Lecture Notes for ECE 440: Transmission of Information

Digital Communications – Spring 2022

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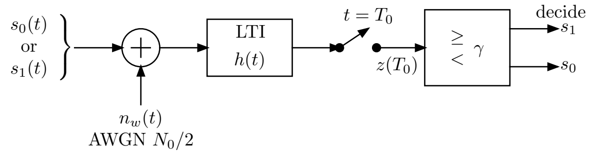
1 Digital Communications

1.1 Binary Single Shot in Noise



Above block diagram shows model for reception of a single information bit in AWGN. The receiver consists of a LTI filter followed by a sampler and a threshold device.

Assumptions are:

- Signals are of finite energy

$$E_i = \int s_i^2(t)\mathrm{d}t < \infty \quad (i = 0, 1)$$

- Channel noise $n_w(t)$ is AWGN and independent of the choice of signal $s_i(t)$ that was transmitted.

1.1.1 Statistical Model

Sampler output $z(T_0)$ is a Gaussian r.v. written as the sum of a signal part and a noise part:

$$\begin{aligned} z(T_0) &= h * s_i(T_0) + h * n_w(T_0) \\ &= \hat{s}_i(T_0) + \hat{n}(T_0) \end{aligned}$$

$i = 0, 1$. Therefore, with $\mu_i = \hat{s}_i(T_0)$ and

$$\begin{aligned} \sigma^2 &= \mathrm{Var}[\hat{n}(T_0)] \\ &= \frac{N_0}{2} \int |H(f)|^2 \mathrm{d}f \;=\; \frac{N_0}{2} \int h^2(t)\mathrm{d}t \end{aligned}$$

the statistical model for the decision r.v. is

- Assuming s_0 was transmitted:

$$z(T_0) \sim \mathcal{N}(\mu_0, \sigma^2).$$

- Assuming s_1 was transmitted:

$$z(T_0) \sim \mathcal{N}(\mu_1, \sigma^2).$$

Assume (WLOG) that $\mu_1 > \mu_0$.

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1.1.2 Performance

Performance criterion is to minimize the probability of error. There are two types of errors:

- Probability of error given s_0 was transmitted:

$$\begin{aligned} P_{e,0}(\gamma) &= \Pr[z(T_0) \geq \gamma | s_0 \text{ trans.}] \\ &= Q\left(\frac{\gamma - \mu_0}{\sigma}\right) \end{aligned}$$

- Probability of error given s_1 was transmitted:

$$\begin{aligned} P_{e,1}(\gamma) &= \Pr[z(T_0) < \gamma | s_1 \text{ trans.}] \\ &= 1 - Q\left(\frac{\gamma - \mu_1}{\sigma}\right) \\ &= Q\left(\frac{\mu_1 - \gamma}{\sigma}\right) \end{aligned}$$

1.1.3 Minimax and Bayesian

It is impossible to simultaneously minimize $P_{e,0}$ and $P_{e,1}$ over γ. More information is needed to have a well-posed minimization problem. There are two approaches:

Minimax: γ_m^* is the threshold which minimizes the maximum error (minimize the worst case)

$$P_{e,m}(\gamma) = \max\{P_{e,0}(\gamma), P_{e,1}(\gamma)\}.$$

The minimax error is $P_{e,m}^* = P_{e,m}(\gamma_m^*)$.

Bayesian: Have prior probabilities $\pi_0 = \Pr[s_0 \text{ trans.}]$, $\pi_1 = \Pr[s_1 \text{ trans.}]$, $\pi_0 + \pi_1 = 1$. γ_b^* is the threshold which minimizes the average probability of error

$$P_{e,b}(\gamma) = \pi_0 P_{e,0}(\gamma) + \pi_1 P_{e,1}(\gamma).$$

The Bayes error is $P_{e,b}^* = P_{e,b}(\gamma_b^*)$.

Solutions (Case $\mu_1 > \mu_2$): Following are optimal threshold and the minimum criterion for the two cases.

- For minimax $\gamma_m^* = (\mu_0 + \mu_1)/2$ and

$$P_{e,m}^* = Q\left(\frac{\mu_1 - \mu_0}{2\sigma}\right).$$

- For Bayes

$$\gamma_b^* = \gamma_m^* + \frac{\sigma^2}{\mu_1 - \mu_0}\ln(\pi_0/\pi_1)$$

and

$$
\begin{aligned}
P^*_{e,b} &= \pi_0 Q\left(\frac{\sigma}{\mu_1-\mu_0}\ln(\pi_0/\pi_1) + \frac{\mu_1-\mu_0}{2\sigma}\right) \\
&+ \pi_1 Q\left(\frac{\mu_1-\mu_0}{2\sigma} - \frac{\sigma}{\mu_1-\mu_0}\ln(\pi_0/\pi_1)\right)
\end{aligned}
$$

1.1.4 The Matched Filter

Optimization over choice of filter and sampling time: It is obvious that $P^*_{e,m}$ is smaller the larger is the argument of the Q function, i.e., $(\mu_1-\mu_0)/2\sigma$. The same is true for $P^*_{e,b}$ (although this would require a little proof). Maximizing $(\mu_1-\mu_0)/2\sigma$ over h involves the following observations.

- Schwarz Inequality: For finite energy signals f and g define inner product by

  $$\langle f, g\rangle \stackrel{\text{def}}{=} \int f(t)g(t)\mathrm{d}t$$

  and norm $\|f\| = \sqrt{\langle f, f\rangle}$. Then $|\langle f, g\rangle| \leq \|f\|\|g\|$. Furthermore, if $\|g\| > 0$ then equality holds in the inequality if and only if there exits a number λ such that $f(t) = \lambda g(t)$ for (almost) all t.

- With $s(t) \stackrel{\text{def}}{=} [s_1(t) - s_0(t)]/2$ and $s_{T_0}(t) \stackrel{\text{def}}{=} s(T_0 - t)$

  $$\frac{\mu_1-\mu_0}{2\sigma} = \sqrt{\frac{2\langle s_{T_0}, h\rangle^2}{N_0\|h\|^2}} \leq \sqrt{\frac{2\|s_{T_0}\|^2}{N_0}} = \sqrt{\frac{2\|s\|^2}{N_0}}$$

  and equality holds (assuming s_1 and s_0 are not identically equal) if and only if h is chosen to be the matched filter

  $$h(t) = \lambda s_{T_0}(t) = \lambda s(T_0 - t)$$

  for an arbitrary non-zero constant λ.

- If the matched filter is used then

  $$P^*_{e,m} = Q\left(\sqrt{\frac{2\|s\|^2}{N_0}}\right)$$

  which does not depend on T_0 (a similar expression can be obtained for $P^*_{e,b}$).

- Sampling time T_0 is arbitrary so long as $h(t) = \lambda s(T_0 - t)$. Thus, T_0 is usually picked so that the matched filter is causal.

1.1.5 Signal Design

Assuming a matched filter and AWGN we can rewrite the expression for SNR in terms of the energy in each signal and the correlation between the two signals. It is a simple exercise to show

$$\begin{aligned} \|s\|^2 &= \frac{1}{4}\left[\|s_1\|^2 + \|s_0\|^2 - 2\langle s_0, s_1\rangle\right] \\ &= \frac{1}{2}\bar{\mathcal{E}}(1-\rho) \end{aligned}$$

where $\bar{\mathcal{E}} = (\|s_0\|^2 + \|s_1\|^2)/2$ and $\rho = \langle s_0, s_1\rangle/\bar{\mathcal{E}}$. Then

$$P_{e,m}^{*} = Q\left(\sqrt{\frac{\bar{\mathcal{E}}(1-\rho)}{N_0}}\right).$$

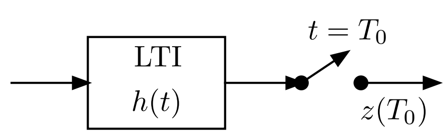
Note that $-1 \leq \rho \leq 1$. Two important cases are:

- orthogonal signals: $\rho = 0$.
- antipodal signals: $\rho = -1$.

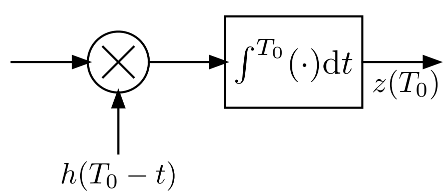
1.1.6 MF/Sampler vs. Correlator

The following are equivalent implementations of the optimal receiver front end.

filter / sampler



correlator



A Old Notes with Proofs

Baseband Transmission of Binary Data

Binary data trans. system: $M=2$ basic waveforms or pulses used to convey information from transmitter $\rightarrow$ receiver

$$\phi_0(t) \quad \text{and} \quad \phi_1(t)$$

Baseband: most of the energy of these two signals is below some freq. W.

Let $b_0, b_1, \ldots, b_{N-1}$ be a sequence of N binary digits ie $b_k \in \{0,1\}$. Then if bits arrive @ rate $1/T$ transmitter:

$$\{b_k\} \xrightarrow[@\,1/T]{} \boxed{\text{baseband modulator}} \longrightarrow x(t) = \sum_{k=0}^{N-1} \phi_{b_k}(t-kT)$$

For now assume both $\phi_0(t)$ and $\phi_1(t)$ are time limited to an interval of duration T ie

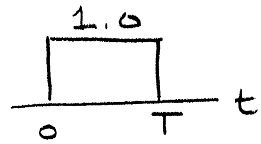
$$\phi_i(t) = 0 \quad \text{if} \quad t<0 \ \text{or} \ t>T$$

for $i=0,1$. Then for say $nT \le t < (n+1)T$ have

$$x(t) = \phi_{b_n}(t-nT)$$

$\Longrightarrow$ If receiver can distinguish between the waveforms $\phi_0(\cdot)$ and $\phi_1(\cdot)$, then the bit stream can be recovered.

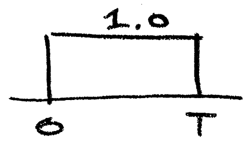
$\Longrightarrow$ However, receiver does not observe waveform $x(t)$ only a corrupted version of it.

Simplest Example $\phi_0(t) \equiv 0, \quad \phi_1(t) =$ 

$\Longrightarrow$ These two are actually orthogonal on the interval $[0,T]$ since

$$\int_0^T \phi_0(t)\,\phi_1(t)\,dt = 0$$

An orthogonal signal set

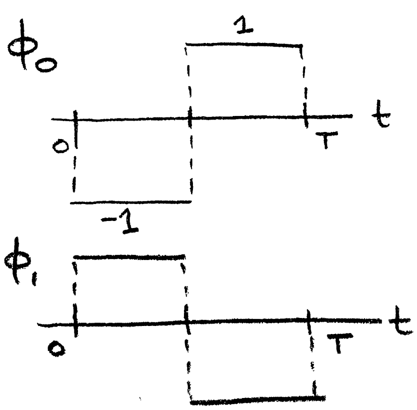
Use $p_T(t) =$ 

to represent generic rectangular pulse.

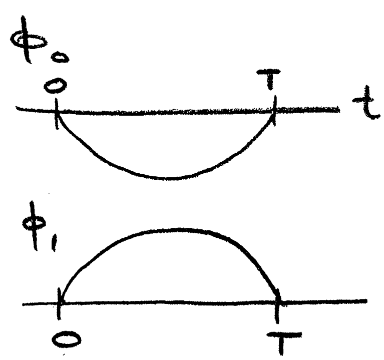
Another Example $\phi_0(t) = -p_T(t), \quad \phi_1(t) = p_T(t).$

An antipodal signal set

Two More Examples



antipodal



$\phi_1(t) = \sin\left(\frac{\pi t}{T}\right) p_T(t)$

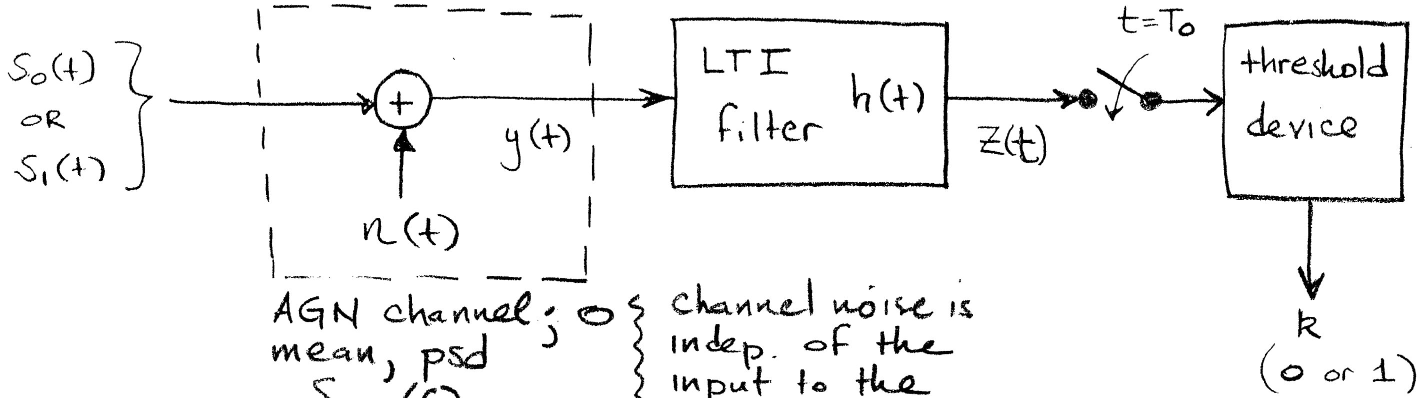
antipodal

Finally note also that all of our signals have

$$\int \phi_i^2(t)\,dt < \infty \qquad \text{ie finite energy.}$$

Linear Receivers

(Considering transmission of a single info. bit)



AGN channel; 0 mean, psd $S_{n,n}(f)$.

channel noise is indep. of the input to the channel.

energy: $$E_i = \int_0^T s_i^2(t)\,dt \qquad i=0,1$$

Depending upon which signal is actually sent

$$y(t) = s_i(t) + n(t)$$ signal + noise

Filter is linear so that $z = h * y$ can be written

$$z(t) = \hat{s}_i(t) + \hat{n}(t)$$

where

$$\hat{s}_i(t) = [h * s_i](t) = \int h(t-\tau)\, s_i(\tau)\, d\tau$$

$$\hat{n}(t) = [h * n](t) = \int h(t-\tau)\, n(\tau)\, d\tau$$

$\hat{n}(t) \longrightarrow$ Also a WSS Gaussian process.

Since $\{\hat{n}(t)\}$ is WSS and Gaussian it is completely specified (in a statistical sense) by

$$\mu_{\hat{n}} = E\{\hat{n}(t)\} \qquad R_{\hat{n}}(\tau) = E\{\hat{n}(t)\hat{n}(t+\tau)\}$$

$$\mu_{\hat{n}} = 0 \quad \left(\text{since } n(t) \text{ has zero mean}\right).$$

Output of sampler $z(T_0)$ a random variable

Threshold Device $z(T_0) \underset{<}{\geq} \gamma$ $\begin{Bmatrix}\text{decide 1 sent}\\ \text{decide 0 sent}\end{Bmatrix}$ OR $\begin{Bmatrix}\text{decide 0 sent}\\ \text{decide 1 sent}\end{Bmatrix}$

the threshold

the only two poss. decision rules

We have chosen a particular structure for the receiver in this binary communication problem. Still need:

1) optimize over LTI filters $h(\cdot)$.
2) " " sampling time T_0.
3) " " threshold γ.
4) pick correct form for the decision rule.

$\Longrightarrow$ If possible, justify the structure.

To optimize anything must have a performance metric: prob. of error.

$P_{e,0}$; $P_{e,1}$

If signal $s_i(t)$ is sent $\Longrightarrow$ $z(t) = \underbrace{\hat{s}_i(t)}_{\text{deterministic waveform}} + \underbrace{\hat{n}(t)}_{\text{WSS process.}}$

From basics of "WSS random processes through LTI systs.":

$$\mu_{\hat{n}} = 0 \qquad R_{\hat{n}} = \tilde{h} * h * R_n$$

where $\tilde{h}(t) \triangleq h(-t)$ $\boxed{\text{Assuming real-valued here}}$.

For the psd:

$$S_{\hat{n}}(f) = |H(f)|^2 S_n(f)$$

<u>Important Special Case</u>: $S_n(f) = \frac{N_0}{2}$ white

$$\Longrightarrow \quad S_{\hat{n}}(f) = \frac{N_0}{2} |H(f)|^2$$

Define $f = \tilde{h} * h$ whence

$$R_{\hat{n}}(\tau) = [f * R_n](\tau) = \int f(\tau-\lambda) \frac{N_0}{2} \delta(\lambda)\, d\lambda$$

$$= \frac{N_0}{2} f(\tau)$$

Note:

$$f(\tau) = \int \tilde{h}(\tau-\lambda) h(\lambda)\, d\lambda = \int h(\lambda-\tau) h(\lambda)\, d\lambda$$

$$= \int h(\lambda') h(\lambda'+\tau)\, d\lambda$$ ← the autocorrelation of $h(\cdot)$

Properties of the random variable $z(T_0) = \hat{s}_i(T_0) + \hat{n}(T_0)$

Clearly $z(T_0)$ is a Gaussian random variable with

$$E\{z(T_0)\} = \hat{s}_i(T_0) = \int h(T_0 - \tau)\, s_i(\tau)\, d\tau$$

$$\text{Var}\{z(T_0)\} = \text{Var}\{\hat{n}(T_0)\}$$

$$= R_{\hat{n}}(0)$$

$$= [\tilde{h} * h * R_n](0) = \int f(-\lambda) R_n(\lambda)\, d\lambda$$

$$= \iint h(u+\lambda) h(u)\, du\, R_n(\lambda)\, d\lambda = \int |H(f)|^2 S_n(f)\, df$$

Important Observation

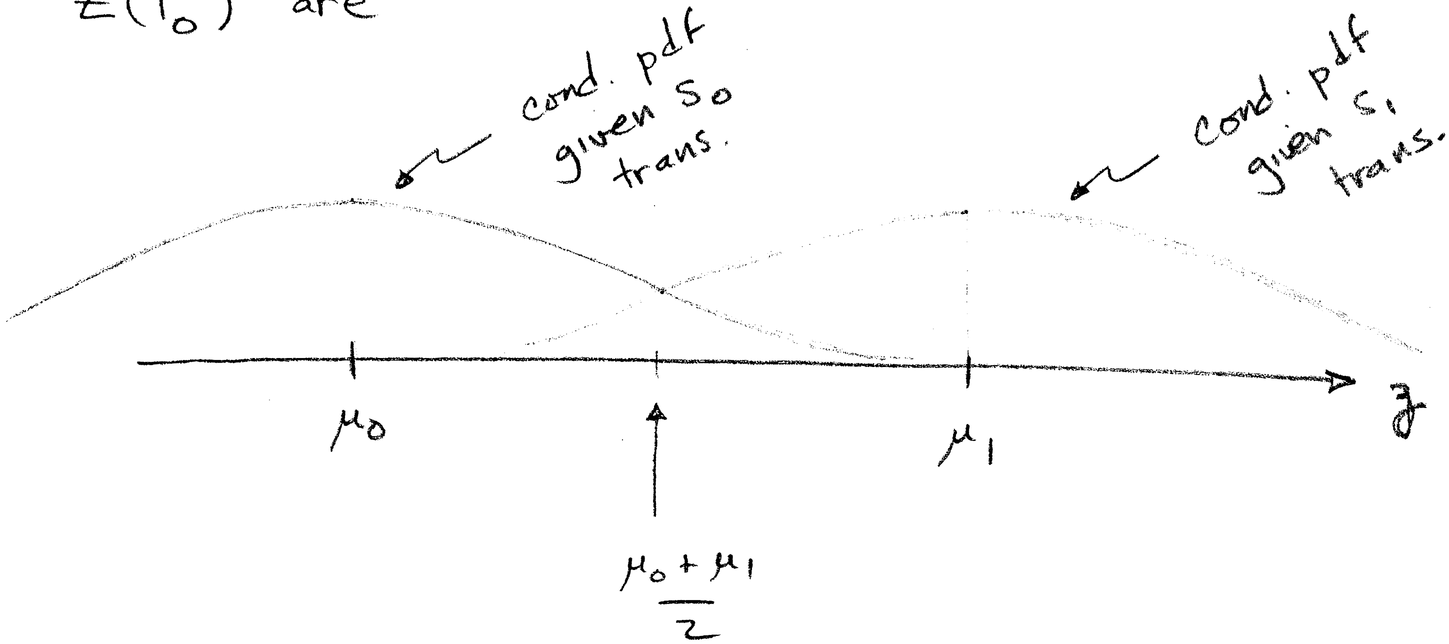
$\text{Var}\{z(T_0)\}$ does not depend upon which signal was actually transmitted. $\boxed{\text{Does not dep. on } T_0 \text{ either}}$

$$\sigma^2 \triangleq \int |H(f)|^2 S_n(f)\, df$$

$$\mu_i \triangleq \hat{s}_i(T_0)$$

Given that $\left\{\begin{matrix} s_0(t) \\ s_1(t) \end{matrix}\right\}$ is transmitted the cond. dist. of $z(T_0)$ is Gaussian with mean $\left\{\begin{matrix} \mu_0 \\ \mu_1 \end{matrix}\right\}$ and var σ^2

Suppose (WLOG) $\mu_1 = \hat{s}_1(T_0) > \mu_0 = \hat{s}_0(T_0)$ then the two conditional pdfs of the output $z(T_0)$ are

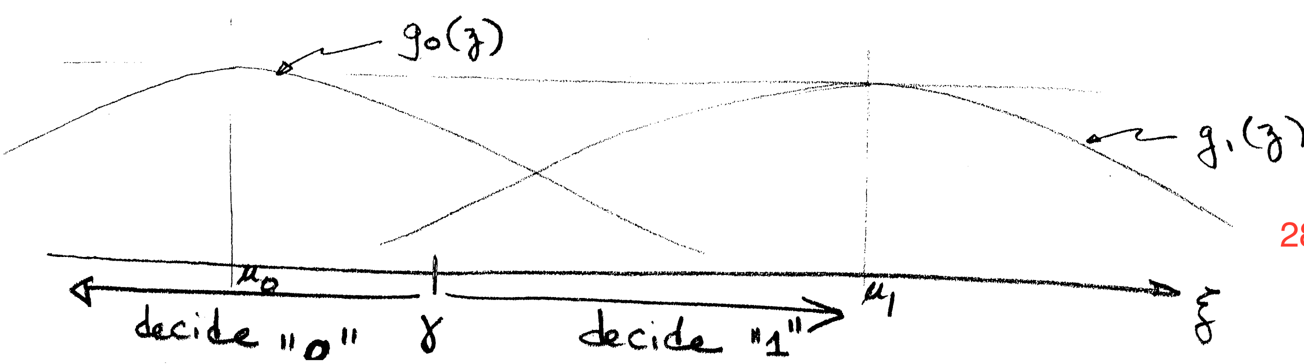


The fact that these two pdfs are different is what enables the decision device to (statistically) decide between bits 0 and 1.

<u>Clear that for the case $\mu_1 > \mu_0$</u>

$$g_i(z) \triangleq \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-\mu_i)^2/2\sigma^2}$$

$$z(T_0) \begin{array}{c} \geq \\ < \end{array} \gamma \quad \begin{array}{l} \text{decide "1"} \\ \text{decide "0"} \end{array}$$



$$P_{e,0} = P\left(z(T_0) \geq \gamma \,\middle|\, s_0 \text{ trans}\right) = \int_{\gamma}^{\infty} g_0(z)\,dz$$

$$= 1 - G_0(\gamma)$$

G_0 ← cdf corresp. to $g_0(z)$.

$$P_{e,1} = P\left(z(T_0) < \gamma \,\middle|\, s_1 \text{ trans}\right) = \int_{-\infty}^{\gamma} g_1(z)\,dz$$

$$= G_1(\gamma).$$

For ease of calculation, usually express Gaussian error probabilities in terms of standard normal dist. (ie mean zero, variance one)

$$\Phi(x) \triangleq \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2}\,d\lambda \qquad Q(x) \triangleq 1 - \Phi(x)$$

Say $Z_0 \sim N(\mu_0, \sigma^2) \Longrightarrow P(Z_0 \geq \gamma) = P\left(\frac{Z_0 - \mu_0}{\sigma} \geq \frac{\gamma - \mu_0}{\sigma}\right)$

$$= Q\left(\frac{\gamma - \mu_0}{\sigma}\right)$$

$Z_1 \sim N(\mu_1, \sigma^2) \Longrightarrow P(Z_1 < \gamma) = P\left(\frac{Z_1 - \mu_1}{\sigma} < \frac{\gamma - \mu_1}{\sigma}\right)$

$$= \Phi\left(\frac{\gamma - \mu_1}{\sigma}\right)$$

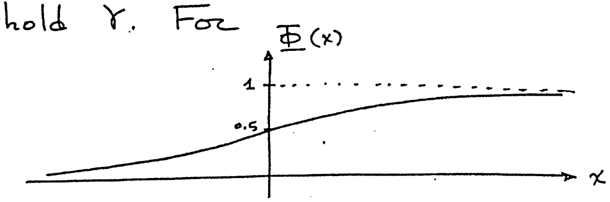
Recall $P_{e,0} = Q\left(\frac{\gamma-\mu_0}{\sigma}\right) = 1 - \Phi\left(\frac{\gamma-\mu_0}{\sigma}\right)$

$$= \Phi\left(\frac{\mu_0-\gamma}{\sigma}\right)$$

$$P_{e,1} = \Phi\left(\frac{\gamma-\mu_1}{\sigma}\right)$$

$$= Q\left(\frac{\mu_1-\gamma}{\sigma}\right).$$

Not clear from above how to optimize over the threshold γ. For



∴ To make $\Phi(x)$ small should make x small. (ie very negative)

$\Rightarrow$ γ large pos. makes $P_{e,0}$ small
$\Rightarrow$ γ "large" neg. " $P_{e,1}$ small

} this for the picture where $\mu_0 < \mu_1$

$\Rightarrow$ Conflicting requirements .... need something more to properly optimize over threshold.

Two possibilities :

A) Minimax $\quad Pe_{,m} \triangleq \max\left\{Pe_{,0}, Pe_{,1}\right\}$

B) Average or Bayes $\quad Pe_{,b} \triangleq \pi_0 Pe_{,0} + \pi_1 Pe_{,1}$ where $\pi_0 + \pi_1 = 1$, $\pi_i \geq 0$

π_i = a priori prob. that $s_i(t)$ is trans.

$\Longrightarrow$ An optimal threshold γ is one that minimizes the criteria above ie

$$\left\{\begin{matrix}\gamma_a^* \\ \text{OR} \\ \gamma_b^*\end{matrix}\right\} = \operatorname{argmin}_\gamma \left\{\begin{matrix}Pe_{,m} \\ \text{OR} \\ Pe_{,b}\end{matrix}\right\}$$

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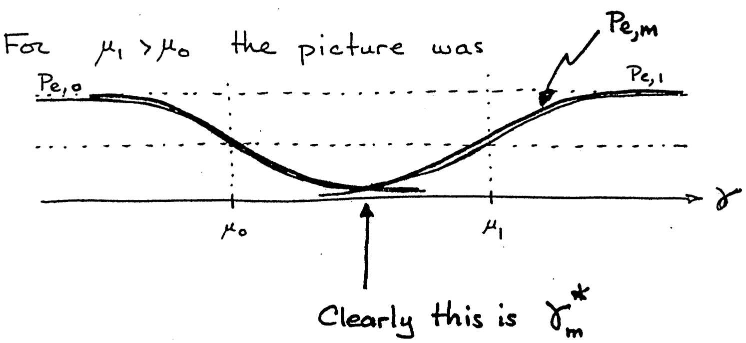
Solution to Minimax Problem

For $\mu_1 > \mu_0$ the picture was



$\therefore \gamma_m^*$ is the solution to

$$\Phi\left(\frac{\mu_0 - \gamma}{\sigma}\right) = \Phi\left(\frac{\gamma - \mu_1}{\sigma}\right)$$

and from the symmetry of the functions (or just sub. the guess) find solution

$$\gamma_m^* = \frac{\mu_0 + \mu_1}{2} = \frac{\hat{s}_1(T_0) + \hat{s}_0(T_0)}{2}$$

Convince yourself that answer does not change if we take opposite $(\mu_0 > \mu_1)$ assumption for picture drawing.

$$P^*_{e,m} = \text{min. value of } P_{e,m} \text{ over } \gamma$$

$$\Rightarrow P^*_{e,m} = \Phi\left(\frac{\mu_0-\mu_1}{2\sigma}\right) = Q\left(\frac{\mu_1-\mu_0}{2\sigma}\right).$$

$\boxed{\text{for the case } \mu_1 > \mu_0}$

<u>Solution to Bayes Problem</u>

$$P_{e,b} = \pi_0\,\Phi\left(\frac{\mu_0-\gamma}{\sigma}\right) + \pi_1\,\Phi\left(\frac{\gamma-\mu_1}{\sigma}\right).$$

$\Rightarrow$ Can see that $P_{e,b}(\gamma)$ is a differentiable function of γ. Thus may solve

$$\frac{dP_{e,b}}{d\gamma} = 0 \quad \text{for } \gamma^*_b$$

$\Rightarrow$ Note $\Phi\left(\frac{\gamma-\mu_1}{\sigma}\right) = F_X(\gamma)$ where $X \sim N(\mu_1, \sigma^2)$

and so

$$\frac{d}{d\gamma}\Phi\left(\frac{\gamma-\mu_1}{\sigma}\right) = f_X(\gamma)$$

Also

$$\Phi\left(\frac{\mu_0-\gamma}{\sigma}\right) = 1 - \Phi\left(\frac{\gamma-\mu_0}{\sigma}\right)$$

$\therefore$

$$\frac{dP_{e,b}}{d\gamma} = -\pi_0\frac{1}{\sqrt{2\pi}\,\sigma}e^{-(\gamma-\mu_0)^2/2\sigma^2} + \pi_1\frac{1}{\sqrt{2\pi}\,\sigma}e^{-(\gamma-\mu_1)^2/2\sigma^2} \overset{\text{SET}}{=} 0$$

$$\Rightarrow -\pi_0 e^{-(\gamma-\mu_0)^2/2\sigma^2} + \pi_1 e^{-(\gamma-\mu_1)^2/2\sigma^2} = 0$$

$$\cancel{e^{-(\gamma-\mu_0)^2/2\sigma^2}}\left[-\pi_0 + \pi_1 \exp\left(-\frac{(\gamma-\mu_1)^2}{2\sigma^2} + \frac{(\gamma-\mu_0)^2}{2\sigma^2}\right)\right] = 0$$

$$\frac{2(\mu_1-\mu_0)\gamma + \mu_0^2 - \mu_1^2}{2\sigma^2}$$

$$\therefore \exp(\;) = \frac{\pi_0}{\pi_1}$$

$$\frac{2(\mu_1-\mu_0)\gamma + \mu_0^2 - \mu_1^2}{2\sigma^2} = \log(\pi_0/\pi_1).$$

$$\vdots$$

$$\underbrace{\gamma}_{\gamma^*_b} = \underbrace{\frac{\sigma^2}{\mu_1-\mu_0}\log(\pi_0/\pi_1)}_{\text{See what happens when } \pi_0=\pi_1=1/2.} + \underbrace{\frac{\mu_1+\mu_0}{2}}_{\gamma^*_m}$$

Still must optimize over filter impulse response $h(\cdot)$ and sampling time T_0. For this need expressions for $P_{e,m}$ and $P_{e,b}$ as case may be.

Minimax Case $\gamma_m^* = \dfrac{\mu_1 + \mu_0}{2}$

$$\Rightarrow P_{e,m}^* = P_{e,0} = P_{e,1} = \Phi\left(\frac{\mu_0 - \mu_1}{2\sigma}\right)$$

$\boxed{\text{ie min. value of } P_{e,m}}$

$$= 1 - \Phi\left(\frac{\mu_1 - \mu_0}{2\sigma}\right) = Q\left(\frac{\mu_1 - \mu_0}{2\sigma}\right).$$

Clearly, the larger is $\dfrac{\mu_1 - \mu_0}{2\sigma}$ the smaller is the error probability:

$$\frac{\mu_1 - \mu_0}{2\sigma} = \frac{\hat{s}_1(T_0) - \hat{s}_0(T_0)}{2\sqrt{\int |H(f)|^2 S_n(f)\, df}}$$

$$= \frac{\int h(T_0 - \tau)\left[s_1(\tau) - s_0(\tau)\right] d\tau}{2\sqrt{\int |H(f)|^2 S_n(f)\, df}}$$

Bayes Case

$$\gamma_b^* = \beta + \gamma_m^*$$

$$\frac{\mu_0 - \gamma}{\sigma} = \frac{\mu_0 - \gamma_m^*}{\sigma} - \frac{\beta}{\sigma}. \Rightarrow \frac{\gamma_b^* - \mu_0}{\sigma} = \frac{\mu_1 - \mu_0}{2\sigma} + \frac{\beta}{\sigma}$$

$$\frac{\gamma - \mu_1}{\sigma} = \frac{\gamma_m^* - \mu_1}{\sigma} + \frac{\beta}{\sigma} \Rightarrow \frac{\mu_1 - \gamma_b^*}{\sigma} = \frac{\mu_1 - \mu_0}{2\sigma} - \frac{\beta}{\sigma}$$

$$= \frac{\mu_0 - \mu_1}{2\sigma}$$

$$\frac{\beta}{\sigma} = \frac{1}{\sigma}\frac{\sigma^2}{\mu_1 - \mu_0}\log\left(\pi_0/\pi_1\right) = \frac{\sigma}{\mu_1 - \mu_0}\log\left(\pi_0/\pi_1\right).$$

$$\therefore P_{e,b}^* = \pi_0 P_{e,0} + \pi_1 P_{e,1}$$ $\boxed{\text{ie min. average prob. of error}}$

$$= \pi_0\, Q\left(\frac{\sigma}{\mu_1 - \mu_0}\log\left(\frac{\pi_0}{\pi_1}\right) + \frac{\mu_1 - \mu_0}{2\sigma}\right)$$

$$+ \pi_1\, Q\left(\frac{\mu_1 - \mu_0}{2\sigma} - \frac{\sigma}{\mu_1 - \mu_0}\log\left(\frac{\pi_0}{\pi_1}\right)\right).$$

Special Case : AWGN Channel

Define : $s_{T_0}(t) = \frac{1}{2}\left[s_1(T_0 - t) - s_0(T_0 - t)\right]$

$S_n(f) = N_0/2$ for all f

$$\left.\begin{aligned} \langle f, g \rangle &\triangleq \int f(t) g(t)\, dt \\ \|f\|^2 &\triangleq \langle f, f \rangle \end{aligned}\right\} \text{inner prod. notation}$$

Then for calculation of $\frac{\mu_1 - \mu_0}{2\sigma}$ have

$$\mu_1 - \mu_0 = \hat{s}_1(T_0) - \hat{s}_0(T_0) = \int h(T_0 - \tau)\left[s_1(\tau) - s_0(\tau)\right] d\tau$$

$$= \int h(\tau)\left[s_1(T_0 - \tau) - s_0(T_0 - \tau)\right] d\tau$$

$$= 2 \langle s_{T_0}, h \rangle$$

$$\sigma^2 = \frac{N_0}{2} \int |H(f)|^2 df = \frac{N_0}{2} \int |h(\tau)|^2 d\tau$$

$$= \frac{N_0}{2} \|h\|^2$$

$$\frac{\mu_1 - \mu_0}{2\sigma} = \frac{\langle s_{T_0}, h \rangle}{\sqrt{\frac{N_0}{2} \|h\|^2}} = \sqrt{\frac{\langle s_{T_0}, h \rangle^2}{\frac{1}{2} N_0 \|h\|^2}}$$

$$\triangleq \text{SNR}$$

Claim $\text{SNR} \le \sqrt{\dfrac{\|s\|^2}{\frac{1}{2} N_0}} \triangleq \sqrt{\dfrac{2\mathcal{E}_s}{N_0}}$

with equality if and only if

$h(t) = c\, s(T_0 - t)$

$s(t) \triangleq s_1(t) - s_0(t)$

(c is an arbitrary constant).

LECTURE 10

SCHWARZ INEQUALITY

FOR ANY FUNCTIONS f AND g ON (a,b)

$$\left[\int_a^b f(u)\,g(u)\,du\right]^2 \le \int_a^b f^2(u)\,du \int_a^b g^2(v)\,dv.$$

FURTHERMORE IF $\int_a^b g^2(v)\,dv > 0$ THEN EQUALITY HOLDS IN THE ABOVE INEQUALITY IFF THERE IS A REAL NUMBER λ SUCH THAT $f(u) = \lambda\, g(u)$ FOR (ALMOST) ALL u.

NOTATION: $\|h\| = \sqrt{\int_a^b h^2(u)\,du}$

$$(f,g) = \int_a^b f(u)\,g(u)\,du$$

SCHWARZ INEQUALITY: $(f,g)^2 \le \|f\|^2 \|g\|^2$

so $|(f,g)| \le \|f\|\,\|g\|$

THE FUNCTIONS f AND g MUST BE SQUARE INTEGRABLE; i.e.,
$\int_a^b f^2(u)\,du < \infty$, $\int_a^b h^2(u) < \infty$

GIVEN $f:(a,b) \to \mathbb{R}$,
WHAT DOES $f(u) = 0$
FOR ALMOST ALL u MEAN?

FOR OUR PURPOSES IT MEANS

$$\int_a^b [f(u)]^2\,du = 0.$$

HENCE $f = \lambda g$ a.e.

OR $f(u) = \lambda\, g(u)$ FOR ALMOST ALL u MEANS

$$\int_a^b [f(u) - \lambda g(u)]^2\,du = 0$$

PROOF: FOR ANY REAL NUMBER λ

$$\|f-\lambda g\|^2 = \int_a^b (f(u)-\lambda g(u))^2\,du \geq 0$$

WITH EQUALITY IFF $f(u)=\lambda g(u)$ FOR ALMOST ALL u ($f=\lambda g$ a.e.)

SUPPOSE $f \neq \lambda g$ a.e., THEN

$$0 < \|f-\lambda g\|^2 = \int_a^b [f(u)-\lambda g(u)]^2\,du$$
$$= \int_a^b f^2(u)\,du - 2\lambda \int_a^b f(u)g(u)\,du + \lambda^2 \int_a^b g^2(u)\,du$$
$$= \|f\|^2 - 2\lambda (f,g) + \lambda^2 \|g\|^2 \qquad (1)$$

THIS HOLDS FOR ANY CHOICE OF λ

SUPPOSE $\|g\|^2 \neq 0$ (IF $\|g\|^2=0$ THE INEQUALITY IS TRIVIALLY TRUE BECAUSE $(f,g)=0$ AND $\|f\|\,\|g\|=0$)

LET $\lambda = (f,g)/\|g\|^2$ SO THAT (1) BECOMES

$$0 < \|f\|^2 - 2\frac{(f,g)}{\|g\|^2}(f,g) + \frac{(f,g)^2}{\|g\|^4}\|g\|^2$$
$$= \|f\|^2 - \frac{(f,g)^2}{\|g\|^2}$$
$$\Rightarrow \|f\|^2 > \frac{(f,g)^2}{\|g\|^2} \Rightarrow \underline{\|f\|^2\|g\|^2 > (f,g)^2}$$

HENCE $f \neq \lambda g$ a.e. AND $\|g\| > 0$ IMPLIES $(f,g)^2 < \|f\|^2\,\|g\|^2$

NOW SUPPOSE $f=\lambda g$ a.e. THEN

$$(f,g)^2 = \left[\int_a^b f(u)\,g(u)\,du\right]^2$$
$$= \left[\int_a^b \lambda g(u)\,g(u)\,du\right]^2$$
$$= \lambda^2 \left[\int_a^b g^2(u)\,du\right]^2 = \lambda^2\|g\|^4$$

$$\|f\|^2\|g\|^2 = \|\lambda g\|^2\|g\|^2 = \lambda^2\|g\|^4$$

SO $\underline{(f,g)^2 = \|f\|^2\,\|g\|^2}$

WE HAVE PROVED THAT

$$(f,g)^2 \leq \|f\|^2\,\|g\|^2 \quad \text{ALL } f,g$$

AND IF $\|g\| \neq 0$

$$f \neq \lambda g \text{ a.e} \Rightarrow (f,g)^2 < \|f\|^2\|g\|^2$$
$$f = \lambda g \text{ a.e.} \Rightarrow (f,g)^2 = \|f\|^2\|g\|^2$$

Back to Matter of SNR

$$(SNR)^2 = \frac{\left(\int_{-\infty}^{\infty} h(T_0-\tau)\, s(\tau)\, d\tau\right)^2}{\frac{1}{2} N_0 \int_{-\infty}^{\infty} h^2(t)\, dt}$$

LET $u = T_0 - \tau$ SO

$$\left(\int_{-\infty}^{\infty} h(T_0-\tau)\, s(\tau)\, d\tau\right)^2 = \left(\int_{-\infty}^{\infty} h(u)\, s(T_0-u)\, du\right)^2$$

$$= \left(\int_{-\infty}^{\infty} h(u)\, s_{T_0}(u)\, du\right)^2$$

$$s_{T_0}(u) = s(T_0-u) \quad , -\infty < u < \infty$$

NOTICE $\int_{-\infty}^{\infty} h(u)\, s_{T_0}(u)\, du = (h, s_{T_0})$.

SO $$(SNR)^2 = \frac{(h, s_{T_0})^2}{\frac{N_0}{2}\|h\|^2}$$

BUT, BY THE SCHWARZ INEQUALITY

$$(h, s_{T_0})^2 \le \|h\|^2 \, \|s_{T_0}\|^2$$

NOTE $\|s_{T_0}\|^2 = \int_{-\infty}^{\infty} s^2(T_0-u)\, du = \int_{-\infty}^{\infty} s^2(\tau)\, d\tau$

$= \mathcal{E}_s \ne 0$ FOR SIGNALS OF INTEREST

HENCE $(h, s_{T_0})^2 = \|h\|^2 \, \|s_{T_0}\|^2$

IFF $h = \lambda s_{T_0}$

IT FOLLOWS THAT

$$(SNR)^2 \le \frac{\|h\|^2 \|s_{T_0}\|^2}{\frac{N_0}{2}\|h\|^2} = \frac{\|s_{T_0}\|^2}{N_0/2}$$

$$= \frac{\mathcal{E}_s}{N_0/2} = \frac{2\mathcal{E}_s}{N_0}.$$

WITH EQUALITY IFF $h(u) = \lambda s(T_0-u)$ FOR ALMOST ALL u

$$\therefore \max_h SNR = \sqrt{\frac{2\mathcal{E}_s}{N_0}}.$$

$$\min_h P_{e,n} = Q(\max_h SNR)$$

$$= Q\left(\sqrt{2\mathcal{E}_s/N_0}\right)$$

AND THE OPTIMUM FILTER IS

$$h(u) = \lambda\, s(T_0-u)$$

FOR ANY $\lambda \ne 0$

$\boxed{\text{Matched Filter}}$

~~RECALL WE MUST ALSO HAVE~~

~~$\hat{s}_0(T_0) > \hat{s}_1(T_0)$~~

~~FOR ANTIPODAL SIGNALS THIS IS EQUIVALENT TO $(s*h)(T_0) > 0$~~

IF THE MATCHED FILTER IS USED

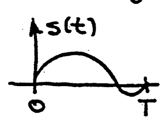
$$P^*_{e,m} = \Phi\left(\sqrt{2\mathcal{E}_s/N_0}\right)$$

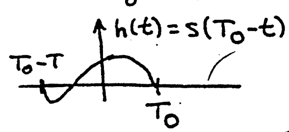
WHICH DOES NOT DEPEND ON THE SAMPLING TIME T_0.

T_0 IS ARBITRARY AS LONG AS

$$h(t) = s(T_0 - t)$$

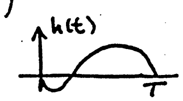
SUPPOSE $s(t)$ IS TIME-LIMITED TO $[0, T]$; i.e., $s(t) = 0$ for $t < 0$ OR $t > T$. THEN $h(t) = s(T_0 - t) = 0$ FOR $t > T_0$ OR $t < T_0 - T$.





THE MATCHED FILTER IS <u>CAUSAL</u> IF $T_0 - T \geq 0$ $(T_0 \geq T)$

USUALLY LET $T_0 = T$



SO $h(t) = s(T - t) = 0$ $t < 0$ OR $t > T$

Have solved the optimization problem for minimax criterion and choice of $h(\cdot), T_0$. Found

$$P^*_{e,m} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \longleftarrow \boxed{\text{after minimizing over } h(\cdot).}$$

achieved for impulse response $h(t) = s(T_0 - t)$ where T_0 is arbitrary.

The Bayesian Problem

$$P^*_{e,b} = \pi_0 Q\left(\frac{\sigma}{\mu_1-\mu_0}\log(\pi_0/\pi_1) + \frac{\mu_1-\mu_0}{2\sigma}\right) + \pi_1 Q\left(\frac{\mu_1-\mu_0}{2\sigma} - \frac{\sigma}{\mu_1-\mu_0}\log(\pi_0/\pi_1)\right)$$

For the AWGN channel case we had

$$\frac{\mu_1-\mu_0}{2\sigma} = \sqrt{\frac{\langle s_{T_0}, h\rangle^2}{\frac{1}{2}N_0\|h\|^2}} = \frac{\langle s_{T_0}, h\rangle}{\sqrt{\frac{1}{2}N_0}\,\|h\|}$$

Define notation: $\lambda \triangleq \frac{1}{2}\left[\hat{s}_1(T_0) - \hat{s}_0(T_0)\right] = \langle s_{T_0}, h\rangle$

$$\lambda' \triangleq \frac{\lambda}{\|s_{T_0}\|\,\|h\|} \quad \text{whence } -1 \le \lambda' \le 1$$

$$\Longrightarrow \frac{\mu_1-\mu_0}{2\sigma} = \frac{\lambda}{\sqrt{\frac{1}{2}N_0}\,\|h\|}\frac{\|s_{T_0}\|}{\|s_{T_0}\|} = \lambda'\frac{\|s_{T_0}\|}{\sqrt{\frac{1}{2}N_0}}$$

$$\| \\ \lambda/\sigma$$

$\therefore$ Can write

$$P^*_{e,b} = \pi_0 Q\left(\frac{\lambda}{\sigma} + \frac{\sigma}{2\lambda}\log(\pi_0/\pi_1)\right) + \pi_1 Q\left(\frac{\lambda}{\sigma} - \frac{\sigma}{2\lambda}\log(\pi_0/\pi_1)\right)$$

Also write $\alpha = \|s_{T_0}\| / \sqrt{\frac{1}{2}N_0} \Longrightarrow \frac{\lambda}{\sigma} = \lambda'\alpha$

$$P^*_{e,b} = \pi_0 Q\left(\lambda'\alpha + \frac{1}{2\lambda'\alpha}\log(\pi_0/\pi_1)\right) + \pi_1 Q\left(\lambda'\alpha - \frac{1}{2\lambda'\alpha}\log(\pi_0/\pi_1)\right)$$

$$= \pi_0 Q\left(\lambda'\alpha + \beta/\lambda'\right) + \pi_1 Q\left(\lambda'\alpha - \beta/\lambda'\right)$$

> Claim: $P^*_{e,b}$ decreases as λ' increases.
> ($\therefore$ $P^*_{e,b}$ minimized by choosing $\lambda' = +1$).

Proof of Claim:

Just need to show that $dP^*_{e,b}/d\lambda' < 0$ for $-1 < \lambda' < 1$.

$$\frac{d}{d\lambda'} Q\left(\lambda'\alpha + \beta/\lambda'\right) = Q'\left(\lambda'\alpha + \beta/\lambda'\right)\left[\alpha - \frac{\beta}{\lambda'^2}\right]$$

$$\frac{d}{d\lambda'} Q\left(\lambda'\alpha - \beta/\lambda'\right) = Q'\left(\lambda'\alpha - \beta/\lambda'\right)\left[\alpha + \frac{\beta}{\lambda'^2}\right]$$

$$Q(x) = 1 - \Phi(x) \Rightarrow Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Thus

$$\frac{dP^*_{e,b}}{d\lambda'} = \pi_0\left[\alpha - \frac{\beta}{\lambda'^2}\right]\left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\lambda'\alpha + \beta/\lambda'\right)^2}\right) + \pi_1\left[\alpha + \frac{\beta}{\lambda'^2}\right]\left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\lambda'\alpha - \beta/\lambda'\right)^2}\right).$$

$\therefore$

$$-\sqrt{2\pi}\,\frac{dP^*_{e,b}}{d\lambda'} = e^{-(\lambda'\alpha)^2/2}\, e^{-(\beta/\lambda')^2/2}\left\{\pi_0\left(\alpha - \frac{\beta}{\lambda'^2}\right) e^{-\alpha\beta} + \pi_1\left(\alpha + \frac{\beta}{\lambda'^2}\right) e^{\alpha\beta}\right\}$$

So $\frac{dP^*_{e,b}}{d\lambda'} < 0 \iff G(\lambda') \triangleq \pi_0\left(\alpha - \frac{\beta}{(\lambda')^2}\right) e^{-\alpha\beta} + \pi_1\left(\alpha + \frac{\beta}{(\lambda')^2}\right) e^{\alpha\beta} > 0$

Now $\alpha\beta = \frac{1}{2}\log\left(\pi_0/\pi_1\right) = \log\left(\sqrt{\pi_0/\pi_1}\right)$

$\Rightarrow e^{\alpha\beta} = \sqrt{\pi_0/\pi_1}\ ;\ e^{-\alpha\beta} = \sqrt{\pi_1/\pi_0}$

$$\Rightarrow G(\lambda') = \pi_0\left(\alpha - \beta/(\lambda')^2\right)\sqrt{\pi_1/\pi_0} + \pi_1\left(\alpha + \beta/(\lambda')^2\right)\sqrt{\pi_0/\pi_1}$$

$$= \sqrt{\pi_0\pi_1}\left\{\alpha - \beta/(\lambda')^2 + \alpha + \beta/(\lambda')^2\right\}$$

$$= 2\alpha\sqrt{\pi_0\pi_1} > 0 \quad \text{since } \alpha > 0.$$

Claim implies that $P_{e,m}^*$ is minimized by taking λ' as large as possible subject to the constraint $-1 \le \lambda' \le +1$ ie

$$\lambda' = +1$$

But
$$\lambda' = \frac{\langle s_{T_0}, h \rangle}{\| s_{T_0} \| \cdot \| h \|} = +1$$

$$\Updownarrow$$

$$h(t) = c\, s_{T_0}(t) \quad \text{for some positive constant } c.$$
$$= c\, s(T_0 - t)$$

$\Longrightarrow$ Get same optimal filter as we had in the case of minimax. Same comment regarding T_0 still applies.

Signal Design

Previously we showed that for the AWGN channel and a matched filter:

$$\frac{\mu_1 - \mu_0}{2\sigma} = \sqrt{\frac{2 \mathcal{E}_s}{N_0}}$$

where

$$\mathcal{E}_s = \| s_{T_0} \|^2 = \| s \|^2 \qquad s(t) = \frac{s_1(t) - s_0(t)}{2}$$

$$\| s \|^2 = \frac{1}{4} \| s_1 - s_0 \|^2$$
$$= \frac{1}{4} \langle s_1 - s_0, s_1 - s_0 \rangle = \frac{1}{4} \left\{ \| s_1 \|^2 + \| s_0 \|^2 - 2 \langle s_0, s_1 \rangle \right\}$$
$$= \frac{1}{4} \left\{ \mathcal{E}_1 + \mathcal{E}_0 - 2 \langle s_0, s_1 \rangle \right\}$$
$$= \frac{1}{4} \left(\mathcal{E}_1 + \mathcal{E}_0 \right) \left\{ 1 - \frac{2}{\mathcal{E}_1 + \mathcal{E}_0} \langle s_0, s_1 \rangle \right\}$$
$$= \frac{1}{2} \bar{\mathcal{E}} \left\{ 1 - \frac{\langle s_0, s_1 \rangle}{\bar{\mathcal{E}}} \right\}$$
$$= \frac{1}{2} \bar{\mathcal{E}} \left(1 - r \right)$$

$$\bar{\mathcal{E}} = \frac{\mathcal{E}_0 + \mathcal{E}_1}{2}$$

$$r = \frac{\langle s_0, s_1 \rangle}{\bar{\mathcal{E}}}$$

$$\frac{2E_s}{N_0} = \frac{\cancel{2}}{N_0}\frac{1}{\cancel{2}}\bar{E}(1-r)$$

$$\therefore P_{e,m}^{*} = Q\left(\sqrt{\frac{\bar{E}(1-r)}{N_0}}\right) \quad \text{when matched filter is used.}$$

Note that $-1 \le r \le 1$. Can minimize over r by choosing

$$r = -1$$

ie antipodal signals are best.

B BPSK and QPSK