Problem 1. [35 points]

You are the designer of a communications receiver. The transmitter designer has told you that the system is for binary communications and that a signal \( s_0(t) \) is used to send a ‘0’ bit, and \( s_1(t) \) is used to send a ‘1’ bit. The signals are shown below. Note that the voltage axes are labelled in micro-volts (i.e., \( 10^{-6} \) volts) and the time axes are labelled in micro-seconds. For power / energy calculations assume that the voltages are developed across a nominal 1 Ohm resistor.

\[ s_0(t) \]

\[ s_1(t) \]

The signal you receive is \( Y(t) = s_i(t) + X(t) \) where \( X(t) \) is white Gaussian noise of power spectral density height equal to \( N_0/2 = -198.6 \) dBJ, and \( i = 0 \) or \( = 1 \) depending on the bit that is actually sent.

There are three parts to this question.

(a) [15 pts.] Assuming that the bits are equally likely, draw the architecture of the minimum probability of error receiver for this problem. You do not have to derive it from first principles, but you must explain it and specify the value of every parameter, filter, sampler, and/or threshold. Your answer will be a block diagram that takes input \( Y(t) \) and produces a decision on which bit (‘0’ or ‘1’) that the receiver thinks was sent.

From class we know that the optimal architecture is a matched filter followed by a sampler and threshold device.

\[ Y(t) \]

To specify: \( h(t) \), \( T_0 \), \( \gamma \) and the form of the decision.
Filter: Should be a matched filter

\[ h(t) = s_1(T_0 - t) - s_0(T_0 - t) \quad \ldots \quad \text{can have an arbitrary scale factor.} \]

Sample Time: Should pick \( t = T_0 \) to sample and say \( T_0 \geq 1.2 \) for causality.

Threshold and Decision: Need the signal terms at the sampler output to choose threshold and the sense of the test

\[ \hat{s}_0(T_0) = s_0 * h(T_0) = \int s_0(\tau) h(t-\tau) \, d\tau \bigg|_{t=T_0}^{t=120\,\mu s} \]
\[ = \int_{0}^{120\,\mu s} s_0(\tau) [s_1(\tau) - s_0(\tau)] \, d\tau = \int_{0}^{120\,\mu s} s_0(\tau) s_1(\tau) \, d\tau - \int_{0}^{120\,\mu s} s_0(\tau) \, d\tau \]

Evaluate the pieces:

\[ \int_{0}^{120\,\mu s} s_0^2(\tau) \, d\tau = \int_{0}^{120\,\mu s} (1\,\mu V)^2 \, \frac{1}{2} \cdot 100\,\mu s \]
\[ \int_{0}^{120\,\mu s} s_0(\tau) s_1(\tau) \, d\tau = 0 \quad \text{since} \ s_0 \ \text{and} \ s_1 \ \text{are orthogonal.} \]
\[ S_1(T_0) = s_1 \ast h(T_0) = \int_0^{120 \mu s} s_1(t) [s_1(t) - s_0(t)] \, dt \]

\[ = \int_0^{120 \mu s} s_1^2(t) \, dt - \int_0^{120 \mu s} s_1(t) s_0(t) \, dt = \int_0^{120 \mu s} s_1^2(t) \, dt \]

\[ = \frac{(1 \mu V)^2}{1 \Omega} \cdot 120 \mu s \]

\[ S_0(T_0) = -E_0 = -1 \times 10^{-16} \, J \]

\[ S_1(T_0) = \varepsilon_1 = +1.2 \times 10^{-16} \, J \]

The threshold is

\[ \gamma_m = \frac{-E_0 + \varepsilon_1}{2} = 1 \times 10^{-17} \, J \]

And the test should be

\[ Z(T_0) \geq Y = 1 \times 10^{-17} \, J \quad \text{say } H_1 \]

\[ < \]

\[ \text{say } H_0 \]
Problem 1. (cont'd.)

(b) [10 pts.] Now in class we showed that the minimum probability of error for the receiver of part (a) was given by the Q-function expression:

\[ P_e = Q\left(\sqrt{\frac{E_b(1-r)}{N_0}}\right), \]

where \( E_b \) is the average energy per bit and \( r = \langle s_0, s_1 \rangle / E_b \) is the normalized correlation between the two signals. For the signals above, evaluate this probability of error and give an approximate numerical value.

\[
E_b = \frac{E_0 + E_1}{2} = \frac{1 \times 10^{-16} + 1.2 \times 10^{-16}}{2} = 1.1 \times 10^{-16} \text{ J}
\]

\( r = 0 \) since the signals are orthogonal

\[
\left(\frac{N_0/2}{E_b}\right)_{\text{dB}} = -19.86 \text{ dB} \Rightarrow 10 \log_{10} \left(\frac{N_0/2}{E_b}\right) = -19.86
\]

\[ \Rightarrow \frac{N_0}{2} = \frac{-19.86}{10} = 1.986 \text{ J} \]

\[ \Rightarrow N_0 = 2 \times 1.986 \text{ J} = 2.76 \times 10^{-20} \text{ J} \]

\[
\text{Arg of } \frac{Q}{\sqrt{\frac{E_b(1-r)}{N_0}}}
\]

\[
= \sqrt{\frac{1.1 \times 10^{-16}}{2.76 \times 10^{-20}}} = 63.12
\]

\( Q(63.12) \approx 0 \)
Problem 1. (cont'd.)

Name: ____________________________

(c) [10 pts.] Assume that the signal \( Y(t) \) described in part (a) is measured at the point shown in the block diagram above. Given the values for the assumed input noise source and the noise figure and gain of the pre-amplifier, find the noise figure and equivalent noise temperature of the system you designed in part (a). This amounts to reconciling the power spectral density height given for \( X(t) \) with the other noise information from the block diagram.

\[
\text{Preamp: } \quad G_{\text{dB}} = 10 \text{dB} \implies G = 10^{10/10} = 1.41
\]

\[
NF_{\text{dB}} = 1.5 \text{dB} \implies NF = 10^{1.5/10} = 10
\]

\[
T_{PA} = 290 \text{K} (NF - 1) = (290 \text{K})(1.41) = 118.9 \text{K}
\]

The noise sources are

\[
(40 \text{K} + 118.9 \text{K}) \text{10} + T_{\text{system}} = \text{temp corresp. to } \frac{N_0}{2} = -198.6 \text{ dB } \frac{\text{J}}{\text{Hz}}
\]

\[
T_{\text{system}} = 2000 \text{K} - 1589 \text{K} = 411 \text{K}
\]

\[
F_{\text{system}} = \frac{411}{290} + 1 = 2.42
\]

\[
= 3.83 \text{ dB}
\]
Problem 2. [15 points]

The input $X(t)$ to the system shown above is a zero mean, Gaussian random process, with triangularly shaped autocorrelation function:

$$R_X(r) = \begin{cases} 
1 - \frac{|r|}{5} & 0 \leq |r| \leq 5 \\
0 & \text{else}
\end{cases}$$

By adjusting the value of the delay $T$ how much can you change the average duty cycle of the random 0 – 1 square wave at the output of the system? Note that the final block is a memoryless non-linearity.

$X(t) \sim N(0, 1)$ with auto-correlation $R_X(r)$. Input to memoryless non-linearity is $Z(t) = X(t) - X(t-T)$, which must be Gaussian since its a linear function of a Gaussian r.p. Clearly $Z(t)$ has zero mean. Its variance is

$$E\{Z^2(t)\} = E\left\{ (X(t) - X(t-T))^2 \right\}$$

$$= E\{X^2(t)\} - E\{X(t-T)X(t)\} - E\{X(t)X(t-T)\}$$

$$+ E\{X^2(t-T)\}$$

$$= R_X(0) - R_X(T) - R_X(T) + R_X(0)$$

$$= 2 \left[ R_X(0) - R_X(T) \right] = 2 \left[ 1 - R_X(T) \right]$$

$$= \begin{cases} 
\frac{2.4T}{5} & 0 \leq |T| \leq 5 \\
2 & |T| > 5
\end{cases}$$
Problem 2. (cont'd.)

\[ \text{Var}\{Z(t)^2\} \]

\[ \begin{matrix}
\begin{array}{c}
\text{E}\{Y(t)\} = P\{Y(t) = 1\} = 1 - P\{-1 < Z(t) \leq 1\} \\
= 1 - P\left\{ \frac{-1}{\sqrt{\frac{4}{1T^4}}} < \frac{Z(t)}{\sqrt{\frac{4}{1T^4}}} \leq \frac{1}{\sqrt{\frac{4}{1T^4}}} \right\} \quad 0 \leq 1T^4 \leq 5 \\
= 2 \Phi \left( \frac{1}{\sqrt{\frac{4}{1T^4}}} \right) \\
\end{array}
\end{matrix} \]

Thus for \( T \) near zero \( E\{Y(t)\} \approx 0 \) and when \( T \) reaches 5 sec.

\[ \begin{matrix}
\begin{array}{c}
E\{Y(t)\} = 2 \Phi \left( \frac{1}{\sqrt{\frac{4}{1T^4}}} \right) = 2 \Phi \left( \frac{1}{\sqrt{\frac{4}{1T^4}}} \right) \\
\approx 2 \cdot 0.242 \\
\approx 0.48 \\
\text{Duty cycles from zero to 48%}.
\end{array}
\end{matrix} \]
Problem 3. [15 points] Name: ____________________

Below is a part of a Matlab code which simulates the nonlinear baseband model of a PLL with phase detector characteristic \( f(\psi) = \sin(\psi) \). Modify the code to implement the new phase detector characteristic \( f(\psi) \) shown below. The following Matlab commands may be useful:

- `floor(x)`, which rounds a real number \( x \) to the nearest integer less than or equal to \( x \), e.g., `floor(-4.2) = -5`, and `floor(4.2) = 4`.
- `ceil(x)`, which rounds a real number \( x \) to the nearest integer greater than or equal to \( x \), e.g., `ceil(-4.2) = -4`, and `ceil(4.2) = 5`.

![Nonlinear BB model showing phase detector \( f(\cdot) \)](image)

```matlab
%% Beginning of Simulation Loop
for i = 1:npts
    if i < nsettle
        fin(i) = 0;
        phin = 0;
    else
        fin(i) = fdel;
        phin = 2*pi*fdel*T*(i-nsettle);
    end
    s1 = phin - vco_out;
    s2 = sin(s1);
    s3 = Kt*s2;
    filt_in = a*s3;
    filt_out = filt_out_last + (T/2)*(filt_in + filt_in_last);
    filt_in_last = filt_in;
    filt_out_last = filt_out;
    vco_in = s3 + filt_out;
    vco_out = vco_out_last + (T/2)*(vco_in + vco_in_last);
    vco_in_last = vco_in;
    vco_out_last = vco_out;
    phie(i) = s1;
    fvco(i) = vco_in/(2*pi);
    freqerror(i) = fin(i) - fvco(i);
end
%% End of Simulation Loop
```
$$s_1 = \psi \implies \text{want } s_2 = f(\psi) \text{ for the given characteristic } f. \text{ Note that it is periodic in } \psi \text{ of period } 2\pi. \text{ Also note that expression for } f \text{ is simplest for}$$

$$-\frac{\pi}{2} \leq \psi < \frac{3\pi}{2}$$

Case $$-\frac{\pi}{2} \leq \psi < \frac{\pi}{2}$$

$$f(\psi) = m\psi \quad \text{where} \quad m = \frac{2}{\pi}$$

$$\implies f(\psi) = \frac{2}{\pi} \psi \quad \text{for} \quad -\frac{\pi}{2} \leq \psi < \frac{\pi}{2}$$

Case $$\frac{\pi}{2} \leq \psi < \frac{3\pi}{2}$$

$$f(\psi) = -\frac{2}{\pi} \psi + b \quad \text{(negative of slope from first case)}$$

Solve for $$b$$

$$f(\frac{\pi}{2}) = -\frac{2}{\pi} \cdot \frac{\pi}{2} + b = 1$$

$$\implies b = 2$$

$$\implies f(\psi) = -\frac{2}{\pi} \psi + 2 \quad \text{for} \quad \frac{\pi}{2} \leq \psi < \frac{3\pi}{2}$$

Computing $$f(\psi)$$ for an arbitrary $$\psi$$.

Let $$\lfloor x \rfloor = \text{greatest integer } \leq x \quad \text{(floor function)}$$

$$\lfloor -4.2 \rfloor = -5 \quad ; \quad \lfloor 4.2 \rfloor = 4$$

Note

$$\psi = \lfloor \frac{\psi}{2\pi} \rfloor 2\pi + \tilde{\psi}$$
and claim that the remainder
\[ \tilde{\psi} = \psi - \left\lfloor \frac{\psi}{2\pi} \right\rfloor 2\pi \]
satisfies \( 0 \leq \tilde{\psi} < 2\pi \). To see this
\[ \frac{\tilde{\psi}}{2\pi} = \frac{\psi}{2\pi} - \left\lfloor \frac{\psi}{2\pi} \right\rfloor \]
\[ \Rightarrow 0 \leq \frac{\psi}{2\pi} < 1 \Rightarrow \text{claim holds.} \]
\[ \therefore \text{New matlab code} \]
\[
\text{s1 = phin - vco_out;}
\text{s1_tilde = s1 - floor(s1/(2*pi))*2*pi;}
\text{if s1_tilde >= 3*pi/2}
\text{\quad s1_tilde = s1_tilde - 2*pi;}
\text{end if}
\text{if -pi/2 <= s1_tilde < pi/2}
\text{\quad s2 = (2/pi)*s1_tilde;}
\text{else}
\text{\quad s2 = -(2/pi)*s1_tilde + 2;}
\text{end if}
\text{s3 = K_t*s2;}
\]
Problem 4. [55 points]

Note that this problem is a follow on to a problem from Exam 2 where we considered the coherent binary detection problem:

\[ H_0 : s_0(t) = \sqrt{2}a_0(t) \cos(2\pi f_c t + \phi) \text{ is transmitted} \]

versus

\[ H_1 : s_1(t) = \sqrt{2}a_1(t) \cos(2\pi f_c t + \phi) \text{ is transmitted} \]

where \( a_0(t) \) and \( a_1(t) \) are as indicated below. Note that the tic marks are spaced by \( T_c \) and that the signals start at \( t = 0 \) and end at \( t = T = 7T_c \).

The receiver architecture for the previous version of the problem consisted of a mixer followed by an LTI filter with non-optimal impulse response \( h(t) = p_T(t) \), a sampler that sampled at multiples of \( T_c \), and a threshold device as drawn below. In the new problem we will modify the threshold device. The noise \( N(t) \) is AWGN with power spectral density height equal to \( N_0/2 \). You may assume that the carrier frequency is large compared to the chipping rate (i.e., \( f_c \gg 1/T_c \)).

One of the main tasks of the previous version of the problem was to find the graphs of the signal parts at the output of the suboptimal filter \( h(t) = p_T(t) \). The answer was as shown in the next figure.
The point of this new problem is to consider changing the receiver architecture to that shown below where the simple threshold device is replaced by a processor that can work with multiple samples at the same time.

The diagram shows the signal processing steps:

1. $s(t)$ is input to the system.
2. The signal is multiplied by $\sqrt{2}\cos(2\pi f_c t + \phi)$.
3. The processed signal $N(t)$ is then passed through an LTI system with impulse response $h(t)$.
4. The output of the LTI system is $Z_k = kT_c$ for $k = 0, 1, 2, \ldots$.
5. The output $Z_k$ is then sent to a vector input decision device.
6. The decision device outputs $H_0$ or $H_1$.

Single sample threshold device from previous problem is replaced by this vector decision device.

There are three parts to this question.
Problem 4. (cont’d.)

(a) [20 pts.] We also found the marginal distributions of the random variables $Z_k$ under both hypotheses. To recall, these were

$$Z_k \sim \begin{cases} 
N(a_0 * h(kT_c), \frac{7}{2}N_0T_c) & \text{under } H_0 \\
N(a_1 * h(kT_c), \frac{7}{2}N_0T_c) & \text{under } H_1 
\end{cases}$$

But in order to design the vector decision device in this problem we need to know the correlation of the samples $Z_k$ and $Z_l$ at different times. Find these correlations and explain.

Let $N_k$ denote the noise part of $Z_k$. That is $N_k = \int \sqrt{2} \mathcal{N}(z) \cos(2\pi f_c z + \phi) h(t-z) \, dz \bigg|_{t = kT_c}$

In the previous problem we found

$$\mathbf{E}\{N_k^2\} = \frac{7}{2}N_0T_c$$

and to compute the correlation $\mathbf{E}\{N_k N_k\}$ we will need

$$2 \int \int \mathbf{E}\{N(z) N(u)\} \cos(2\pi f_c z + \phi) \cos(2\pi f_c u + \phi) h(kT_c-z) h(lT_c-u) \, dz \, du.$$

$$= \frac{N_0}{2} \delta(z-u)$$

$$= N_0 \int \frac{\cos^2(2\pi f_c z + \phi) h(kT_c-z) h(lT_c-z)}{\frac{1}{2} (1 + \cos(2\pi 2f_c z + 2\phi))} \, dz$$

$$\approx \frac{N_0}{2} \int h(kT_c-z) h(lT_c-z) \, dz$$

This is proportional to the deterministic auto correlation of $h(t)$.

$$\frac{A}{-1} = R_{hh}(\lambda) = \int_{-\infty}^{\infty} h(z+\lambda) h(z) \, dz$$
In the integral change variables \( S = \lambda T_c - \tau \rightarrow -\tau = -\lambda T_c + s \)
\[ ds = -d\tau \]

\[
E\{N_k N_{k'}^2\} = \frac{N_o}{2} \int_{-\infty}^{\infty} h(\lambda T_c - \lambda T_c + s) h(s) \, ds = \left. \frac{N_o R_{hh}(\lambda)}{2} \right|_{\lambda = (k-k')T_c}
\]

The impulse response \( h(t) \) is rectangular so its autocorrelation is triangular

\[
\frac{N_o R_{hh}(\lambda)}{2} = \begin{cases} \frac{7}{2}N_o T_c \left(1 - \frac{1}{7T_c}\right) & 0 \leq |\lambda| \leq 7T_c \\ 0 & \text{else} \end{cases}
\]

\[
E\{N_k N_{k'}^2\} = \begin{cases} \frac{7}{2}N_o T_c \left(1 - \frac{|k-k'|}{7}\right) & 0 \leq |k-k'| \leq 7 \\ 0 & \text{else} \end{cases}
\]
(b) [20 pts.] Outline the design of the vector decision device starting from the likelihood ratio test. You do not need to simplify the test statistic but you must be very clear about the design of the unsimplified test. For example, how wide should the parallel-out shift register be in order to capture all relevant statistical information? What is the value of the threshold in the vector input likelihood ratio test for the case here considered of equally likely hypotheses and minimum probability of error criterion?

Let \( \mu_0 \) and \( \mu_1 \) denote the corresponding mean vectors for each hypothesis. The pdfs are of the form:

\[
f_i(z) = (2\pi)^{-L/2} |\Sigma|^{-1/2} \exp \left\{ -\frac{1}{2} (z - \mu_i)^T \Sigma^{-1} (z - \mu_i) \right\}
\]

where

\[
z = \left[ z_0, z_1, \ldots, z_L \right]^T
\]

and

\[
\mu_0 = \left[ a_0 \ast h(s), a_0 \ast h(T_c), \ldots, a_0 \ast h(LT_c) \right]^T
\]

\[
\mu_1 = \left[ a_1 \ast h(s), a_1 \ast h(T_c), \ldots, a_1 \ast h(LT_c) \right]^T
\]

\[
\Sigma = \left[ E\{N_k N_L^2\} \right]_{k,l=0}^L
\]

\[
= \frac{1}{L} \begin{bmatrix}
1 & \frac{1}{7} & \frac{5}{7} & \ldots & \frac{1}{7} & 0 & \ldots & 0 \\
\frac{1}{7} & \frac{1}{7} & \frac{5}{7} & \ldots & \frac{1}{7} & 0 & \ldots & 0 \\
\frac{5}{7} & \frac{1}{7} & \ldots & \frac{5}{7} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{7} & 0 & \ldots & 0 & \ldots & \frac{5}{7} & \ldots & \frac{1}{7} \\
0 & \frac{1}{7} & \ldots & 0 & \ldots & \frac{1}{7} & \ldots & \frac{5}{7} \\
\end{bmatrix}_{(L+1) \times (L+1)}
\]

which is the model for the vector observation

\[
z = \begin{bmatrix} \mu_0 \\ \mu_1 \end{bmatrix} + N
\]

\( N \sim N(0, \Sigma) \)
The generic L.R.T. is of the form
\[
\frac{f_1(z)}{f_0(z)} \begin{cases} (H_1) \\ (H_0) \end{cases} \begin{cases} > \quad z = 1 \rightarrow \text{threshold equals one for uniform costs and equal priors.} \\ < \end{cases}
\]
\[
= \exp \left\{ -\frac{1}{2} \left[ (z - \mu_1)^\top \Sigma^{-1} (z - \mu_1) - (z - \mu_0)^\top \Sigma^{-1} (z - \mu_0) \right] \right\}
\]
\[
= \exp \left\{ (\mu_1 - \mu_0)^\top \Sigma^{-1} (z - \frac{\mu_0 + \mu_1}{2}) \right\}
\]

Take the natural log of both sides of the test to have an equiv. test
\[
(\mu_1 - \mu_0)^\top \Sigma^{-1} z > \ln(1) + \frac{1}{2} (\mu_1 - \mu_0)^\top \Sigma (\mu_1 + \mu_0)
\]
\[
\frac{B}{B^0} \begin{cases} (H_1) \\ (H_0) \end{cases}
\]

There are more simplifications that can be found, though we don’t need them here.

More interesting would be to compute the performance. For this need statistical description of

BZ under H_0 and H_1. (First, compute a bound, then return).
Problem 4. (cont'd.)

(c) [15 pts.] Find an obvious (but tight) bound on the performance of the optimal vector decision device designed in (b). Can the receiver of (b) actually meet the bound?

We know that a matched filter sampler at $t = 7T_c$ and a simple threshold test is optimal and we have the formula

$$P_e = \Phi \left( \frac{\sqrt{E_b(1-r)}}{N_0} \right)$$

for the average performance (avg. bit error prob.) in terms of

$$E_b = \frac{\varepsilon_0 + \varepsilon_1}{2}, \quad r = \langle s_0, s_1 \rangle / E_b$$

This formula did not apply to the previous version of this problem because the filter was not matched. It is clearly a bound on the performance of our new multi-sample receiver.

Here

$$\varepsilon_0 = \int_0^{7T_c} a_o^2(t) \, dt = \varepsilon_1 = \int_0^{7T_c} a_1^2(t) \, dt$$

$$= A^2 7T_c$$

$$\Rightarrow E_b = \frac{7A^2 T_c}{7T_c} = A^2$$

$$\langle s_0, s_1 \rangle = \int_0^{7T_c} a_o(t) a_1(t) \, dt = \int_0^{7T_c} a_1(t) a_1(t) \, dt$$

$$= -A^2 T_c$$

$$\Rightarrow r = \frac{-A^2 T_c / A^2 7T_c}{7T_c} = -\frac{1}{7}$$

$$P_e = \Phi \left( \sqrt{\frac{8A^2 T_c}{N_0}} \right)$$

$$-5-$$
\[ E_o \{ B^2 \} = B E_o \{ Z^2 \} = B \mu_o = (\mu_1 - \mu_0) \Sigma^{-1} \mu_0 \triangleq \tilde{\mu}_0 \]
\[ E_1 \{ B^2 \} = (\mu_1 - \mu_0) \Sigma^{-1} \mu_1 \triangleq \tilde{\mu}_1 \]
\[ \text{Var}_o \{ B^2 \} = E_o \{ (B^2 - B \mu_o)^2 \} = E_o \{ (B(Z - \mu_o))^2 \} \]
\[ = E \{ (BN)^2 \} = E \{ BNN^T B \} = B \Sigma B^T \]
\[ = (\mu_1 - \mu_o) \Sigma^{-1} \Sigma \Sigma^{-1} (\mu_1 - \mu_0) = (\mu_1 - \mu_o) \Sigma^{-1} (\mu_1 - \mu_0) \]
\[ \triangleq d^2 \rightarrow \text{Turns out this is also equal to} \]
\[ \text{Var}_1 \{ B^2 \} \]

\[ P_{e,o} = P \left( B^2 > \frac{1}{2} B (\mu_1 + \mu_o) \mid H_o \right) \]
\[ = P \left( \frac{B^2 - B \mu_o}{d} > \frac{1}{2} B (\mu_1 + \mu_o) - B \mu_o \mid H_o \right) \]
\[ = P \left( \frac{B^2 - B \mu_o}{d} > \frac{1}{2} B (\mu_1 - \mu_o) \mid H_o \right) \]
\[ \frac{1}{2} d^{1/2} = d^{1/2} \]

\[ \therefore P_{e,o} = Q \left( \frac{d^{1/2}}{2} \right) \]

Of course it also turns out that \( P_{e,o} = P_{e,1} \).
Re:
Can the receiver meet the (cont'd.) bound?

\[ d = \sqrt{(\mu_1 - \mu_0)^T \Sigma^{-1} (\mu_1 - \mu_0)} \]

Try to simplify. Write

\[ \mu_1 - \mu_0 = (ZAT_e) \begin{bmatrix} 
0 & 0 & 0 & 0 & 2 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 2 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 \\
-1 & -1 & 0 & 0 & 2 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 \\
-1 & -1 & -1 & 0 & 2 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 \\
-1 & -1 & -1 & -1 & 0 & 2 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 2 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 & -1 & 1 & 1 & 1 & 1 & 1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
\end{bmatrix} \]

\[ \Sigma = \frac{3}{2} N_o T_e \quad \text{where} \quad C = \text{Toeplitz matrix with 1st row} \]

\[ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\end{bmatrix} \]

Then

\[ \frac{1}{2} = \frac{1}{2} \sqrt{(ZAT_e)^2 \cdot \frac{1}{\frac{3}{2} N_o T_e} \cdot \frac{T C^{-1} T}{N_o}} \]

\[ = \sqrt{\frac{3}{2} \frac{A^2 T_e}{N_o} \cdot \frac{T C^{-1} T}{N_o}} \]

Can experiment with \( \frac{T C^{-1} T}{N_o} \) by letting L increase and get the attached plot. (of course can't do this on the test.)
Comparing with MF bound we see a loss diminishing with observation length:

<table>
<thead>
<tr>
<th>factor multiplying ( \frac{A^2 T_c}{N_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>MF</td>
</tr>
<tr>
<td>vect. proc. @ length 100</td>
</tr>
<tr>
<td>@ length 15</td>
</tr>
</tbody>
</table>

It gets very close.
Problem 5. [45 points]

Consider the 4-ary QASK constellation shown below, which has been defined in terms of a pair of orthonormal signals \( \phi_0(t) \) and \( \phi_1(t) \). Assume these signals are received in AWGN of power spectral density height \( N_0/2 \). Assume equally likely symbols and maximum likelihood receiver.

Let \( Z_0 \) and \( Z_1 \) be correlator outputs for \( \phi_0 \) and \( \phi_1 \), respectively.

Then \( Z_0 \perp Z_1 \) and Gaussian of var \( \sigma^2 = N_0/2 \) and means depending on the symbol transmitted.

There are three parts to this question.

(a) [15 pts.] Find the average probability of a symbol error in terms of the Gaussian Q function, \( D_{\text{min}} \), and \( N_0 \). Explain.

The average prob. of a symbol error is equal to the conditional prob. of a symbol error for any of the four symbols.

\[
P(\text{symb. rec. | so sent}) = P(Z_0 > 0, Z_1 > 0 \mid \text{so sent})^2
= P(Z_0 > 0 \mid \text{so sent})P(Z_1 > 0 \mid \text{so sent}) = P(Z_0 > 0 \mid \text{so sent})^2
\]

Now
\[
P(Z_0 > 0 \mid \text{so sent}) = P\left(\frac{Z_0 - D_{\text{min}}/2}{\sqrt{N_0/2}} > \frac{-D_{\text{min}}/2}{\sqrt{N_0/2}}\right) = Q\left(\frac{-D_{\text{min}}/2}{\sqrt{N_0/2}}\right)
= \Phi\left(\frac{D_{\text{min}}/2}{\sqrt{N_0/2}}\right) = 1 - Q\left(\frac{D_{\text{min}}/2}{\sqrt{N_0/2}}\right)
\]

\[
P_{\text{correct}} = \left[1 - Q\left(\sqrt{\frac{D_{\text{min}}}{2N_0}}\right)^2\right]^2
= 1 - 2Q\left(\sqrt{\frac{D_{\text{min}}}{2N_0}}\right) + Q^2\left(\sqrt{\frac{D_{\text{min}}}{2N_0}}\right)
\]
Problem 5. (cont’d.)

\[ P_e = 1 - P_{\text{correct}} \]
\[ = 2 \Phi \left( \sqrt{\frac{D_{\text{min}}^2}{2N_0}} \right) - \Phi \left( \sqrt{\frac{D_{\text{min}}^2}{2N_0}} \right) \]

Often want to write answer in terms of the average bit energy or average symbol energy.

\[ ||s||^2 = \frac{D_{\text{min}}^2}{4} + \frac{D_{\text{min}}^2}{4} = \frac{D_{\text{min}}^2}{2} = E_s \]
(all symbols have same energy here)

A symbol carries 2 bits. So average bit energy is

\[ E_b = \frac{E_s}{2} = \frac{D_{\text{min}}^2}{4} \]

\[ \Rightarrow P_e = 2 \Phi \left( \sqrt{\frac{2E_b}{N_0}} \right) - \Phi \left( \sqrt{\frac{2E_b}{N_0}} \right) \]
Problem 5. (cont’d.)

(b) [15 pts.] Now assuming the same constellation and the same assumptions but with the bit labeling shown below find the average probability of a bit error for

1. the first bit, and
2. the second bit.

This is called Gray Labelling.

Let \( N \sim N(0, \frac{N_0}{2}) \) which is projection of noise comp. along any orthonormal axis.

\[ D_{\text{min}}/2 \]

\[ \phi_1 \]

\[ \phi_0 \]

\[ 01 \]

\[ 00 \]

\[ 11 \]

\[ 10 \]

\[ -D_{\text{min}}/2 \]

\[ -D_{\text{min}}/2 \]

\[ D_{\text{min}}/2 \]

\[ -D_{\text{min}}/2 \]

\[ D_{\text{min}}/2 \]

\[ 1. \text{ First Bit} \]

\[ P(1\text{st bit incorrect } | \text{ 00 sent}) = P(N > \frac{D_{\text{min}}}{2}) \]

\[ = P\left( \frac{N}{\sqrt{N_0/2}} > \frac{D_{\text{min}}/2}{\sqrt{N_0/2}} \right) = Q\left( \sqrt{\frac{D_{\text{min}}}{2N_0}} \right) \]

\[ = P(1\text{st bit incorrect } | \text{ 01 sent}) \]

\[ \text{or sent, 10 sent} \]

\[ \therefore P_{b1} = \text{average prob. of bit error in bit one} \]

\[ Q\left( \sqrt{\frac{D_{\text{min}}}{2N_0}} \right) = Q\left( \sqrt{\frac{2E_b}{N_0}} \right) \]

\[ 2. \text{ Second Bit} \]

Looking at the geometry its obvious that

\[ P_{b2} = P_{b1} \] for Gray Labelling.

-3-
Problem 5. (cont’d.)

(c) [15 pts.] Repeat (b) for the bit labeling shown below.

1. First Bit

Note that the first bit labelling is identical to previous cases, and therefore

\[ P_{b1} = Q \left( \sqrt{\frac{D_{\text{min}}^2}{2N_0}} \right) \]

2. Second Bit

From symmetry, we can see that

\[ P(2nd \text{ bit incorrect} | \infty) = P(2nd \text{ bit incorrect} | \text{any one of the symbols} \text{ sent}) \]

\[ = P(\text{choose 01} | \text{00 was sent}) + P(\text{choose 10} | \text{00 was sent}) \]

\[ = P(Z_0 < 0, Z_1 > 0 | \text{00 was sent}) + P(Z_0 > 0, Z_1 < 0 | \text{00 was sent}) \]

\[ = P(Z_0 < 0 | 00)P(Z_1 > 0 | 00) + P(Z_0 > 0 | 00)P(Z_1 < 0 | 00) \]
Problem 5. (cont’d.)

As before
\[ P(Z_0 > 0 | \theta_0) = P(Z_1 > 0 | \theta_0) = 1 - \Phi \left( \sqrt{\frac{D_{\text{min}}^2}{2N_0}} \right) \]

\[ P(Z_0 < 0 | \theta_0) = P(Z_1 < 0 | \theta_0) \]

\[ = P \left( \frac{Z_0 - D_{\text{min}} l_2}{\sqrt{N_0 l_2}} < \frac{-D_{\text{min}} l_2}{\sqrt{N_0 l_2}} \right) = \Phi \left( \frac{-D_{\text{min}} l_2}{\sqrt{N_0 l_2}} \right) \]

\[ = \Phi \left( \frac{D_{\text{min}} l_2}{\sqrt{N_0 l_2}} \right) \]

\[ \therefore P_{b2} = 2 \Phi \left( \sqrt{\frac{D_{\text{min}}^2}{2N_0}} \right) \left[ 1 - \Phi \left( \sqrt{\frac{D_{\text{min}}^2}{2N_0}} \right) \right] \]

\[ P_{b2} = 2 \Phi \left( \sqrt{\frac{2E_b}{N_0}} \right) \left[ 1 - \Phi \left( \sqrt{\frac{2E_b}{N_0}} \right) \right] \]
Problem 6. [30 points]

In a homework problem on binary, equal-energy FSK in fading the received signal was modeled as

\[ Y(t) = \sqrt{2}V \cos(\omega_it + \phi_i) + N(t), \quad 0 \leq t \leq T, \]

for \( i = 0 \) (bit '0' sent), \( i = 1 \) (bit '1' sent), and \( N(t) \) a white Gaussian noise of power spectral density height \( N_0/2 \). The amplitude term \( V \) was modeled as a Gaussian random variable independent of the noise and the actual signal transmitted. Here we model \( V \sim \mathcal{N}(\mu, \sigma^2) \).

The conditional error probability given \( V = v \) was shown to be

\[ P_e(v) = \frac{1}{2} e^{-v^2T/2N_0}. \]

Average over the distribution of \( V \) to find

\[ \bar{P}_e = \mathbb{E}\{P_e(V)\} \]

and simplify as much as possible.

\[
\mathbb{E}\{P_e(V)\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{v^2}{2\sigma^2}} P_e(v, v) \, dv
\]

\[
= \frac{1}{2\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} \exp\left[ -\frac{T}{2N_0} v^2 - \frac{1}{2\sigma^2} (v-v_0)^2 \right] \, dv
\]

Expanding the argument of the exponential in order to use the completing of the square method to evaluate the integral:

\[
-\frac{T}{2N_0} v^2 - \frac{1}{2\sigma^2} (v^2 - 2v_0 v + v_0^2) = -\left( \frac{T}{2N_0} + \frac{1}{2\sigma^2} \right) v^2 + \frac{v_0}{\sigma^2} v - \frac{v_0^2}{2\sigma^2}
\]

\[
= -A v^2 + B v - C = -A \left[ v^2 - \frac{B}{A} v \right] - C
\]

\[
= -A \left[ v^2 - \frac{B}{2A} v + \frac{B^2}{4A^2} - \frac{B^2}{4A^2} \right] - C
\]

\[
= -A \left( v - \frac{B}{2A} \right)^2 + \frac{B^2}{4A} - C
\]

\[ -1 \]
Problem 6. (cont’d.)

Can write
\[
E\{P_e(V)\} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{B^2}{4A}} \int_{-\infty}^{\infty} e^{-A(v - \frac{B}{2A})^2} dv
\]

\[
A = \frac{1}{2} \left[ \frac{T}{N_0} + \frac{1}{\sigma^2} \right]
\]

and define \( \bar{\sigma} \) by
\[
\frac{1}{\bar{\sigma}^2} = \frac{T}{N_0} + \frac{1}{\sigma^2} \Rightarrow \bar{\sigma}^2 = \left[ \frac{T}{N_0} + \frac{1}{\sigma^2} \right]^{-1} = \frac{N_0 \sigma^2}{\sigma^2 T + N_0}
\]

\[
\Rightarrow A = \frac{1}{2\bar{\sigma}^2} \Leftrightarrow \bar{\sigma}^2 = \frac{1}{2A} \Leftrightarrow \bar{\sigma} = \frac{1}{\sqrt{2A}}
\]

Now should be able to write
\[
E\{P_e(V)\} = \frac{1}{2\bar{\sigma}} e^{-\frac{B^2}{4A}} \int_{-\infty}^{\infty} e^{-\frac{(v - \frac{B}{2A})^2}{2\bar{\sigma}^2}} dv
\]

Simplifying
\[
\frac{B^2}{4A} = \frac{u^2}{\sigma^4} \frac{1}{4} \bar{\sigma}^2 = \frac{u^2 \bar{\sigma}^2}{2\sigma^4}
\]

\[
= \frac{u^2}{2\sigma^4} \cdot \frac{N_0 \sigma^2}{\sigma^2 T + N_0} = \frac{u^2 N_0}{2\sigma^2 (\sigma^2 T + N_0)}
\]

\[
C = \frac{u^2}{2\sigma^2}
\]

\[
\frac{\bar{\sigma}}{2\sigma} = \frac{1}{2\sigma} \sqrt{\frac{N_0 \sigma^2}{\sigma^2 T + N_0}} = 1
\]
Problem 6. (cont’d.)

\[ \overline{P_e} = E \left\{ \overline{P_e(V)} \right\} \]

\[ = \frac{1}{2\sigma} \sqrt{\frac{N_0 \sigma^2}{\sigma^2 T + N_0}} \exp \left\{ \frac{\nu^2 N_0}{2\sigma^2 (\sigma^2 T + N_0)} - \frac{\nu^2}{2\sigma^2} \right\} \]

\[ = \frac{1}{2\sigma} \sqrt{\frac{N_0 \sigma^2}{\sigma^2 T + N_0}} \exp \left\{ \frac{\nu^2}{2\sigma^2} \left[ \frac{N_0}{\sigma^2 T + N_0} - 1 \right] \right\} \]

\[ = \frac{1}{2\sigma} \sqrt{\frac{N_0 \sigma^2}{\sigma^2 T + N_0}} \exp \left\{ -\frac{\nu^2}{2\sigma^2} \frac{\sigma^2 T}{\sigma^2 T + N_0} \right\} \]

\[ = \frac{1}{2\sigma} \sqrt{\frac{N_0 \sigma^2}{\sigma^2 T + N_0}} \exp \left\{ -\frac{\nu^2}{2} \frac{T}{\sigma^2 T + N_0} \right\} \]
Problem 7. [15 points]

Suppose a radio channel of bandwidth equal to 20 MHz, i.e., the channel passes frequencies \( f \) of the form

\[
f_c - 10 \text{ MHz} \leq |f| \leq f_c - 10 \text{ MHz}
\]

for some center frequency \( f_c \gg 10 \text{ MHz} \). We wish to use this channel for BPSK signalling where the transmitted signal is of the form

\[
s(t) = \sqrt{2A} \sum_n d_n g(t - nT) \cos(2\pi f_c t)
\]

and \( g(t) \) is a baseband square-root raised cosine pulse with bandwidth expansion factor \( \beta = 0.4 \). Assume that the receiver uses filtering matched to the pulse \( g(t) \) and that receiver-side symbol timing is perfect.

What is the maximum bit rate that can be supported provided that ISI free reception is required?

The channel looks like a bandpass filter

\[
\text{When downconverted and matched filter the resulting baseband pulse will be a raised cosine with a baseband spectrum of form}
\]

\[
\frac{1+\beta}{2T} \leq 10 \text{ MHz} \quad \Rightarrow \quad \frac{T}{1} \leq \frac{2 \cdot 10 \text{ MHz}}{1.4} = 14.29 \text{ Mbps}
\]