1}{\sqrt{\alpha}} & 1 < \alpha < 2 \end{cases}$$