General Instructions:

• You have 50 minutes to complete the exam.
• Write your name on every page of the exam.
• Do not write on the backs of the pages. If you need more paper, it will be provided to you upon request.
• The exam is closed book and closed notes, except that you may use one 8.5 by 11 inch crib sheet.
• Calculators are allowed.
• Your work must be explained to receive full credit.
• Point values for each problem are as indicated. The exam totals 100 points.
• All plots must be carefully drawn with axes labeled.

Do not open the exam until you are told to begin.
Description and Block Diagram for Problems 1 and 2

The block diagram shown is a possible implementation of a differentially encoded communication system.

The encoder accepts a binary (i.e., 0 or 1) string $B_k$ at the input and produces a binary string $D_k$ at the output. The output string contains an extra digit $D_{-1}$, which is set as an initial condition of the encoder. The symbol “⊕” in the encoder denotes modulo 2 binary addition (i.e., exclusive or).

The decoder takes a binary string input $\hat{D}_k$ and produces a binary string output $\hat{B}_k$ containing one fewer digit than the input. The digits are lined up so that ideally $\hat{B}_k = B_k$ for $k = 0, 1, 2, \ldots$. 
Problem 1. [25 pts. total] Consider the differentially encoded BPSK system described on the previous page focusing on the Encoder and Decoder blocks.

(a) [4 pts.] The table below gives an example input string $B_k$. Assuming that the encoder initial condition is $D_{-1} = 0$ as shown, fill in the values for the encoded bits $D_k$ in the indicated row of the table.

<table>
<thead>
<tr>
<th>Time index $k$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_k$</td>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_k$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{D}_k$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{B}_k$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) [7 pts.] The bits $D_k$ starting from $k = -1$ are mapped to symbols, sent through the channel, and then de-mapped to produce the estimated bit sequence $\hat{D}_k$. Assuming that no bit errors occur in transmission, fill in the row in the above table corresponding to $\hat{D}_k$.

Then give a mathematical formula for the decoder and write down the estimated bit sequence $\hat{B}_k$ in the above table.
Problem 1. (cont’d.)

(c) [7 pts.] Repeat part (b) for the table shown below but now assume that the channel is such that the bits $\hat{D}_k$ are the complements of the corresponding bits $D_k$. Use exactly the same decoder as you found in part (b). The table is repeated below. You will need to fill the $D_k$ row in with the same values as found in part (a).

Explain why the result you find is important in BPSK systems, which use either the squaring loop or the Costas loop for carrier phase recovery.

<table>
<thead>
<tr>
<th>Time index k</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
</tr>
<tr>
<td>$B_k$</td>
</tr>
<tr>
<td>$\text{-}$</td>
</tr>
</tbody>
</table>
Problem 1. (cont’d.)

(d) [7 pts.] Repeat part (b) only now assuming that a single bit error is made in the channel at time index $k = 3$, i.e., $\hat{D}_k = D_k$ for $k \neq 3$ and $\hat{D}_3 \neq D_3$. The position of the bit error is indicated in the table below with a small box.

Comment on the result in light of what was found in the homework problem about the probability of error performance of differentially encoded BPSK in comparison to regular BPSK.

<table>
<thead>
<tr>
<th></th>
<th>Time index $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_k$</td>
<td>- 0 1 1 0 1 0 0 1</td>
</tr>
<tr>
<td>$D_k$</td>
<td>0</td>
</tr>
<tr>
<td>$\hat{D}_k$</td>
<td>[ ]</td>
</tr>
<tr>
<td>$\hat{B}_k$</td>
<td>-</td>
</tr>
</tbody>
</table>
Problem 2. [35 pts. total] This problem concerns just the encoder of the DBPSK system given before. Suppose that the input bit string $B_k$ is independent and identically distributed (i.i.d.) with

$$P(B_k = 1) = p \quad \text{and} \quad P(B_k = 0) = 1 - p.$$ 

(a) [15 pts.] Assuming that the encoder initial state is $D_{-1} = 0$ find the marginal probability distribution of the encoder output for all time $k$, i.e., find

$$q_k \overset{\text{def}}{=} P(D_k = 1)$$

for $k \geq 0$. \textit{Hint:} Find a first order difference equation for $q_k$ and solve it.

(b) [10 pts.] For general $p$, are the random variables $\{D_k : k \geq 0\}$ identically distributed? Are they statistically independent? You must prove or give a counter example.

(c) [10 pts.] For the special case of $p = 1/2$, are the random variables $\{D_k : k \geq 0\}$ identically distributed? Are they statistically independent? You must prove or give a counter example.
Problem 2. (cont’d.)
Problem 2. (cont’d.)
Problem 2. (cont’d.)
Problem 3. [40 pts. total] Consider the binary baseband communication and basic pulse \( g(t) \), which are shown in the figures below. The noise \( N(t) \) is AWGN with power spectral density height equal to \( N_0/2 \). The signals under the two hypotheses are \( s_0(t) = \alpha_0 g(t) \) and \( s_1(t) = \alpha_1 g(t) \) where \( \alpha_0 > \alpha_1 \). In this problem the performance criterion is average probability of error (i.e., it is a Bayes problem) where the priors on the two signals are \( \pi_0 = \pi_1 = 1/2 \).

Let hypothesis \( H_i \) correspond to the transmission of signal \( s_i \) for \( i = 0, 1 \). Assume that the LTI filter \( h(t) \) is chosen to be the matched filter for pulse shape \( g(t) \) and sampling time \( t = T \).

(a) [16 pts.] Find the probability density functions \( f_i(z) \), \( i = 0, 1 \), for the decision statistic \( Z \) under the two hypotheses including writing the parameters of the pdfs in terms of \( \alpha_0, \alpha_1, N_0, \) and \( T \).

(b) [10 pts.] Starting from the likelihood ratio test for this Bayesian hypothesis test show that an equivalent test is of the form

\[
Z > \gamma \quad \text{decide} \quad H_0
\]

\[
Z \leq \gamma \quad \text{decide} \quad H_1
\]

and find the threshold \( \gamma \) for minimum average probability of error.

(c) [14 pts.] Find the average probability of error in terms of \( \alpha_0, \alpha_1, N_0, \) and \( T \).
Problem 3. (cont’d.)
Problem 3. (cont’d.)
Problem 3. (cont’d.)