General Instructions:

- You have 50 minutes to complete the exam.
- Write your name on every page of the exam.
- Do not write on the backs of the pages. If you need more paper, it will be provided to you upon request.
- The exam is closed book and closed notes.
- Calculators are allowed.
- Your work must be explained to receive full credit.
- Point values for each problem are as indicated. The exam totals 100 points.
- All plots must be carefully drawn with axes labeled.

Do not open the exam until you are told to begin.
Problem 1. [20 pts. total] In the block diagram above $W(t)$ is a wide sense stationary, Gaussian random process with zero mean and a flat power spectral density of height $N_0/2$ (in other words, it is a white random process). The two signals $s_0(t)$ and $s_1(t)$ driving the correlator multipliers are nonzero only over the time interval $0 \leq t \leq T$. A pair of random variables $W_0$ and $W_1$ are output from the correlators.

(a) [5 pts.] Name the type of joint probability distribution of $W_0$ and $W_1$ and explain.

- $W_0$ and $W_1$ are jointly Gaussian random variables.
- Multiplying a Gaussian r.p. by a deterministic time function results in a Gaussian r.p. In fact $s_0(t)W(t)$ and $s_1(t)W(t)$ are jointly Gaussian r.p.s.
- Linear operations, such as integration, on Gaussian rps produce Gaussian random variables.

(b) [15 pts.] From first principles derive the parameters needed to characterize the joint probability distribution of $W_0$ and $W_1$. They should be written in terms of the power spectral density height $N_0/2$, the correlation between the signals $s_0(t)$ and $s_1(t)$, and their energies.

The joint prob. distribution of $W_0$ and $W_1$ is Gaussian and therefore completely specified by

- means $E_{W_0}, E_{W_1}$
- variances $\text{Var}_{W_0}, \text{Var}_{W_1}$
- correlation $E_{W_0 W_1}$

Proceed to find these.
Problem 1. (cont'd.)

Means: \[ \mathbb{E} W_i = \mathbb{E} \int_0^T W(t) s_i(t) \, dt = \int_0^T \mathbb{E}\{W(t)\} s_i(t) \, dt \]
\[ = 0 \quad \text{since } W(t) \text{ has zero mean} \]
\[ i=0,1 \]

Variances: \[ \text{Var} W_i = \mathbb{E}\{W_i^2\} = \mathbb{E}\left\{ \int_0^T W(t) s_i(t) \, dt \int_0^T W(s) s_i(s) \, ds \right\} \]
\[ = \mathbb{E} \int_0^T \int_0^T W(t) W(s) s_i(t) s_i(s) \, dt \, ds \]
\[ = \int_0^T \int_0^T \mathbb{E}\{W(t)W(s)\} s_i(t) s_i(s) \, dt \, ds \]
\[ = \int_0^T \int_0^T \frac{N_0}{2} \delta(t-s) s_i(t) s_i(s) \, dt \, ds \]
\[ = \frac{N_0}{2} \int_0^T s_i^2(t) \, dt = \frac{N_0}{2} \| s_i \|^2 \quad i=0,1 \]

Correlation: \[ \mathbb{E} W_0 W_1 = \mathbb{E}\left\{ \int_0^T W(t) s_0(t) \, dt \int_0^T W(s) s_1(s) \, ds \right\} \]
\[ = \int_0^T \int_0^T \frac{N_0}{2} \delta(t-s) s_0(t) s_1(s) \, dt \, ds \]
\[ = \frac{N_0}{2} \int_0^T s_0(t) s_1(t) \, dt = \frac{N_0}{2} \langle s_0, s_1 \rangle. \]
Problem 2. [45 pts. total] The receiver above is used to solve the hypothesis test:

\[ H_0 : Y(t) = s_0(t) + W(t) \]
\[ \text{vs.} \]
\[ H_1 : Y(t) = s_1(t) + W(t) \]

where \( W(t) \) is a white, Gaussian random process with zero mean and auto-correlation \( R_W(\tau) = 0.5N_0\delta(\tau) \), and signals \( s_0(t) \) and \( s_1(t) \) are given by:

\[
\begin{align*}
\text{s}_0(t) & \quad \text{s}_1(t) \\
1.0 & \quad 1.0 \\
0.5 & \quad 0.5 \\
0.0 & \quad 0.0 \\
-0.5 & \quad -0.5 \\
-1.0 & \quad -1.0 \\
0 & 0 \quad T/4 \quad T/2 \quad 3T/4 \quad T \\
T/4 & T/2 \quad 3T/4 \quad T
\end{align*}
\]

(a) [10 pts.] Characterize the joint distribution of \( Y_0 \) and \( Y_1 \), i.e., name it and write down its parameters, under hypothesis \( H_0 \).

Following the argument from Problem 1 and using linearity (signal & noise are added together under either hypothesis) we know that the joint dist. of \( Y_0 \) and \( Y_1 \) is Gaussian.

Thus to completely specify it we need only means, variances, and correlations. The means are due to the signals and so depend on which hypothesis is in force. The rest only depends on noise.
Problem 2. (cont'd.)

Let \( W_0 \) and \( W_1 \) be the noise components in the r.vs \( Y_0 \) and \( Y_1 \), i.e.

\[
W_0 = \int_0^T W(t) s_0(t) \, dt, \quad W_1 = \int_0^T W(t) s_1(t) \, dt
\]

as in Prob. 1 we have \( EW_0 = EW_1 = 0 \) and

\[
E\{W_0^2\} = \text{Var} W_0 = \frac{N_0}{2} \|s_0\|^2,
\]
\[
E\{W_1^2\} = \text{Var} W_1 = \frac{N_0}{2} \|s_1\|^2
\]
\[
E\{W_0 W_1\} = \frac{N_0}{2} \langle s_0, s_1 \rangle
\]

For the pulses \( s_0(\cdot) \) and \( s_1(\cdot) \) given we have

\[
\|s_0\|^2 = \int_0^T s_0^2(t) \, dt = 1 \cdot \frac{T}{4} + \left(\frac{1}{2}\right)^2 \frac{T}{4} + \left(\frac{1}{2}\right)^2 \frac{T}{4} + \left(-\frac{1}{2}\right)^2 \frac{T}{4} = 5T/8
\]
\[
\|s_1\|^2 = \int_0^T s_1^2(t) \, dt = 1 \cdot \frac{T}{4} + 1 \cdot \frac{T}{4} = T/2
\]
\[
\langle s_0, s_1 \rangle = 1 \cdot \frac{T}{4} + \frac{1}{2} \cdot \frac{T}{4} = 3T/8
\]

Under \( H_0 \)

\[
EY_0 = \int_0^T s_0^2(t) \, dt = 5T/8
\]
\[
EY_1 = \int_0^T s_0(t) s_1(t) \, dt = 3T/8
\]

\[
Y_0 \sim N\left(\frac{5T}{8}, \frac{N_0}{2} \frac{5T}{8}\right), \quad Y_1 \sim N\left(\frac{3T}{8}, \frac{N_0}{2} \frac{T}{2}\right)
\]

and the cross-cov is

\[
\text{Cov}\{Y_0, Y_1\} = \frac{N_0}{2} \frac{3T}{8}
\]
Problem 2. (cont’d.)

(b) [10 pts.] Repeat part (a) under the assumption of hypothesis $H_1$.

The only change is to the mean.

Under $H_1$

\[
E\{Y_0\} = \int s_1(t) s_0(t) \, dt = \frac{3T}{8}
\]

\[
E\{Y_1\} = \int_0^T s_1^2(t) \, dt = \frac{T}{2}
\]

\[
Y_0 \sim N\left(\frac{3T}{8}, \frac{N_0}{2} \frac{5T}{8}\right) \quad \text{Cov}\{Y_0, Y_1\} = \frac{N_0}{2} \frac{3T}{8}
\]

\[
Y_1 \sim N\left(\frac{T}{2}, \frac{N_0}{2} \frac{1}{2}\right)
\]
Problem 2. (cont'd.)

(c) [15 pts.] For the receiver given find the probability of error assuming $H_0$, i.e., $P_{e_0}$, the probability that the receiver chooses $H_1$ when $H_0$ is really true. Express your answer in terms of the Gaussian $Q$ function, $T$, and $N_0$.

Assuming $H_0$ is true the receiver makes an error if it decides that $H_1$ was true. The event is

$$\{ Y_0 < Y_1 \}$$

$$\therefore P_{e_0} = P(Y_0 < Y_1 \mid H_0) = P(Y_1 - Y_0 > 0 \mid H_0)$$

Now under $H_0$: $Y_1 - Y_0 = \frac{3T}{8} + W_1 - \left( \frac{5T}{8} + W_0 \right)$

$$= -\frac{2T}{8} + W_1 - W_0$$

$$\therefore P_{e_0} = P(W_1 - W_0 > \frac{2T}{8})$$

$W_1 - W_0$ is Gaussian of mean zero. But to calculate its variance we will need to account for correlation:

$$\text{Var}\{W_1 - W_0\} = E\{(W_1 - W_0)^2\} = E\{W_1^2\} + E\{W_0^2\} - 2E\{W_0 W_1\}$$

$$= \frac{N_0}{2} \cdot \frac{T}{2} + \frac{N_0}{2} \cdot \frac{5T}{8} - 2 \cdot \frac{N_0}{2} \cdot \frac{3T}{8} = \frac{N_0}{2} \cdot \frac{T}{2} \left( 1 + \frac{5}{4} - 2 \cdot \frac{3}{4} \right)$$

$$= \frac{N_0 T}{4} \cdot \frac{3}{4} = \frac{3N_0 T}{16}$$

Then

$$P_{e_0} = P\left( \frac{W_1 - W_0}{\sqrt{3N_0 T/16}} > \frac{2T/8}{\sqrt{3N_0 T/16}} \right) = Q\left( \frac{2T/8}{\sqrt{3N_0 T/16}} \right)$$

$$= Q\left( \sqrt{\frac{T}{3N_0}} \right)$$
Problem 2. (cont’d.)

(d) [10 pts.] Evaluate the error probability from (c) using the table below for the case where $T = 9N_0$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$Q(x)$</th>
<th>$x$</th>
<th>$Q(x)$</th>
<th>$x$</th>
<th>$Q(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5</td>
<td>1.5</td>
<td>0.066807</td>
<td>3.0</td>
<td>0.0013499</td>
</tr>
<tr>
<td>0.1</td>
<td>0.46017</td>
<td>1.6</td>
<td>0.054799</td>
<td>3.1</td>
<td>0.00096760</td>
</tr>
<tr>
<td>0.2</td>
<td>0.42074</td>
<td>1.7</td>
<td>0.044565</td>
<td>3.2</td>
<td>0.00068714</td>
</tr>
<tr>
<td>0.3</td>
<td>0.38209</td>
<td>1.8</td>
<td>0.035930</td>
<td>3.3</td>
<td>0.00048342</td>
</tr>
<tr>
<td>0.4</td>
<td>0.34458</td>
<td>1.9</td>
<td>0.028717</td>
<td>3.4</td>
<td>0.00033693</td>
</tr>
<tr>
<td>0.5</td>
<td>0.30854</td>
<td>2.0</td>
<td>0.022750</td>
<td>3.5</td>
<td>0.00023263</td>
</tr>
<tr>
<td>0.6</td>
<td>0.27425</td>
<td>2.1</td>
<td>0.017864</td>
<td>3.6</td>
<td>0.00015911</td>
</tr>
<tr>
<td>0.7</td>
<td>0.24196</td>
<td>2.2</td>
<td>0.013903</td>
<td>3.7</td>
<td>0.00010780</td>
</tr>
<tr>
<td>0.8</td>
<td>0.21186</td>
<td>2.3</td>
<td>0.010724</td>
<td>3.8</td>
<td>7.2348 × 10⁻⁵</td>
</tr>
<tr>
<td>0.9</td>
<td>0.18406</td>
<td>2.4</td>
<td>0.0081975</td>
<td>3.9</td>
<td>4.8096 × 10⁻⁵</td>
</tr>
<tr>
<td>1.0</td>
<td>0.15866</td>
<td>2.5</td>
<td>0.0062097</td>
<td>4.0</td>
<td>3.1671 × 10⁻⁵</td>
</tr>
<tr>
<td>1.1</td>
<td>0.13567</td>
<td>2.6</td>
<td>0.0046612</td>
<td>4.1</td>
<td>2.0658 × 10⁻⁵</td>
</tr>
<tr>
<td>1.2</td>
<td>0.11507</td>
<td>2.7</td>
<td>0.0034670</td>
<td>4.2</td>
<td>1.3346 × 10⁻⁵</td>
</tr>
<tr>
<td>1.3</td>
<td>0.096800</td>
<td>2.8</td>
<td>0.0025551</td>
<td>4.3</td>
<td>8.5399 × 10⁻⁶</td>
</tr>
<tr>
<td>1.4</td>
<td>0.080757</td>
<td>2.9</td>
<td>0.0018658</td>
<td>4.4</td>
<td>5.4125 × 10⁻⁶</td>
</tr>
</tbody>
</table>

If $T = 9N_0$

$$P_{e_0} = Q \left( \sqrt{\frac{9N_0}{3N_0}} \right) = Q \left( \sqrt{3} \right) \approx Q (1.73)$$

$$\approx 0.04$$
Problem 3. [35 pts. total] The input \( X(t) \) to the system shown above is a zero mean, Gaussian random process, with trianularly shaped autocorrelation function:

\[
R_X(\tau)
\]

For \( T = 2.5 \) find the mean value \( \mathbb{E}\{Y(t)\} \) of the random \( 0 - 1 \) square wave at the output of the system. This mean value may be interpreted as the average duty cycle of the square wave \( Y(t) \). Explain why. Finally, sketch a representative sample function for \( Y(t) \).

\[
Z(t) = X(t) - X(t-T)
\]

Since \( Z(t) \) is a linear combination of Gaussian random variables, it too is Gaussian and therefore \( Z(t) \) is a Gaussian random process. Since

\[
\mathbb{E}\{Z(t)\} = \mathbb{E}\{X(t)\} - \mathbb{E}\{X(t-T)\} = 0 - 0
\]

its mean is also zero.

Now

\[
\text{Var}\{Z(t)\} = \mathbb{E}\{Z^2(t)\}
\]

\[
= \mathbb{E}\{(X(t) - X(t-T))^2\}
\]

\[
= \mathbb{E}\{X^2(t)\} + \mathbb{E}\{X^2(t-T)\} - 2\mathbb{E}\{X(t)X(t-T)\}
\]

\[
= R_X(0) + R_X(0) - 2R_X(T)
\]

\[
= 2[R_X(0) - R_X(T)]
\]
Problem 3. (cont'd.)

Then for $T = 2.5$ have $R_x(2.5) = 0.5$. Similarly, $R_x(0) = 1$.

$\implies \text{Var} \{Z(t)\} = 2 \left(1 - 0.5\right) = 1$

$\therefore$ For any time $t$: $Z(t) \sim N(0, 1)$

Now $E\{Y(t)\} = P\{Y(t) = 1\}$

$= P\{|Z(t)| > 1\}$

$= 1 - P\{|Z(t)| \leq 1\}$

$= 1 - P\{-1 < Z(t) \leq 1\}$

$= 1 - [\Phi(1) - \Phi(-1)]$

$= 1 - \Phi(1) + \Phi(-1)$

$= 2 \Phi(1) = 2 \cdot 0.15866$

$\uparrow$

from Table.

$\therefore E\{Y(t)\} = P\{Y(t) = 1\} \approx 0.32$
Problem 3. (cont’d.)

The duty cycle interpretation works because $\gamma(t)$ is a binary wave.

\[ \ldots \quad \begin{array}{cccc} & & & \\ \ldots & \ldots & \ldots & \ldots \\ & & & \end{array} \rightarrow t \]

In the long run it is equal to 1 about 32% of the time.