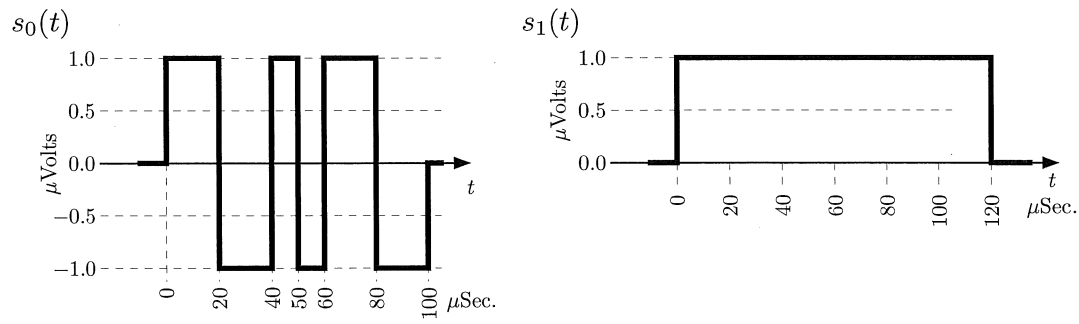
Problem 1. [35 points] Name: Solution

You are the designer of a communications receiver. The transmitter designer has told you that the system is for binary communications and that a signal $s_0(t)$ is used to send a '0' bit, and $s_1(t)$ is used to send a '1' bit. The signals are shown below. Note that the voltage axes are labelled in micro-volts (i.e., 10^{-6} volts) and the time axes are labelled in micro-seconds. For power / energy calculations assume that the voltages are developed across a nominal 1 Ohm resistor.

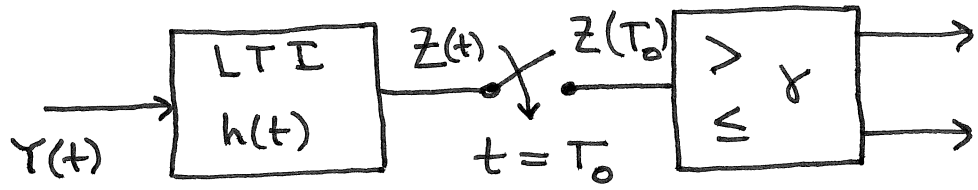


The signal you receive is $Y(t) = s_i(t) + X(t)$ where $X(t)$ is white Gaussian noise of power spectral density height equal to $N_0/2 = -198.6$ dBJ, and $i = 0$ or $= 1$ depending on the bit that is actually sent.

There are three parts to this question.

(a) [15 pts.] Assuming that the bits are equally likely, draw the architecture of the minimum probability of error receiver for this problem. You do **not** have to derive it from first principles, but you must explain it and specify the value of every parameter, filter, sampler, and/or threshold. Your answer will be a block diagram that takes input $Y(t)$ and produces a decision on which bit ('0' or '1') that the receiver thinks was sent.

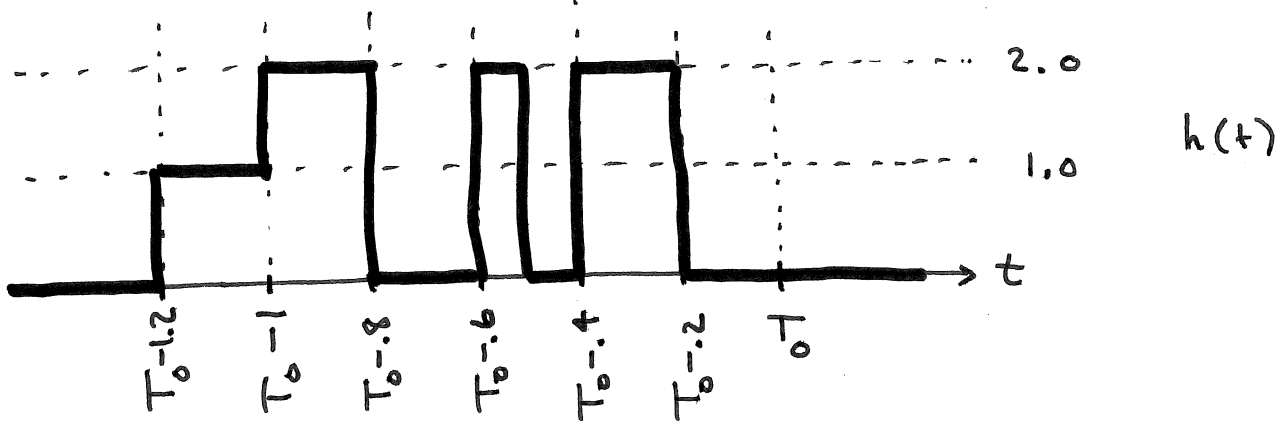
From class we know that the optimal architecture is a matched filter followed by a sampler and threshold device



To specify: $h(t)$, T_0, γ and the form of the decision.

<u>Filter:</u> Should be a matched filter

$$h(t) = s_1(T_0 - t) - s_0(T_0 - t)$$ ... can have an arbitrary scale factor.



<u>Sample Time</u> Should pick $t = T_0$ to sample and say $T_0 \geq 1.2$ for causality.

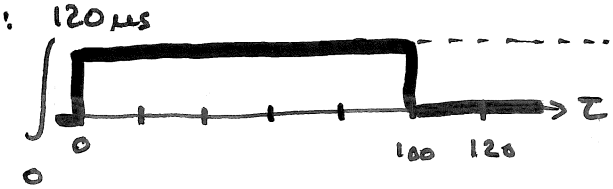
<u>Threshold and Decision</u> Need the signal terms at the sampler output to choose threshold and the sense of the test

$$\hat{s}_0(T_0) = s_0 * h(T_0) = \int s_0(\tau) h(t-\tau)\, d\tau \Big|_{t=T_0}$$

$$= \int_0^{120\mu s} s_0(\tau)\left[s_1(\tau) - s_0(\tau)\right] d\tau = \int_0^{120\mu s} s_0(\tau) s_1(\tau)\, d\tau - \int_0^{120\mu s} s_0^2(\tau)\, d\tau$$

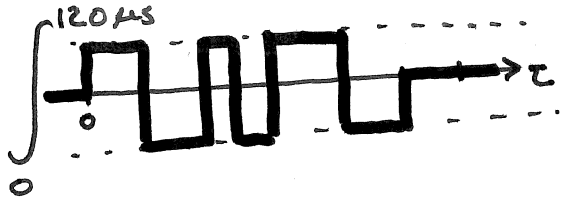
Evaluate the pieces:

$$\int_0^{120\mu s} s_0^2(\tau)\, d\tau =$$



$$d\tau = \frac{(1\mu V)^2}{1\Omega} \cdot 100\,\mu s$$

$$\int_0^{120\mu s} s_0(\tau) s_1(\tau)\, d\tau =$$



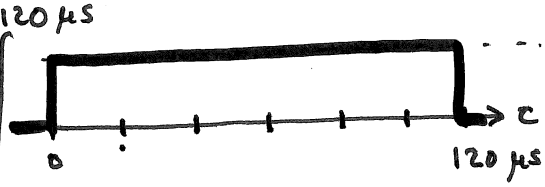
$$d\tau = 0$$ ie s_0 and s_1 are orthog.

Name: _____

$$\hat{s}_1(T_0) = s_1 * h(T_0) = \int_0^{120\mu s} s_1(\tau)\left[s_1(\tau) - s_0(\tau)\right] d\tau$$

$$= \int_0^{120\mu s} s_1^2(\tau)\, d\tau - \int_0^{120\mu s} s_1(\tau) s_0(\tau)\, d\tau = \int_0^{120\mu s} s_1^2(\tau)\, d\tau \qquad \text{since } s_0 \perp s_1$$

$$= \int_0^{120\mu s}$$



$$d\tau = \frac{(1\mu V)^2}{1\Omega} \cdot 120\,\mu s$$

$$\therefore \quad \hat{s}_0(T_0) = -\mathcal{E}_0 = -1 \times 10^{-16}\ J$$

$$\hat{s}_1(T_0) = \mathcal{E}_1 = +1.2 \times 10^{-16}\ J$$

$\Rightarrow$ The threshold is

$$\gamma_m = \frac{-\mathcal{E}_0 + \mathcal{E}_1}{2} = 1 \times 10^{-17}\ J$$

And the test should be

$$Z(T_0) \underset{<}{\geq} \gamma = 1 \times 10^{-17}\ J \qquad \begin{array}{l} \text{say } H_1 \\ \text{say } H_0 \end{array}$$

Problem 1. (cont'd.) Name: _____

(b) [10 pts.] Now in class we showed that the minimum probability of error for the receiver of part (a) was given by the Q-function expression:

$$P_e = Q(\sqrt{E_b(1-r)/N_0}),$$

where E_b is the average energy per bit and $r = \langle s_0, s_1 \rangle / E_b$ is the normalized correlation between the two signals. For the signals above, evalute this probability of error and give an approximate numerical value.

$$E_b = \frac{\mathcal{E}_0 + \mathcal{E}_1}{2} = \frac{1\times 10^{-16} + 1.2\times 10^{-16}}{2} = 1.1\times 10^{-16}\,\text{J}$$

$r = 0$ since the signals are orthogonal

$$\left(N_0/2\right)_{dB} = -198.6\ \text{dBJ} = 10\log_{10}\left(N_0/2\right)$$

$$\Rightarrow \frac{N_0}{2} = 10^{-198.6/10} = 10^{-19.86}\ \text{J}$$

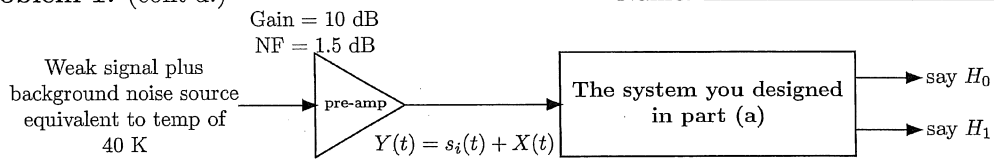
$$\Rightarrow N_0 = 2\times 10^{-19.86}\ \text{J} = 2.76\times 10^{-20}\ \text{J}$$

Arg of Q:

$$\sqrt{\frac{E_b(1-r)}{N_0}} = \sqrt{\frac{1.1\times 10^{-16}}{2.76\times 10^{-20}}} = 63.12$$

$$Q(63.12) \approx 0$$

Problem 1. (cont'd.) Name: _____



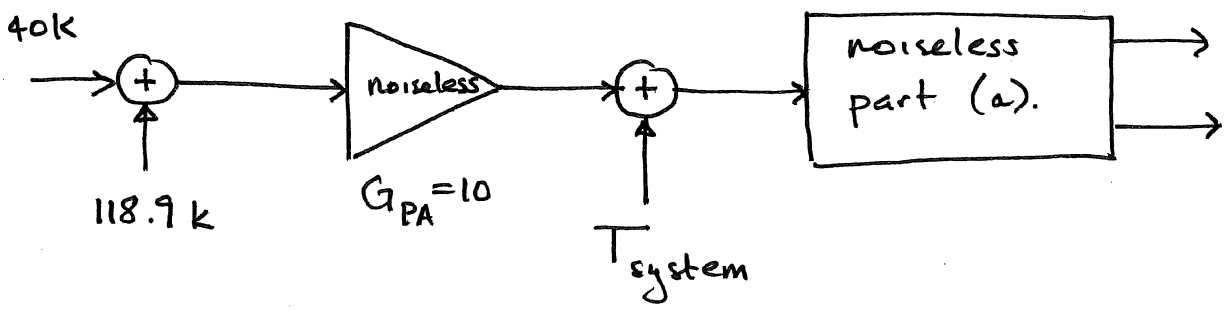
(c) [10 pts.] Assume that the signal $Y(t)$ described in part (a) is measured at the point shown in the block diagram above. Given the values for the assumed input noise source and the noise figure and gain of the pre-amplifier, find the noise figure and equivalent noise temperature of the system you designed in part (a). This amounts to reconciling the power spectral denity height given for $X(t)$ with the other noise information from the block diagram.

Pre amp $G_{dB} = 10\,dB \Rightarrow G = 10$

$NF_{dB} = 1.5\,dB \Rightarrow NF = 10^{1.5/10} = 1.41$

$T_{PA} = 290\,K\,(NF-1) = (290\,K)(.41) = 118.9\,K.$

The noise sources are



$(40K + 118.9\,K)\,10 + T_{system}$ = temp corresp. to $\frac{N_0}{2} = -198.6\,dBJ$

$= 1.38 \times 10^{-20}\,J.$

$= \frac{kT}{2}$

$\Rightarrow T = \frac{2 \cdot (1.38 \times 10^{-20}\,J)}{1.38 \times 10^{-23}\,J/K}$

$= 2000\,K$

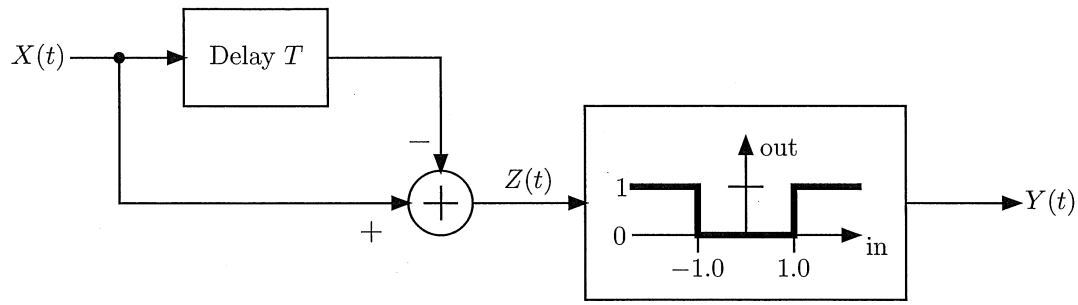
$\therefore T_{system} = 2000\,K - 1589\,K$

$= 411\,K$

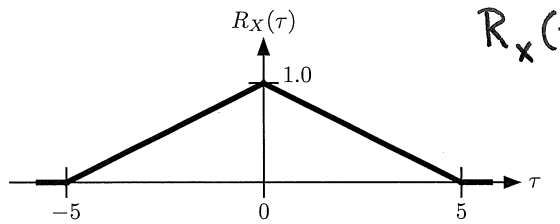
$F_{system} = \frac{411}{290} + 1 = 2.42$

$= 3.83\,dB$

Problem 2. [15 points] Name: Solution



The input $X(t)$ to the system shown above is a zero mean, Gaussian random process, with triangularly shaped autocorrelation function:



$$R_X(\tau) = \begin{cases} 1 - \frac{|\tau|}{5} & 0 \le |\tau| \le 5 \\ 0 & \text{else} \end{cases}$$

By adjusting the value of the delay T how much can you change the average duty cycle of the random $0-1$ square wave at the output of the system? Note that the final block is a memoryless non-linearity.

$X(t) \sim N(0,1)$ with auto-correlation $R_X(\tau)$. Input to memoryless non-linearity is $Z(t) = X(t) - X(t-T)$, which must be Gaussian since its a linear function of a Gaussian r.p. Clearly $Z(t)$ has zero mean. Its variance is

$$E\{Z^2(t)\} = E\{[X(t)-X(t-T)][X(t)-X(t-T)]\}$$

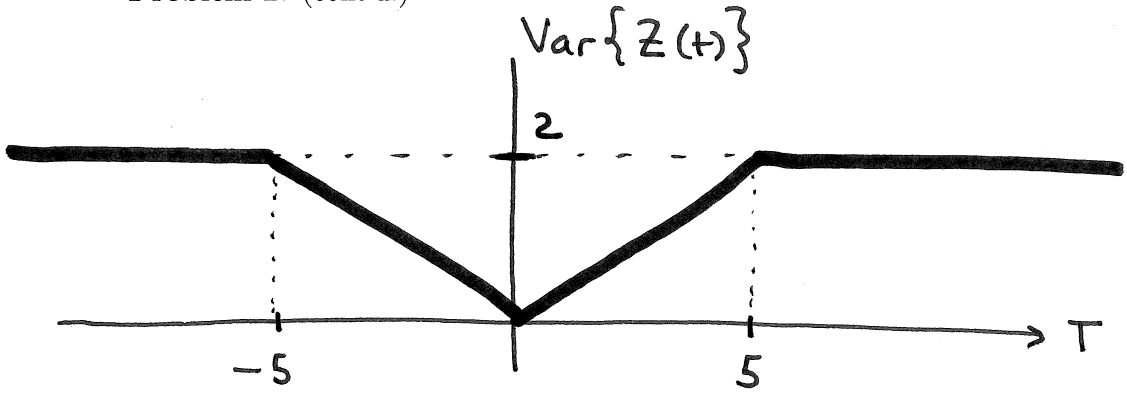
$$= E\{X^2(t)\} - E\{X(t-T)X(t)\} - E\{X(t)X(t-T)\} + E\{X^2(t-T)\}$$

$$= R_X(0) - R_X(T) - R_X(T) + R_X(0)$$

$$= 2\left[R_X(0) - R_X(T)\right] = 2\left[1 - R_X(T)\right]$$

$$= \begin{cases} \frac{2|T|}{5} & 0 \le |T| \le 5 \\ 2 & |T| > 5 \end{cases}$$

Problem 2. (cont'd.) Name: _____



$$E\{Y(t)\} = P\{Y(t)=1\} = 1 - P\{-1 < Z(t) \le 1\}$$

$$= 1 - P\left\{\frac{-1}{\sqrt{.4|T|}} < \frac{Z(t)}{\sqrt{.4|T|}} \le \frac{1}{\sqrt{.4|T|}}\right\} \qquad 0 \le |T| \le 5$$

$$= 2Q\left(\frac{1}{\sqrt{.4|T|}}\right)$$

Thus for T near zero $E\{Y(t)\} \approx 0$ and when T reaches 5 sec.

$$E\{Y(t)\} = 2Q\left(\frac{1}{\sqrt{2}}\right) = 2Q(.71)$$

$$\approx 2 \cdot .242$$

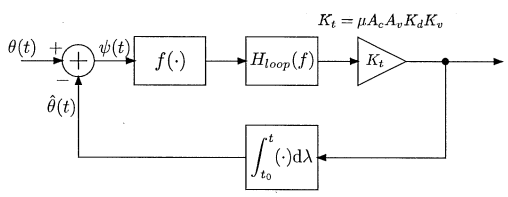
$$\approx 0.48$$

Duty cycles from zero to 48 %.

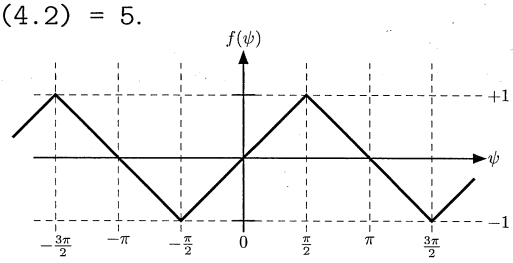
Problem 3. [15 points] Name: _____

Below is a part of a Matlab code which simulates the nonlinear baseband model of a PLL with phase detector characteristic $f(\psi) = \sin(\psi)$. Modify the code to implement the new phase detector characteristic $f(\psi)$ shown below. The following Matlab commands may be useful:

- `floor(x)`, which rounds a real number `x` to the nearest integer less than or equal to `x`, e.g., `floor(-4.2) = -5`, and `floor(4.2) = 4`.
- `ceil(x)`, which rounds a real number `x` to the nearest integer greater than or equal to `x`, e.g., `ceil(-4.2) = -4`, and `ceil(4.2) = 5`.



Nonlinear BB model showing phase detector $f(\cdot)$



Definition of phase detector $f(\cdot)$

```
%% Beginning of Simulation Loop
for i = 1:npts
    if i < nsettle
        fin(i) = 0;
        phin = 0;
    else
        fin(i) = fdel;
        phin = 2*pi*fdel*T*(i-nsettle);
    end

    s1 = phin - vco_out;
    s2 = sin(s1);
    s3 = Kt*s2;

    filt_in = a*s3;
    filt_out = filt_out_last + (T/2)*(filt_in + filt_in_last);
    filt_in_last = filt_in;
    filt_out_last = filt_out;
    vco_in = s3 + filt_out;
    vco_out = vco_out_last + (T/2)*(vco_in + vco_in_last);
    vco_in_last = vco_in;
    vco_out_last = vco_out;
    phierror(i) = s1;
    fvco(i) = vco_in/(2*pi);
    freqerror(i) = fin(i) - fvco(i);
end
% End of Simulation Loop
```

$s1 = \psi \implies$ want $s2 = f(\psi)$ for the given characteristic f. Note that it is periodic in ψ of period 2π. Also note that expression for f is simplest for

$$-\pi/2 \le \psi < 3\pi/2$$

<u>Case</u> $-\pi/2 \le \psi < \pi/2$

$$f(\psi) = m\psi \quad \text{where} \quad m = \frac{2}{\pi}$$

$$\implies f(\psi) = \frac{2}{\pi}\psi \quad \text{for} \quad -\pi/2 \le \psi < \pi/2$$

<u>Case</u> $\pi/2 \le \psi < 3\pi/2$

$$f(\psi) = -\frac{2}{\pi}\psi + b$$ (negative of slope from first case)

Solve for b

$$f(\pi/2) = -\frac{2}{\pi}\frac{\pi}{2} + b = +1$$

$$\implies b = 2$$

$$\implies f(\psi) = -\frac{2}{\pi}\psi + 2 \quad \text{for} \quad \pi/2 \le \psi < 3\pi/2$$

<u>Computing $f(\psi)$ for an arbitrary ψ.</u>

Let $\lfloor x \rfloor$ = greatest integer $\le x$ (floor function)

$$\lfloor -4.2 \rfloor = -5 \; ; \; \lfloor 4.2 \rfloor = 4$$

Note

$$\psi = \lfloor \psi/2\pi \rfloor 2\pi + \tilde{\psi}$$

and claim that the remainder

$$\tilde{\psi} = \psi - \lfloor \psi/2\pi \rfloor 2\pi$$

satisfies $0 \le \tilde{\psi} < 2\pi$. To see this

$$\frac{\tilde{\psi}}{2\pi} = \frac{\psi}{2\pi} - \left\lfloor \frac{\psi}{2\pi} \right\rfloor$$

$$\Longrightarrow \quad 0 \le \frac{\tilde{\psi}}{2\pi} < 1 \quad \Longrightarrow \text{ claim holds.}$$

$\therefore$ New matlab code

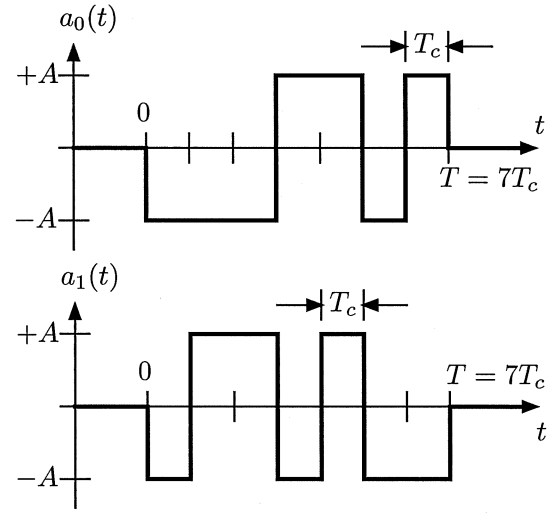
```
s1 = phin - vco_out;
s1_tilde = s1 - floor(s1/(2*pi))*2*pi;
if s1_tilde >= 3*pi/2
    s1_tilde = s1_tilde - 2*pi;
end if
if -pi/2 <= s1_tilde < pi/2
    s2 = (2/pi)*s1_tilde;
else
    s2 = -(2/pi)*s1_tilde + 2;
end if
s3 = Kt*s2;
```

Problem 4. [55 points] Name: Solution

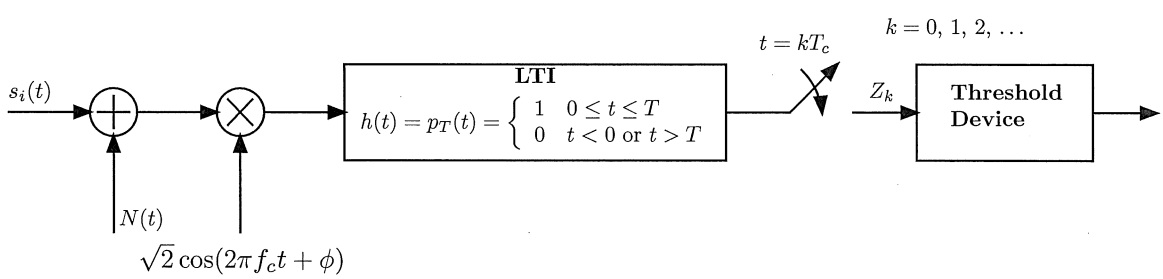
Note that this problem is a follow on to a problem from Exam 2 where we considered the coherent binary detection problem:

$$\begin{aligned} H_0 : \ s_0(t) &= \sqrt{2}a_0(t)\cos(2\pi f_c t + \phi) \quad \text{is transmitted} \\ \text{versus} \\ H_1 : \ s_1(t) &= \sqrt{2}a_1(t)\cos(2\pi f_c t + \phi) \quad \text{is transmitted} \end{aligned}$$

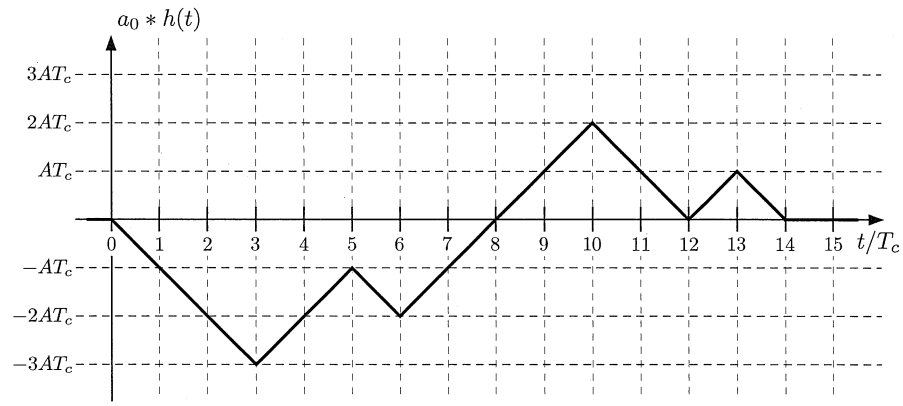
where $a_0(t)$ and $a_1(t)$ are as indicated below. Note that the tic marks are spaced by T_c and that the signals start at $t = 0$ and end at $t = T = 7T_c$.

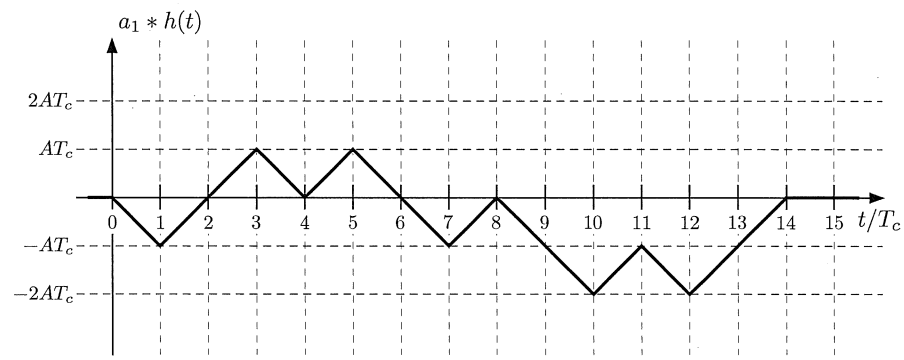


The receiver architecture for the previous version of the problem consisted of a mixer followed by an LTI filter with non-optimal impulse response $h(t) = p_T(t)$, a sampler that sampled at multiples of T_c, and a threshold device as drawn below. In the new problem we will modify the threshold device. The noise $N(t)$ is AWGN with power spectral density height equal to $N_0/2$. You may assume that the carrier frequency is large compared to the chipping rate (i.e., $f_c \gg 1/T_c$).

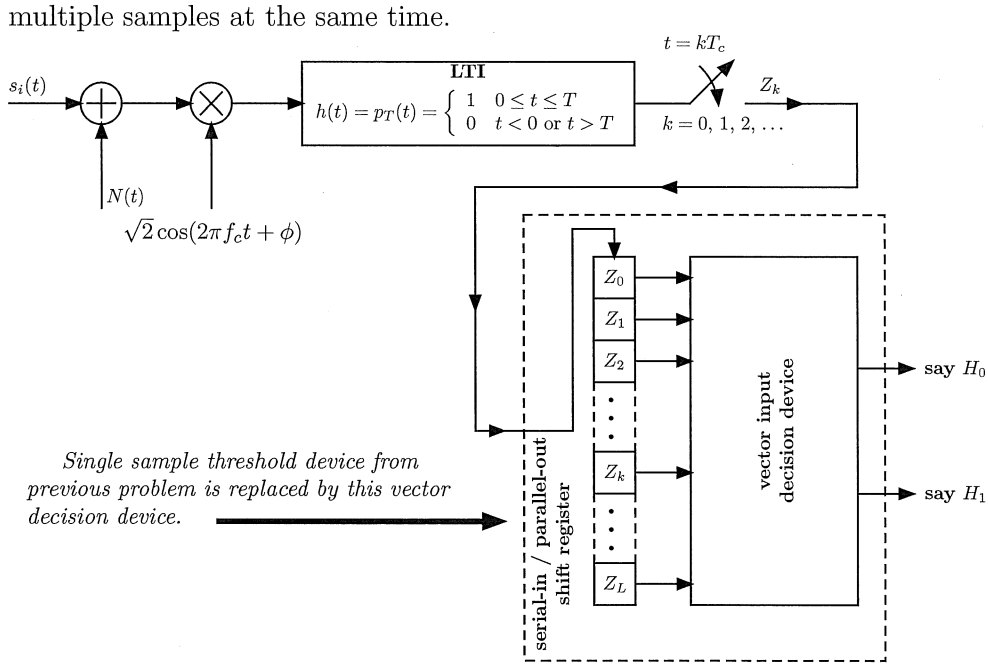


One of the main tasks of the previous version of the problem was to find the graphs of the signal parts at the output of the suboptimal filter $h(t) = p_T(t)$. The answer was as shown in the next figure.





The point of this new problem is to consider changing the receiver architecture to that shown below where the simple threshold device is replaced by a processor that can work with multiple samples at the same time.



There are three parts to this question.

Problem 4. (cont'd.) Name: _____

(a) [20 pts.] We also found the marginal distributions of the random variables Z_k under both hypotheses. To recall, these were

$$Z_k \sim \begin{cases} \mathcal{N}(a_0 * h(kT_c), \frac{7}{2}N_0T_c) & \text{under } H_0 \\ \mathcal{N}(a_1 * h(kT_c), \frac{7}{2}N_0T_c) & \text{under } H_1 \end{cases}$$

But in order to design the vector decision device in this problem we need to know the correlation of the samples Z_k and Z_l at different times. Find these correlations and explain.

Let N_k denote the noise part of Z_k. That is $N_k =$

$$\int \sqrt{2}\, N(\tau) \cos(2\pi f_c \tau + \phi)\, h(t-\tau)\, d\tau \Big|_{t = kT_c}$$

In the previous problem we found

$$E\{N_k^2\} \cong \frac{7}{2} N_0 T_c$$

and to compute the correlation $E\{N_k N_\ell\}$ we will need

$$2 \iint \underbrace{E\{N(\tau)N(\nu)\}}_{= \frac{N_0}{2}\delta(\tau-\nu)} \cos(2\pi f_c \tau + \phi) \cos(2\pi f_c \nu + \phi)\, h(kT_c - \tau)\, h(\ell T_c - \nu)\, d\tau\, d\nu.$$

$$= N_0 \int \underbrace{\cos^2(2\pi f_c \tau + \phi)}_{\frac{1}{2}(1 + \cos(2\pi 2 f_c \tau + 2\phi))}\, h(kT_c - \tau)\, h(\ell T_c - \tau)\, d\tau$$

$$\approx \frac{N_0}{2} \int h(kT_c - \tau)\, h(\ell T_c - \tau)\, d\tau$$

This is proportional to the deterministic auto correlation of $h(t)$.

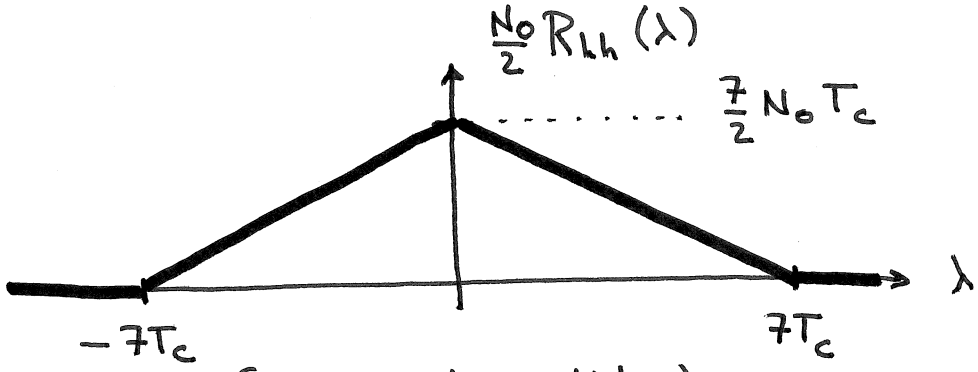
$$\triangleq R_{hh}(\lambda) = \int_{-\infty}^{\infty} h(\tau + \lambda)\, h(\tau)\, d\tau$$

In the integral change variables $s = \ell T_c - \tau \rightarrow -\tau = -\ell T_c + s$

$ds = -d\tau$

$$\Longrightarrow E\{N_k N_\ell\} = \frac{N_0}{2}\int_{-\infty}^{\infty} h(kT_c - \ell T_c + s)\, h(s)\, ds = \frac{N_0}{2} R_{hh}(\lambda)\Big|_{\lambda=(k-\ell)T_c}$$

The impulse response $h(t)$ is rectangular so its auto-correlation is triangular



$$\frac{N_0}{2} R_{hh}(\lambda) = \begin{cases} \frac{7}{2} N_0 T_c \left(1 - \frac{|\lambda|}{7T_c}\right) & 0 \le |\lambda| \le 7T_c \\ 0 & \text{else.} \end{cases}$$

$$E\{N_k N_\ell\} = \begin{cases} \frac{7}{2} N_0 T_c \left(1 - \frac{|k-\ell|}{7}\right) & 0 \le |k-\ell| \le 7 \\ 0 & \text{else.} \end{cases}$$

Problem 4. (cont'd.) Name: _____

(b) [20 pts.] Outline the design of the vector decision device starting from the likelihood ratio test. You do not need to simplify the test statistic but you must be very clear about the design of the unsimplified test. For example, how wide should the parallel-out shift register be in order to capture all relevant statistical information? What is the value of the threshold in the vector input likelihood ratio test for the case here considered of equally likely hypotheses and minimum probability of error criterion?

Let μ_0 and μ_1 denote the corresponding mean vectors for each hypothesis. The pdfs are of the form:

$$f_i(z) = (2\pi)^{-\frac{L+1}{2}} |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(z-\mu_i)^T \Sigma^{-1} (z-\mu_i)\right\}$$

where

$$z = [z_0, z_1, \dots z_L]^T$$

and

$$\mu_0 = [a_0 * h(0),\ a_0 * h(T_c) \ \dots\ a_0 * h(LT_c)]^T$$

$$\mu_1 = [a_1 * h(0),\ a_1 * h(T_c) \ \dots\ a_1 * h(LT_c)]^T$$

$$\Sigma = \left[E\{N_k N_\ell\}\right]_{k,\ell=0}^{L}$$

$$= \frac{7}{2} N_0 T_c \begin{bmatrix} 1 & 6/7 & 5/7 & \cdots & 1/7 & 0 & \cdots & 0 \\ 6/7 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 5/7 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 1/7 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & \end{bmatrix} \quad (L+1)\times(L+1)$$

which is the model for the vector observation

$$z = \begin{Bmatrix} \mu_0 \\ \text{or} \\ \mu_1 \end{Bmatrix} + N$$

$$N \sim N(0, \Sigma)$$

The generic L.R.T. is of the form

$$\frac{f_1(z)}{f_0(z)} \underset{(H_0)}{\overset{(H_1)}{\gtrless}} \tau = 1 \Longleftarrow$$ threshold equals one for uniform costs and equal priors.

$$= \exp\left\{-\frac{1}{2}\left[(z-\mu_1)^T\Sigma^{-1}(z-\mu_1) - (z-\mu_0)^T\Sigma^{-1}(z-\mu_0)\right]\right\}$$

$$= \exp\left\{(\mu_1-\mu_0)^T\Sigma^{-1}\left(z - \frac{\mu_0+\mu_1}{2}\right)\right\}$$

Take the natural log of both sides of the test to have an equiv. test

$$\underbrace{(\mu_1-\mu_0)^T\Sigma^{-1}}_{B} z \underset{(H_0)}{\overset{(H_1)}{\gtrless}} \underbrace{\ln(1)}_{0} + \frac{1}{2}(\mu_1-\mu_0)^T\Sigma^{-1}(\mu_1+\mu_0)$$

There are more simplifications that can be found, though we don't need them here.

More interesting would be to compute the performance. For this need statistical description of

$$BZ$$

under H_0 and H_1. (First, compute a bound, then return).

Problem 4. (cont'd.) Name: _____

(c) [15 pts.] Find an obvious (but tight) bound on the performance of the optimal vector decision device designed in (b). Can the receiver of (b) actually meet the bound?

We know that a matched filter, sampler @ $t = 7T_c = T$, and a simple threshold test is optimal and we have the formula

$$P_e = Q\left(\sqrt{E_b(1-r)/N_0}\right)$$

for the average performance (avg. bit error prob.) in terms of

$$E_b = \frac{\mathcal{E}_0 + \mathcal{E}_1}{2} \quad ; \quad r = \langle s_0, s_1 \rangle / E_b$$

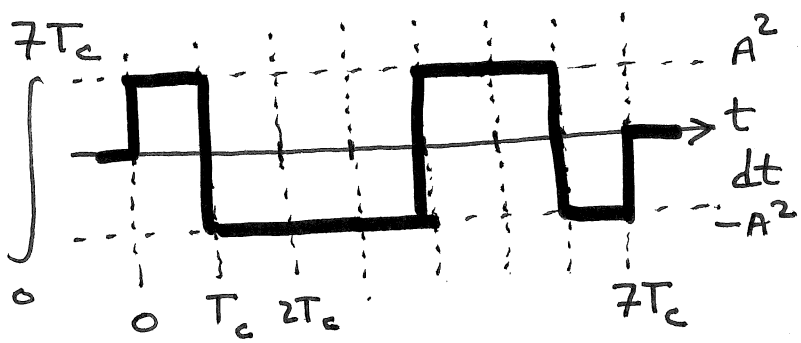
This formula did not apply to the previous version of this problem because the filter was not matched. It is clearly a bound on the performance of our new multi-sample receiver.

Here $$\mathcal{E}_0 = \int_0^{7T_c} a_0^2(t)\,dt = \mathcal{E}_1 = \int_0^{7T_c} a_1^2(t)\,dt$$

$$= A^2 7T_c$$

$$\Rightarrow E_b = 7A^2T_c$$

$$\langle s_0, s_1 \rangle = \int_0^{7T_c} a_0(t)a_1(t)\,dt =$$



$$= -A^2T_c$$

$$\Rightarrow r = -A^2T_c / A^2 7T_c = -\frac{1}{7}$$

$$\Rightarrow \frac{E_b(1-r)}{N_0} = \frac{7A^2T_c\left(1+\frac{1}{7}\right)}{N_0} = 8A^2T_c/N_0$$

$$P_e = Q\left(\sqrt{\frac{8A^2T_c}{N_0}}\right)$$

Re: Can Vector Receiver Meet Bound?

Extra Pages. Name: _____

$$E_0\{BZ\} = BE_0\{Z\} = B\mu_0 = (\mu_1-\mu_0)^T\Sigma^{-1}\mu_0 \triangleq \tilde{\mu}_0$$
$$E_1\{BZ\} = (\mu_1-\mu_0)^T\Sigma^{-1}\mu_1 \triangleq \tilde{\mu}_1$$

$$\mathrm{Var}_0\{BZ\} = E_0\{(BZ-B\mu_0)^2\} = E_0\{(B(Z-\mu_0))^2\}$$
$$= E\{(BN)^2\} = E\{BNN^TB^T\} = B\Sigma B^T$$
$$= (\mu_1-\mu_0)^T\Sigma^{-1}\Sigma\Sigma^{-1}(\mu_1-\mu_0) = (\mu_1-\mu_0)^T\Sigma^{-1}(\mu_1-\mu_0)$$
$$\triangleq d^2 \longrightarrow$$ Turns out this is also equal to $\mathrm{Var}_1\{BZ\}$.

$$P_{e,0} = P\left(BZ > \tfrac{1}{2}B(\mu_1+\mu_0) \mid H_0\right)$$
$$= P\left(\frac{BZ-B\mu_0}{d} > \frac{\frac{1}{2}B(\mu_1+\mu_0)}{d} - \frac{B\mu_0}{d} \;\middle|\; H_0\right)$$
$$= P\left(\frac{BZ-B\mu_0}{d} > \underbrace{\frac{\frac{1}{2}B(\mu_1-\mu_0)}{d}}_{\frac{1}{2}d^2/d \,=\, d/2} \;\middle|\; H_0\right)$$

$$\therefore P_{e,0} = Q(d/2)$$

Of course it also turns out that $P_{e,0} = P_{e,1}$.

Re:
Can the receiver meet the (cont'd.) bound?

Extra Pages. Name: _____

$$d = \sqrt{(\mu_1 - \mu_0)^T \Sigma^{-1} (\mu_1 - \mu_0)}$$

Try to simplify. Write

$$\mu_1 - \mu_0 = (2AT_c) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ -2 \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \triangleq (2AT_c)\,\underline{x} \qquad (15\times 1)$$

$\Sigma = \frac{7}{2} N_0 T_c \, \underline{C}$ where $\underline{C}$ = Toeplitz matrix with 1st row

$$\begin{bmatrix} 1 & 6/7 & 5/7 & 4/7 & 3/7 & 2/7 & 1/7 & 0 \cdots 0 \end{bmatrix}$$

Then

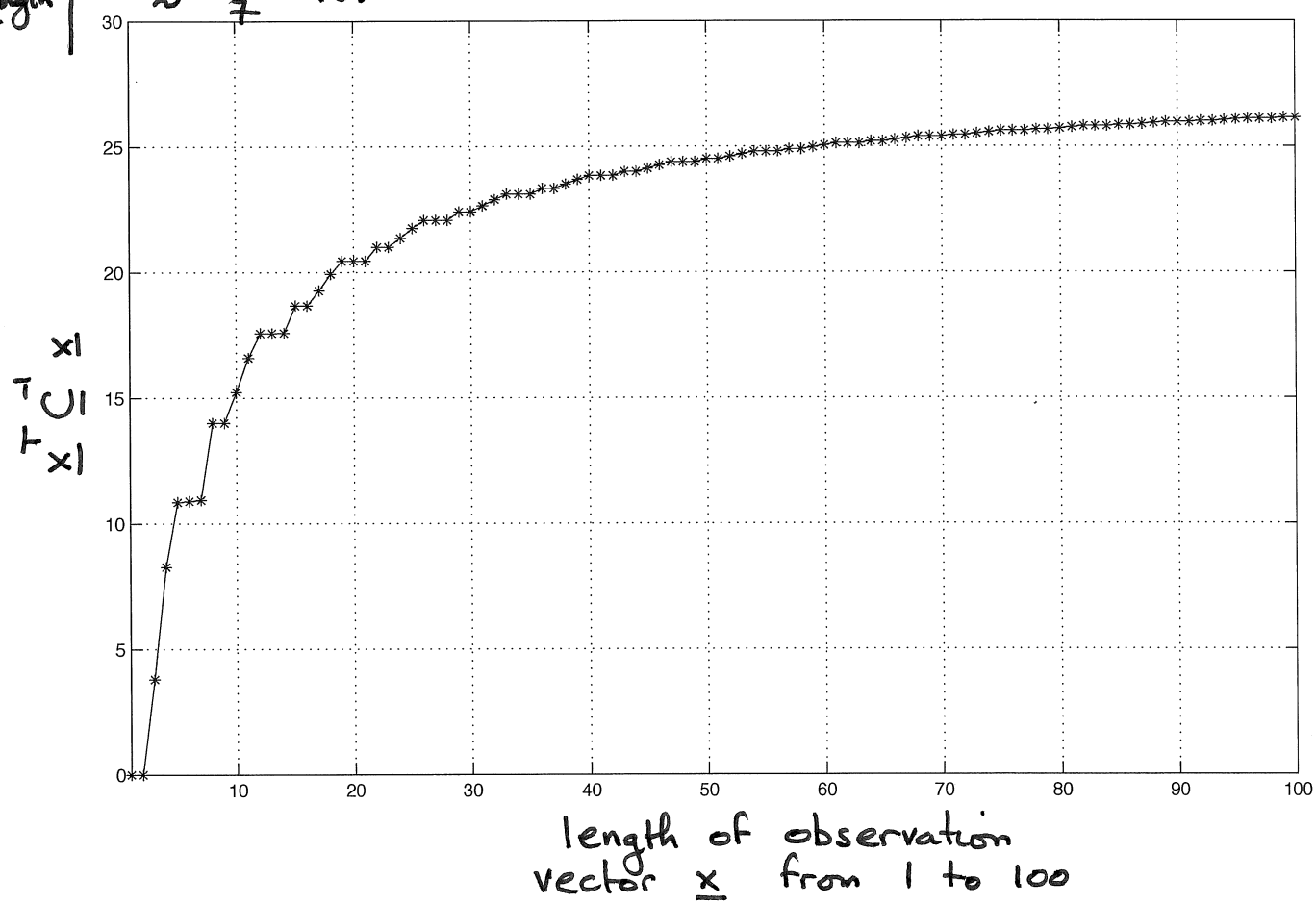
$$\frac{d}{2} = \frac{1}{2}\sqrt{(2AT_c)^2 \cdot \frac{1}{\frac{7}{2} N_0 T_c} \, \underline{x}^T \underline{C}^{-1} \underline{x}}$$

$$= \sqrt{\frac{2}{7} \frac{A^2 T_c}{N_0} \cdot \underline{x}^T \underline{C}^{-1} \underline{x}}$$

Can experiment with $\underline{x}^T \underline{C}^{-1} \underline{x}$ by letting L increase and get the attached plot. (of course can't do this on the test)

Comparing with MF bound we see a loss diminishing with observation length:

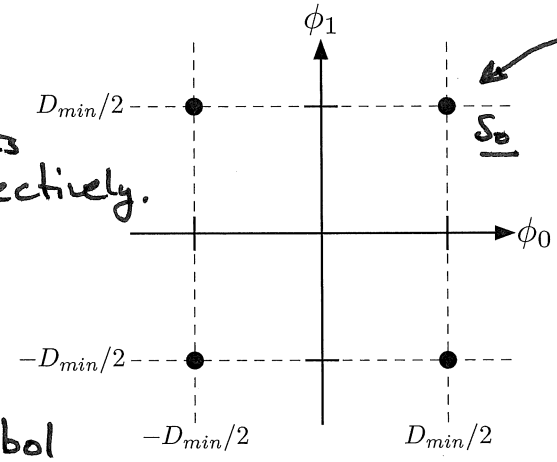
	factor multiplying A^2T_c/N_0
MF	8
vect. proc. @ length 100	$\approx \frac{2}{7} \cdot 27 = 7.71$
@ length 15	$\approx \frac{2}{7} \cdot 18.7 = 5.34$



∴ It gets very close.

Problem 5. [45 points] Name: **Solution**

Consider the 4-ary QASK constellation shown below, which has been defined in terms of a pair of orthonormal signals $\phi_0(t)$ and $\phi_1(t)$. Assume these signals are received in AWGN of power spectral density height $N_0/2$. Assume equally likely symbols and maximum likelihood receiver.



no loss of gen. to assume this is the transmitted symbol. Under this assumption

$Z_0 \sim N\left(\frac{D_{min}}{2}, \frac{N_0}{2}\right)$

$Z_1 \sim N\left(\frac{D_{min}}{2}, \frac{N_0}{2}\right)$

Let Z_0 and Z_1 be correlator outputs for ϕ_0 and ϕ_1, respectively.

Then $Z_0 \perp\!\!\!\perp Z_1$ and Gaussian of var $\sigma^2 = N_0/2$ and means depending on the symbol transmitted.

There are three parts to this question.

(a) [15 pts.] Find the average probability of a symbol error in terms of the Gaussian Q function, D_{min}, and N_0. Explain.

The average prob. of a symbol error is equal to the conditional prob. of a symbol error for any of the four symbols.

$$P(\text{symb. rec. correctly} \mid \underline{s_0} \text{ sent}) = P(Z_0 > 0, Z_1 > 0 \mid \underline{s_0} \text{ sent})$$

$$= P(Z_0 > 0 \mid \underline{s_0} \text{ sent}) P(Z_1 > 0 \mid \underline{s_0} \text{ sent}) = P(Z_0 > 0 \mid \underline{s_0} \text{ sent})^2$$

Now

$$P(Z_0 > 0 \mid \underline{s_0} \text{ sent}) = P\left(\frac{Z_0 - \frac{D_{min}}{2}}{\sqrt{N_0/2}} > \frac{-D_{min}/2}{\sqrt{N_0/2}}\right) = Q\left(\frac{-D_{min}/2}{\sqrt{N_0/2}}\right)$$

$$= \Phi\left(\frac{D_{min}/2}{\sqrt{N_0/2}}\right) = 1 - Q\left(\frac{D_{min}/2}{\sqrt{N_0/2}}\right)$$

$$\therefore P_{correct} = \left[1 - Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)\right]^2$$

$$= 1 - 2Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right) + Q^2\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

$$P_e = 1 - P_{correct}$$
$$= 2Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right) - Q^2\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

Often want to write answer in terms of the average bit energy or average symbol energy.

$$\|\underline{s_0}\|^2 = \frac{D_{min}^2}{4} + \frac{D_{min}^2}{4} = \frac{D_{min}^2}{2} = \mathcal{E}_s$$

(all symbols have same energy here)

A symbol carries 2 bits. So average bit energy is

$$E_b = \mathcal{E}_s/2 = \frac{D_{min}^2}{4}$$

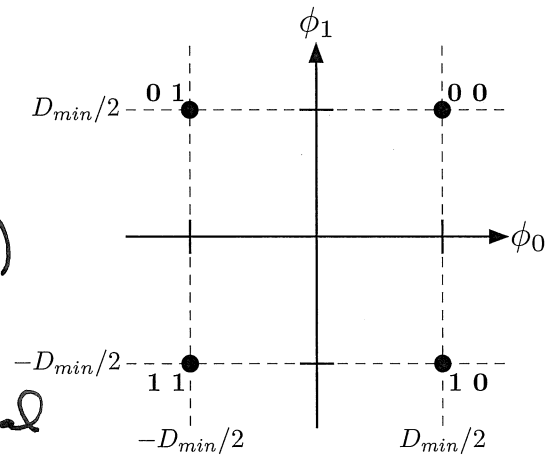
$$\Longrightarrow \quad P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Problem 5. (cont'd.) Name: _____

(b) [15 pts.] Now assuming the same constellation and the same assumptions but with the bit labeling shown below find the average probability of a bit error for

1. the first bit, and
2. the second bit.

This is called Gray Labelling.



Let $N \sim \mathcal{N}(0, N_0/2)$ which is projection of noise comp. along any orthonormal axis.

$\boxed{\text{1. First Bit}}$ $P\left(\begin{matrix}\text{1st bit}\\ \text{incorrect}\end{matrix} \,\middle|\, \text{00 sent}\right) = P\left(N > \frac{D_{min}}{2}\right)$

$$= P\left(\frac{N}{\sqrt{N_0/2}} > \frac{D_{min}/2}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

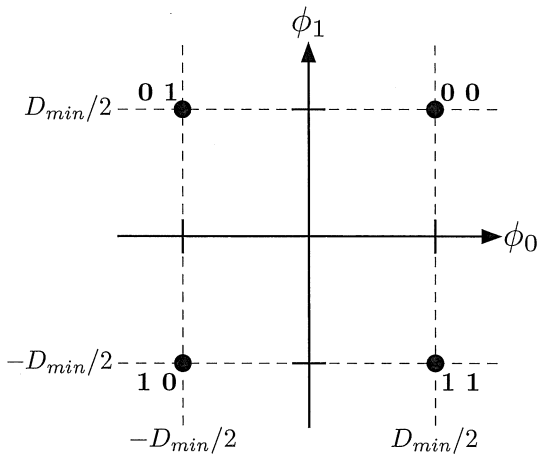
$$= P\left(\begin{matrix}\text{1st bit}\\ \text{incorrect}\end{matrix} \,\middle|\, \begin{matrix}\text{01 sent}\\ \vdots \\ \text{or}\\ \text{11 sent, 10 sent}\end{matrix}\right)$$

$\therefore$ P_{b1} = average prob. of bit error in bit one is $Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$\boxed{\text{2. Second Bit}}$ Looking at the geometry its obvious that $P_{b2} = P_{b1}$ for Gray Labelling.

Problem 5. (cont'd.) Name: _____

(c) [15 pts.] Repeat (b) for the bit labeling shown below.



1. First Bit

Note that first bit labelling is identical to previous case and therefore

$$P_{b1} = Q\left(\sqrt{D_{min}^2/2N_0}\right)$$

2. Second Bit

From symmetry can see that

$$\underbrace{P(\text{2nd bit incorrect} \mid 00)}_{} = P\left(\begin{matrix}\text{2nd bit}\\ \text{incorrect}\end{matrix} \,\middle|\, \begin{matrix}\text{any one of}\\ \text{the symbols}\\ \text{sent}\end{matrix}\right)$$

$$= P\left(\begin{matrix}\text{choose}\\ 01\end{matrix} \,\middle|\, 00 \text{ was sent}\right) + P\left(\begin{matrix}\text{choose}\\ 10\end{matrix} \,\middle|\, 00 \text{ was sent}\right)$$

$$= P\left(Z_0 < 0, Z_1 > 0 \,\middle|\, \begin{matrix}00 \text{ was}\\ \text{sent}\end{matrix}\right) + P\left(Z_0 > 0, Z_1 < 0 \,\middle|\, \begin{matrix}00 \text{ was}\\ \text{sent}\end{matrix}\right)$$

$$= P(Z_0 < 0 \mid 00)P(Z_1 > 0 \mid 00) + P(Z_0 > 0 \mid 00)P(Z_1 < 0 \mid 00)$$

Problem 5. (cont'd.) Name: _____

As before

$$P(Z_0 > 0 \mid 00) = P(Z_1 > 0 \mid 00) = 1 - Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

$$P(Z_0 < 0 \mid 00) = P(Z_1 < 0 \mid 00)$$

$$= P\left(\frac{Z_0 - D_{min}/2}{\sqrt{N_0/2}} < \frac{-D_{min}/2}{\sqrt{N_0/2}}\right) = \Phi\left(\frac{-D_{min}/2}{\sqrt{N_0/2}}\right)$$

$$= Q\left(\frac{D_{min}/2}{\sqrt{N_0/2}}\right)$$

$$\therefore \; P_{b2} = 2Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)\left[1 - Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)\right]$$

$$P_{b2} = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]$$

Problem 6. [30 points] Name: Solution

In a homework problem on binary, equal-energy FSK in fading the received signal was modeled as

$$Y(t) = \sqrt{2}V\cos(\omega_i t + \phi_i) + N(t), \quad 0 \le t \le T,$$

for $i = 0$ (bit '0' sent), $i = 1$ (bit '1' sent), and $N(t)$ a white Gaussian noise of power spectral density height $N_0/2$. The amplitude term V was modeled as a Gaussian random variable independent of the noise and the actual signal transmitted. Here we model $V \sim \mathcal{N}(\nu, \sigma^2)$.

The conditional error probability given $V = v$ was shown to be

$$P_e(v) = \frac{1}{2}e^{-v^2T/2N_0}.$$

Average over the distribution of V to find

$$\bar{P}_e = \mathrm{E}\{P_e(V)\}$$

and simplify as much as possible.

$$E\{P_e(V)\} = \int_{-\infty}^{\infty} \frac{1}{2} e^{-v^2T/2N_0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(v-\nu)^2/2\sigma^2}\, dv$$

$$= \frac{1}{2\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{T}{2N_0}v^2 - \frac{1}{2\sigma^2}(v-\nu)^2\right] dv$$

Expanding the argument of the exponential in order to use the completing of the square method to evaluate the integral:

$$-\frac{T}{2N_0}v^2 - \frac{1}{2\sigma^2}(v^2 - 2\nu v + \nu^2) = -\underbrace{\left(\frac{T}{2N_0} + \frac{1}{2\sigma^2}\right)}_{A} v^2 + \underbrace{\frac{\nu}{\sigma^2}}_{B} v - \underbrace{\frac{1}{2\sigma^2}\nu^2}_{C}$$

$$= -Av^2 + Bv - C = -A\left[v^2 - \frac{B}{A}v\right] - C$$

$$= -A\left[v^2 - \frac{B}{A}v + \frac{B^2}{4A^2} - \frac{B^2}{4A^2}\right] - C$$

$$= -A\left(v - \frac{B}{2A}\right)^2 + \frac{B^2}{4A} - C$$

Problem 6. (cont'd.) Name: _____

$\therefore$ Can write

$$E\{P_e(V)\} = \frac{1}{2\sqrt{2\pi}\,\sigma} e^{B^2/4A} e^{-C} \int_{-\infty}^{\infty} e^{-A(v - B/2A)^2} dv$$

$A = \frac{1}{2}\left[\frac{T}{N_0} + \frac{1}{\sigma^2}\right]$ and define $\tilde{\sigma}$ by

$$\frac{1}{\tilde{\sigma}^2} = \frac{T}{N_0} + \frac{1}{\sigma^2} \implies \tilde{\sigma}^2 = \left[\frac{T}{N_0} + \frac{1}{\sigma^2}\right]^{-1} = \frac{N_0\sigma^2}{\sigma^2 T + N_0}$$

$$\implies A = \frac{1}{2\tilde{\sigma}^2} \iff \tilde{\sigma}^2 = \frac{1}{2A} \iff \tilde{\sigma} = \frac{1}{\sqrt{2A}}$$

Now should be able to write

$$E\{P_e(V)\} = \frac{1}{2\sigma} e^{B^2/4A} e^{-C} \tilde{\sigma} \underbrace{\frac{1}{\sqrt{2\pi}\,\tilde{\sigma}} \int_{-\infty}^{\infty} e^{-(v - B/2A)^2/2\tilde{\sigma}^2} dv}_{=1}$$

Simplifying

$$\frac{B^2}{4A} = \frac{v^2}{\sigma^4}\,\frac{1}{4}\,2\tilde{\sigma}^2 = \frac{v^2\tilde{\sigma}^2}{2\sigma^4}$$

$$= \frac{v^2}{2\sigma^4} \cdot \frac{N_0\sigma^2}{\sigma^2 T + N_0} = \frac{v^2 N_0}{2\sigma^2(\sigma^2 T + N_0)}$$

$$C = \frac{v^2}{2\sigma^2}$$

$$\frac{\tilde{\sigma}}{2\sigma} = \frac{1}{2\sigma}\sqrt{\frac{N_0\sigma^2}{\sigma^2 T + N_0}}$$

$\therefore$ Average prob. of error

$$\overline{P}_e = E\{P_e(V)\}$$

$$= \frac{1}{2\sigma}\sqrt{\frac{N_0\sigma^2}{\sigma^2 T + N_0}}\ \exp\left\{\frac{v^2 N_0}{2\sigma^2(\sigma^2 T + N_0)} - \frac{v^2}{2\sigma^2}\right\}$$

$$= \frac{1}{2\sigma}\sqrt{\frac{N_0\sigma^2}{\sigma^2 T + N_0}}\ \exp\left\{\frac{v^2}{2\sigma^2}\left[\frac{N_0}{\sigma^2 T + N_0} - 1\right]\right\}$$

$$= \frac{1}{2\sigma}\sqrt{\frac{N_0\sigma^2}{\sigma^2 T + N_0}}\ \exp\left\{-\frac{v^2}{2\sigma^2}\,\frac{\sigma^2 T}{\sigma^2 T + N_0}\right\}$$

$$= \frac{1}{2\sigma}\sqrt{\frac{N_0\sigma^2}{\sigma^2 T + N_0}}\ \exp\left\{-\frac{v^2}{2}\,\frac{T}{\sigma^2 T + N_0}\right\}$$

Problem 7. [15 points] Name: Solution

Suppose a radio channel of bandwidth equal to 20 MHz, i.e., the channel passes frequencies f of the form

$$f_c - 10 \text{ MHz } \leq |f| \leq f_c - 10 \text{ MHz}$$

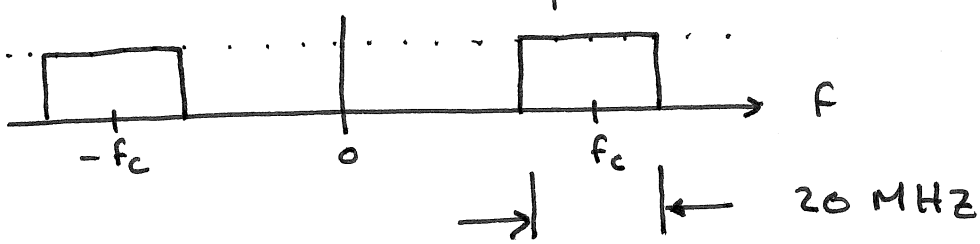
for some center frequency $f_c \gg 10$ MHz. We wish to use this channel for BPSK signalling where the transmitted signal is of the form

$$s(t) = \sqrt{2}A \sum_n d_n g(t - nT) \cos(2\pi f_c t)$$

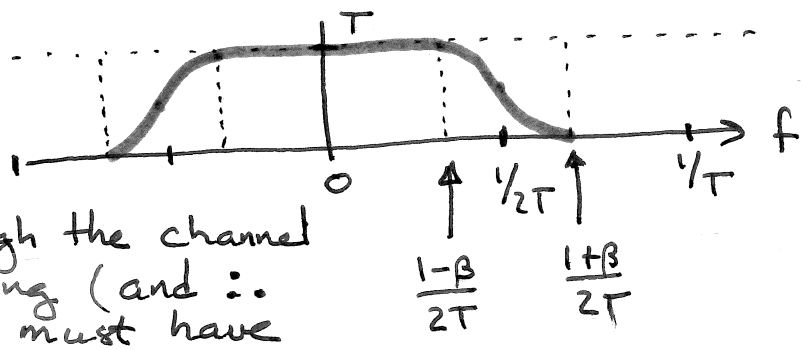
and $g(t)$ is a baseband square-root raised cosine pulse with bandwidth expansion factor $\beta = 0.4$. Assume that the receiver uses filtering matched to the pulse $g(t)$ and that receiver-side symbol timing is perfect.

What is the maximum bit rate that can be supported provided that ISI free reception is required?

The channel looks like a bandpass filter



When downconverted and matched filter the resulting baseband pulse will be a raised cosine with a baseband spectrum of form



To pass through the channel without filtering (and ∴ without ISI) must have

$$\frac{1+\beta}{2T} \leq 10 \text{ MHz} \implies \frac{1}{T} \leq \frac{2 \cdot 10 \text{ MHz}}{1.4} = 14.29 \text{ Mbps}$$