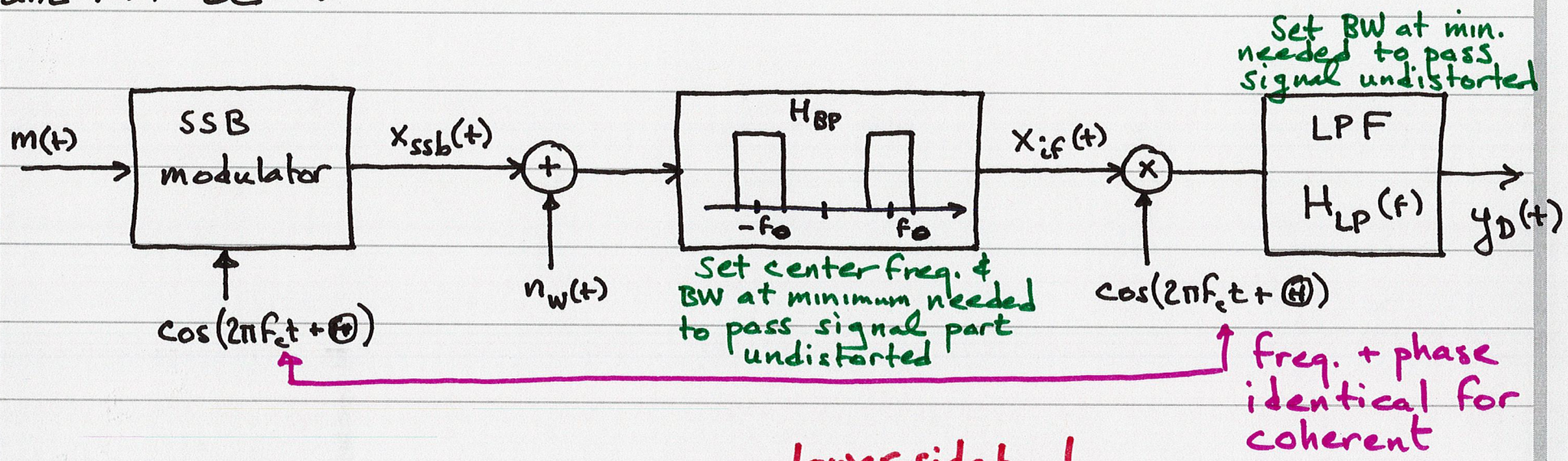
Noise Performance for Single Sideband

Follow an approach similar to that for coherently demodulated DSB and AM-LC ...



$$x_{ssb}(t) = 0.5 A_c m(t) \cos(2\pi f_c t + \Theta) \pm 0.5 A_c \hat{m}(t) \sin(2\pi f_c t + \Theta)$$

lower sideband

upper sideband

Need to compute the power and spectral occupancy of $x_{ssb}(t)$ to choose the BPF and to compute signal power @ IF.

Model $m(t)$ zero mean WSS random process with $R_{mm}(\tau) \leftrightarrow S_{mm}(f)$; Θ unif. on $[0, 2\pi)$; $n_w(t)$ AWGN of psd height $N_0/2$; signal, phase, noise are stat. indep. of each other.

Compute Auto Correlation of $x_{ssb}(t)$

$$R_{xx}(\tau) = E\{x(t)\,x(t+\tau)\}$$

$$= 0.25\,A_c^2\,E\Big\{\Big[m(t)\cos(2\pi f_c t + \Theta) \pm \hat{m}(t)\sin(2\pi f_c t + \Theta)\Big] \cdot$$

$$\cdot \Big[m(t+\tau)\cos(2\pi f_c (t+\tau) + \Theta) \pm \hat{m}(t+\tau)\sin(2\pi f_c (t+\tau) + \Theta)\Big]\Big\}$$

$$= 0.25 A_c^2\,E\{4 \text{ terms}\}$$

$$m(t)\,m(t+\tau)\cos(2\pi f_c t + \Theta)\cos(2\pi f_c (t+\tau) + \Theta) \rightarrow \text{term 1}$$

$$\pm\, m(t)\hat{m}(t+\tau)\cos(2\pi f_c t + \Theta)\sin(2\pi f_c (t+\tau) + \Theta) \rightarrow \text{term 2}$$

$$\pm\, m(t+\tau)\hat{m}(t)\sin(2\pi f_c t + \Theta)\cos(2\pi f_c (t+\tau) + \Theta) \rightarrow \text{term 3}$$

$$+\, \hat{m}(t)\hat{m}(t+\tau)\sin(2\pi f_c t + \Theta)\sin(2\pi f_c (t+\tau) + \Theta) \rightarrow \text{term 4}$$

Auto Correlation $R_{XX}(\tau)$

$m(t)$ and Θ are statistically independent

$$\therefore\ E\{\text{term } 1\} = \underbrace{E\{m(t)m(t+\tau)\}}_{R_{mm}(\tau)}\ \underbrace{E\{\cos(2\pi f_c t+\Theta)\cos(2\pi f_c(t+\tau)+\Theta)\}}_{\circledast}$$

* use trig identity: $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)$

* definition of $E\{\cdot\}$ and pdf of Θ: $f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0<\theta<2\pi \\ 0 & \text{else} \end{cases}$

$$E\{G(\Theta)\} = \int_0^{2\pi} G(\theta)\frac{1}{2\pi}\,d\theta$$

$$E\{\circledast\} = E\left\{\frac{1}{2}\cos(2\pi f_c\tau) + \frac{1}{2}\cos(2\pi(2f_c)t + 2\pi f_c\tau + 2\Theta)\right\}$$

$$= \boxed{\frac{1}{2}E\{\cos(2\pi f_c\tau)\}} + \boxed{\frac{1}{2}E\{\cos(2\pi(2f_c)t + 2\pi f_c\tau + 2\Theta)\}}$$

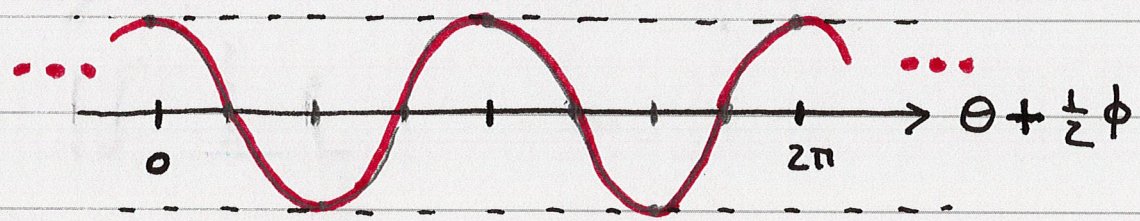
This is zero.

$= \frac{1}{2}\cos(2\pi f_c\tau)$ since argument of expectation is deterministic

$$\boxed{E\{\cos(2\pi(2f_c)t + 2\pi f_c \tau + 2\Theta)\} = 0}$$

Define $\phi = 2\pi(2f_c)t + 2\pi f_c \tau$ a deterministic (non-random) number. Then

$$E\{\cos(\phi + 2\Theta)\} = \int_0^{2\pi} \cos(\phi + 2\theta)\frac{1}{2\pi}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(\phi + 2\theta)\,d\theta$$



$= 0$ because the integral is over two complete cycles of the cosine no matter what is the value of ϕ.

$$\therefore E\{\text{term } 1\} = \tfrac{1}{2}R_{mm}(\tau)\cos(2\pi f_c \tau)$$

Expectations of terms 2, 3, 4 will be similarly computed but will require us to figure out

$E\{m(t)\hat{m}(t+\tau)\}$

$E\{\hat{m}(t)\hat{m}(t+\tau)\}$ etc.

Cross Correlation of $m, \hat{m}$ and Auto Correlation of $\hat{m}$

Theorem on WSS Processes and LTI Systems

If:

$$X(t) \text{ WSS} \longrightarrow \boxed{\begin{array}{c}\text{BIBO Stable} \\ \text{LTI} \\ h(t) \leftrightarrow H(f)\end{array}} \longrightarrow Y(t)$$

; assume real-valued signals for simplicity

Then: $Y(t) = h * X(t)$ is WSS and $X(t), Y(t)$ are jointly WSS with

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)\,dt = \mu_X H(0)$$

$$R_{X,Y}(\tau) = h * R_{X,X}(\tau) \leftrightarrow S_{X,Y}(f) = H(f) S_{X,X}(f)$$

$$R_{Y,Y}(\tau) = \tilde{h} * h * R_{X,X}(\tau) \leftrightarrow S_{Y,Y}(f) = |H(f)|^2 S_{X,X}(f)$$

where $\tilde{h}(t) \triangleq h(-t)$.

$$m(t) \longrightarrow \boxed{\text{Hilbert Transf.} \quad \frac{1}{\pi t} \longleftrightarrow -j\,\mathrm{sgn}(f)} \longrightarrow \hat{m}(t)$$

Don't worry about BIBO as we can always band limit the H.T.

$$\mu_{\hat{m}} = H(0)\mu_m = 0 \quad \text{(also assume } m(t) \text{ is zero mean)}$$

$$R_{m,\hat{m}}(\tau) = h_{Hilbert} * R_{m,m}(\tau) \longleftrightarrow S_{m,\hat{m}}(f) = -j\,\mathrm{sgn}(f)\,S_{m,m}(f)$$

--- in otherwords $R_{m,\hat{m}}(\tau)$ is the Hilbert Transform of $R_{m,m}(\tau)$,

$$R_{m,\hat{m}}(\tau) = \hat{R}_{m,m}(\tau)$$

$$S_{\hat{m},\hat{m}}(f) = \underbrace{\boxed{|H_{Hilbert}(f)|^2}}_{=1} S_{m,m}(f) = S_{m,m}(f)$$

$$\updownarrow$$

$$R_{\hat{m},\hat{m}}(\tau) = R_{m,m}(\tau)$$

Therefore, it will not be hard to show that

$$E\{\text{term } 4\} = E\{\text{term } 1\}$$

Term 2

$$E\{\text{Term } 2\} = R_{m,\hat{m}}(\tau)\,\frac{1}{2}\sin(2\pi f_c \tau)$$

Term 3

$$E\{\text{Term } 3\} = R_{m,\hat{m}}(-\tau)\,\frac{1}{2}\sin(-2\pi f_c \tau) = E\{\text{Term } 2\}$$

because $R_{m,\hat{m}}(\tau)$ is real and odd

why?

$R_{m,\hat{m}}(\tau)$ is real-valued because both $m(t)$ and $\hat{m}(t)$ are real-valued. It is seen to be odd because

$$S_{m,\hat{m}}(f) = -j\,\mathrm{sgn}(f)\,S_{m,m}(f)$$

is seen to be purely imaginary and odd.

Put it Together

$$R_{x,x}(\tau) = 0.25A_c^2\Big\{\underbrace{\tfrac{1}{2}R_{m,m}(\tau)\cos(2\pi f_c\tau)}_{\text{term 1}} \pm \underbrace{\tfrac{1}{2}R_{m,\hat{m}}(\tau)\sin(2\pi f_c\tau)}_{\text{term 2}}$$

$$\pm \underbrace{\tfrac{1}{2}R_{m,\hat{m}}(\tau)\sin(2\pi f_c\tau)}_{\text{term 3}} + \underbrace{\tfrac{1}{2}R_{m,m}(\tau)\cos(2\pi f_c\tau)}_{\text{term 4}}\Big\}$$

$$= 0.25A_c^2\Big\{R_{m,m}(\tau)\cos(2\pi f_c\tau) + R_{m,\hat{m}}(\tau)\sin(2\pi f_c\tau)\Big\}$$

$$= 0.25A_c^2\Big\{R_{m,m}(\tau)\cos(2\pi f_c\tau) \pm \hat{R}_{m,m}(\tau)\sin(2\pi f_c\tau)\Big\}$$

Because $R_{m,\hat{m}}(\tau) = \text{Hilbert Transform}\{R_{m,m}(\tau)\}$

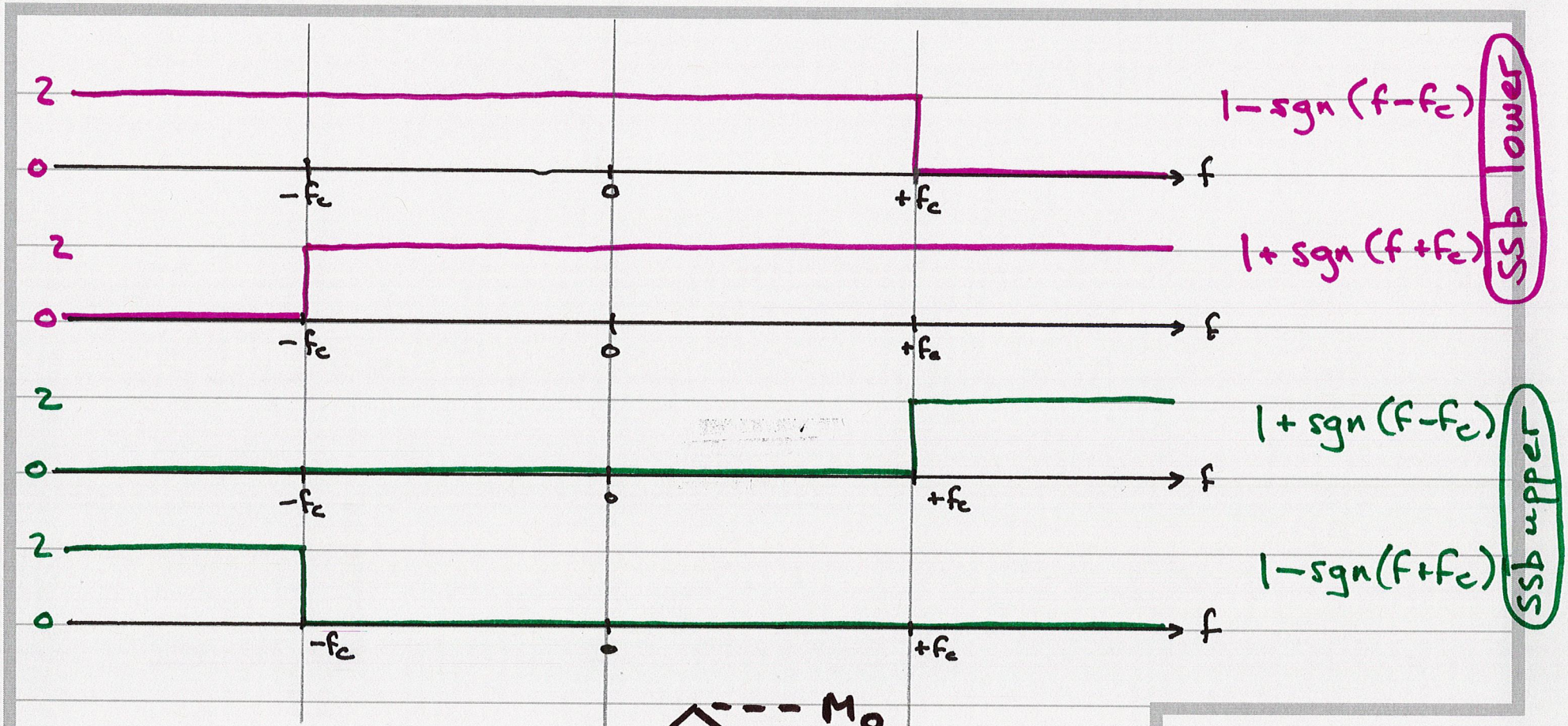
Fourier Transform $S_{xx}(f)$

Use: $R_{m,m}(\tau) \leftrightarrow S_{m,m}(f)$

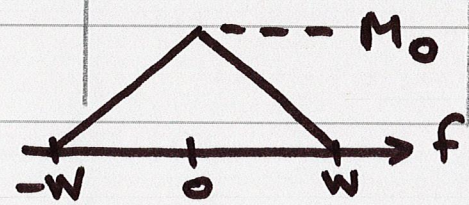
$\hat{R}_{m,m}(\tau) \leftrightarrow \hat{S}_{m,m}(f) = -j\,\text{sgn}(f)S_{m,m}(f)$

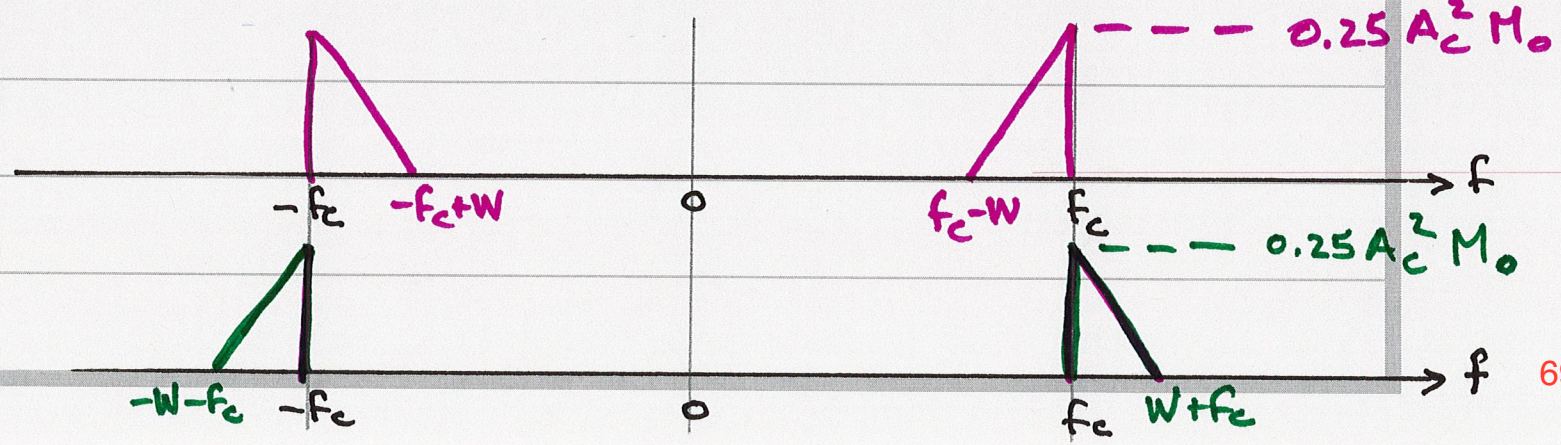
$\cos(2\pi f_c\tau) \leftrightarrow \dfrac{\delta(f-f_c)+\delta(f+f_c)}{2}$

$\sin(2\pi f_c\tau) \leftrightarrow \dfrac{\delta(f-f_c)-\delta(f+f_c)}{j2}$


2
0
$-f_c$
0
$+f_c$
f
$1-\mathrm{sgn}(f-f_c)$
$1+\mathrm{sgn}(f+f_c)$
ssb lower
$1+\mathrm{sgn}(f-f_c)$
$1-\mathrm{sgn}(f+f_c)$
ssb upper

Then if $S_{m,m}(f)$


M_0
f
$-W$
0
W


$0.25 A_c^2 M_0$
$S_{x,x}(f)$
ssb lower
f
$-f_c$
$-f_c+W$
0
f_c-W
f_c
$0.25 A_c^2 M_0$
$S_{x,x}(f)$
ssb upper.
$-W-f_c$
$-f_c$
0
f_c
$W+f_c$

$$R_{m,m}(\tau)\cos(2\pi f_c \tau) \longleftrightarrow \frac{S_{m,m}(f-f_c) + S_{m,m}(f+f_c)}{2}$$

$$\hat{R}_{m,m}(\tau)\sin(2\pi f_c \tau) \longleftrightarrow \frac{-\operatorname{sgn}(f-f_c)S_{m,m}(f-f_c) + \operatorname{sgn}(f+f_c)S_{m,m}(f+f_c)}{2}$$

$$S_{x,x}(f) = 0.125 A_c^2 \left[S_{m,m}(f-f_c) + S_{m,m}(f+f_c) \right]$$

lower

$$\mp 0.125 A_c^2 \left[\operatorname{sgn}(f-f_c)S_{m,m}(f-f_c) - \operatorname{sgn}(f+f_c)S_{m,m}(f+f_c) \right]$$

upper

$$= 0.125 A_c^2 \left(1 \mp \operatorname{sgn}(f-f_c)\right) S_{m,m}(f-f_c)$$

$$+ 0.125 A_c^2 \left(1 \pm \operatorname{sgn}(f+f_c) S_{m,m}(f+f_c)\right)$$

top → ssb lower

bot → ssb upper

Observations from SSB Signal Model

$$\text{Power}\{x(t)\} = R_{x,x}(0) = \int_{-\infty}^{\infty} S_{x,x}(f)\,df$$

$$= 0.25 A_c^2 R_{m,m}(0) = 0.25 A_c^2 P_m$$

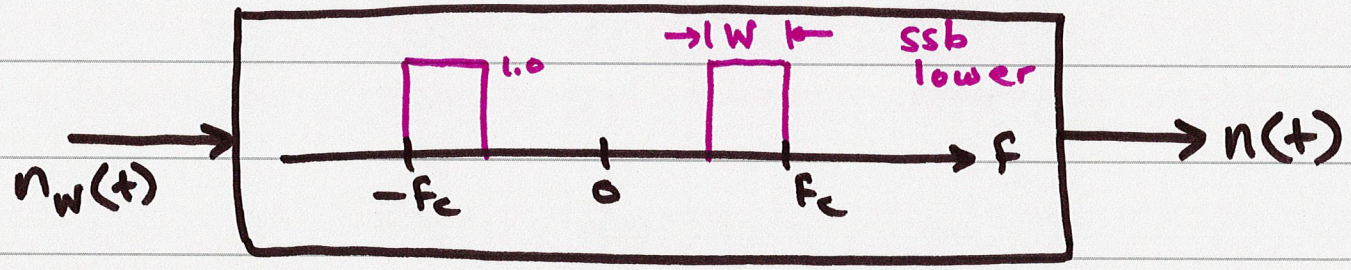
for either upper or lower ssb

Transmission BW needed to pass $x(t)$ undistorted $= W$

Transmission Band : $[-f_c, -f_c+W] \cup [f_c-W, f_c]$ ssb lower

$[-W-f_c, -f_c] \cup [f_c, f_c+W]$ ssb upper

To get noise power at the IF filter output, we pass $n_W(t)$ through the BPF, which is chosen to just pass the info bearing signal.

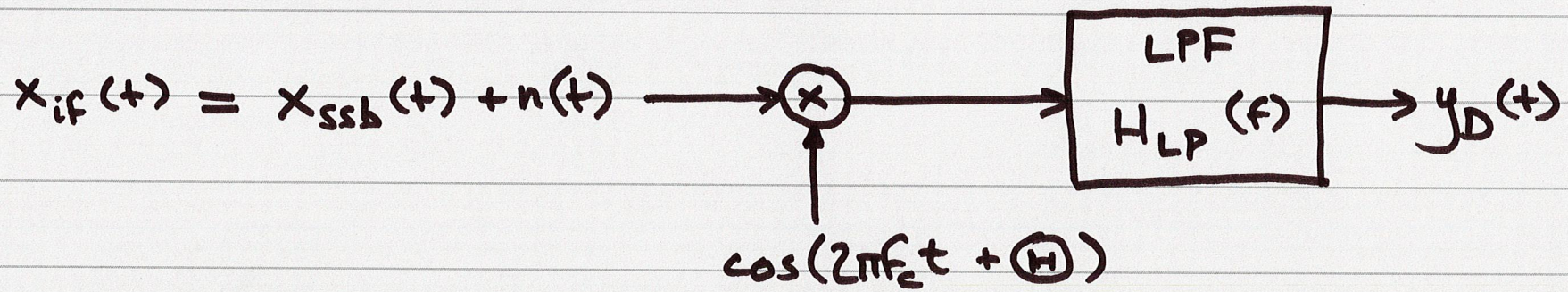


$$\text{noise power} = \frac{N_0}{2} \cdot W + \frac{N_0}{2} W = N_0 W$$

For ssb upper, have the same noise power although the BPF is shifted to pass the upper sideband ...

$$SNR_{IF} = \frac{0.25 A_c^2 P_m}{N_0 W} = \frac{P_T}{N_0 W}$$

Coherent Demodulation of SSB



Can use linearity to conclude:

$$y_D(t) = \underbrace{y_D(t|m)}_{x_{ssb}(t)\cos(2\pi f_c t + \Theta) * h_{LP}(t)} + \underbrace{y_D(t|n_w)}_{n(t)\cos(2\pi f_c t + \Theta) * h_{LP}(t)}$$

Signal Part

$$y_D(t|m) = \left[0.5A_c m(t)\cos(2\pi f_c t + \Theta) \pm 0.5A_c \hat{m}(t)\sin(2\pi f_c t + \Theta)\right] \cdot \cos(2\pi f_c t + \Theta) * h_{LP}(t)$$

Using trig. identities the signal part at the input to the LPF is ...

$$0.5A_c m(t)\left[1 + \frac{\cos(2\pi 2f_c t + 2\Theta)}{2}\right] \pm 0.5A_c \hat{m}(t)\left[\frac{\sin(2\pi 2f_c t + 2\Theta)}{2}\right]$$

The only term to pass the LPF is ...

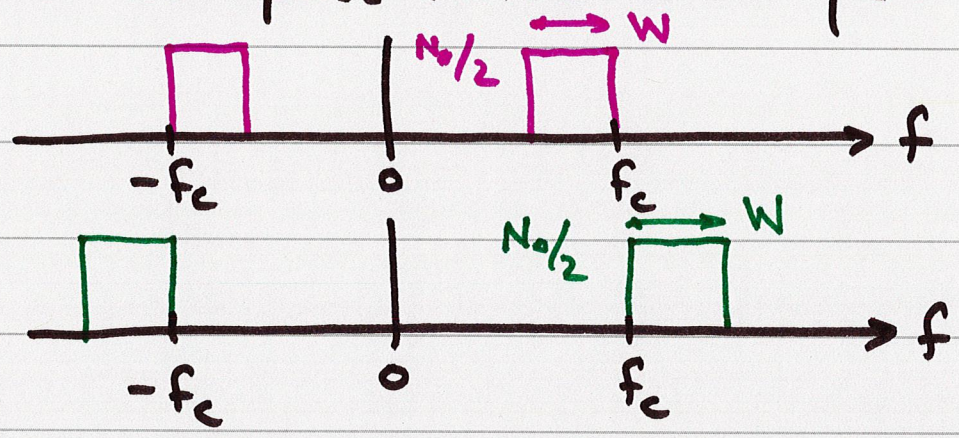
$$y_D(t|m) = 0.25A_c m(t)$$

and

$$\text{Power}\{y_D(t|m)\} = (0.25A_c)^2 P_m$$

Noise Part

$n(t)$ is a bandpass noise with psd $S_{nn}(f)$ equal to



ssb lower

ssb upper.

To figure out the noise power in the output of the LPF we must first find the psd of

$$\tilde{n}(t) \triangleq n(t)\cos(2\pi f_c t + \Theta)$$

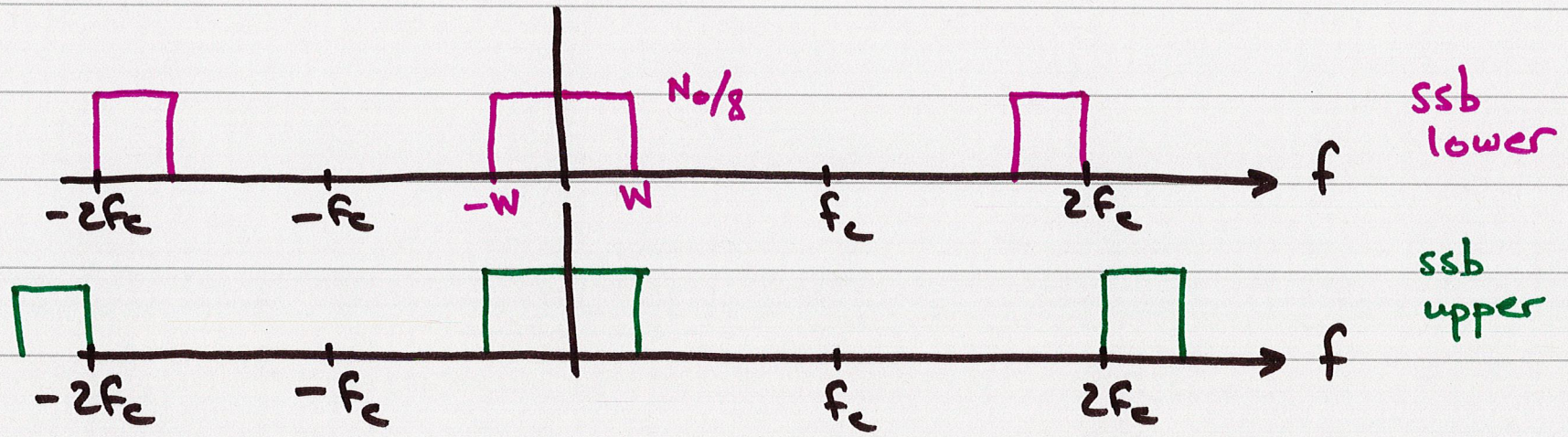
and then apply the LPF $H_{LP}(f)$.

Can show: $R_{\tilde{n},\tilde{n}}(\tau) = 0.5\, R_{n,n}(\tau)\cos(2\pi f_c \tau)$

you should show this.

Then taking the F.T.

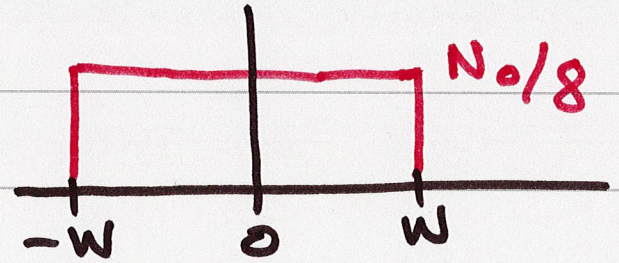
$$S_{\tilde{n},\tilde{n}}(f) = 0.25 S_{n,n}(f-f_c) + 0.25 S_{n,n}(f+f_c)$$



Then $y_D(t|n_w) = \tilde{n} * h_{LP}(t)$ and the psd would be

$$S_{y_D|n_w}(f) = |H_{LP}(f)|^2 S_{\tilde{n},\tilde{n}}(f)$$

$=$



for either lower or upper

$$\therefore \text{Power}\left\{y_D(t|n_w)\right\} = \int_{-\infty}^{\infty} S_{y_D|n_w}(f)\, df$$

$$= 2W \cdot \frac{N_o}{8} = \frac{N_o W}{4}$$

$$SNR_D = \frac{(0.25 A_c)^2 P_m}{0.25\ N_o W} = \frac{0.25 A_c^2 P_m}{N_o W}$$

$$= \frac{P_T}{N_o W}$$

Angle modulation

angle modulated wave: $x(t) = A_c \cos\left[\theta_i(t)\right]$

angle of modulated sinusoidal carrier ... presumably a funct. of $m(t)$.

instantaneous frequency:

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

some possibilities:

1) unmodulated carrier $\theta_i(t) = 2\pi f_c t + \phi_c$

2) phase modulation $\theta_i(t) = 2\pi f_c t + k_p m(t)$

phase sensitivity or deviation (radians/volt).

3) frequency modulation $f_i(t) = f_c + f_d\, m(t)$

frequency sensitivity or deviation (Hz/volt)

$$\Rightarrow \int_{t_0}^{t} d\theta_i(s) = 2\pi \int_{t_0}^{t} f_i(s)\, ds$$

$$\theta_i(t) = 2\pi f_c t + 2\pi f_d \int_{t_0}^{t} m(s)\, ds + \underbrace{\theta_0}_{\theta_i(t_0) - 2\pi f_c t_0}$$

N.B. There are technology limits to how big f_d can be usually of form $f_d \le \eta f_c$ typ. $\eta \sim 0.05 - 0.1$

Usually don't worry about initial phase since its just an arb. ref. point.

Thus:

1) output of a phase modulator

$$x(t) = A_c \cos\left(2\pi f_c t + k_p m(t)\right)$$

2) output of a frequency modulator

$$x(t) = A_c \cos\left(2\pi f_c t + 2\pi f_d \int^{t} m(s)\, ds\right)$$

$\Longrightarrow$ Check out figs 3.22 + 3.23 in Z+T for examples.

Shows case where $m(t)$ is sinusoidal. In the mod waveform:

PM $\mapsto$ inst. freq. is max when $\dot{m}(t)$ is max.

FM $\mapsto$ inst. freq. is max when $m(t)$ is max

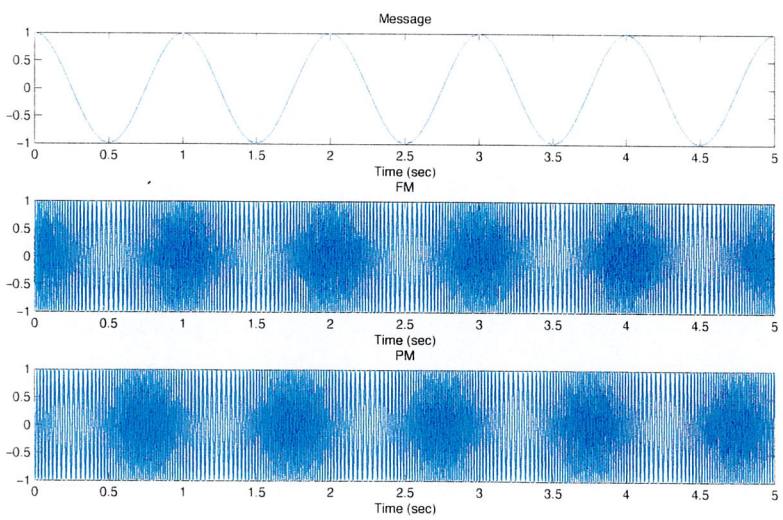


Figure 1: Time domain comparison of FM and PM for sinusoidal message.

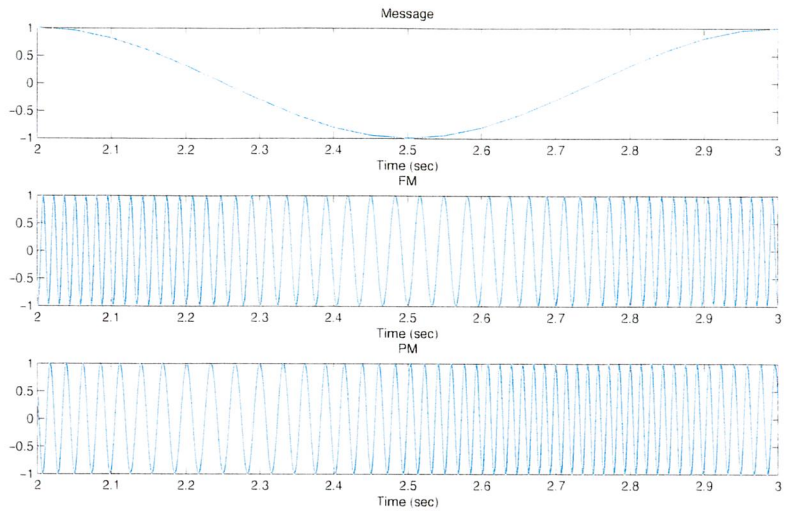


Figure 2: Time domain comparison of FM and PM for sinusoidal message (zoomed).

From comparison of 1) and 2) see that

FM signal is a PM signal where $m(t)$ is replaced by

$$\int_{t_0}^{t} m(s)\,ds$$

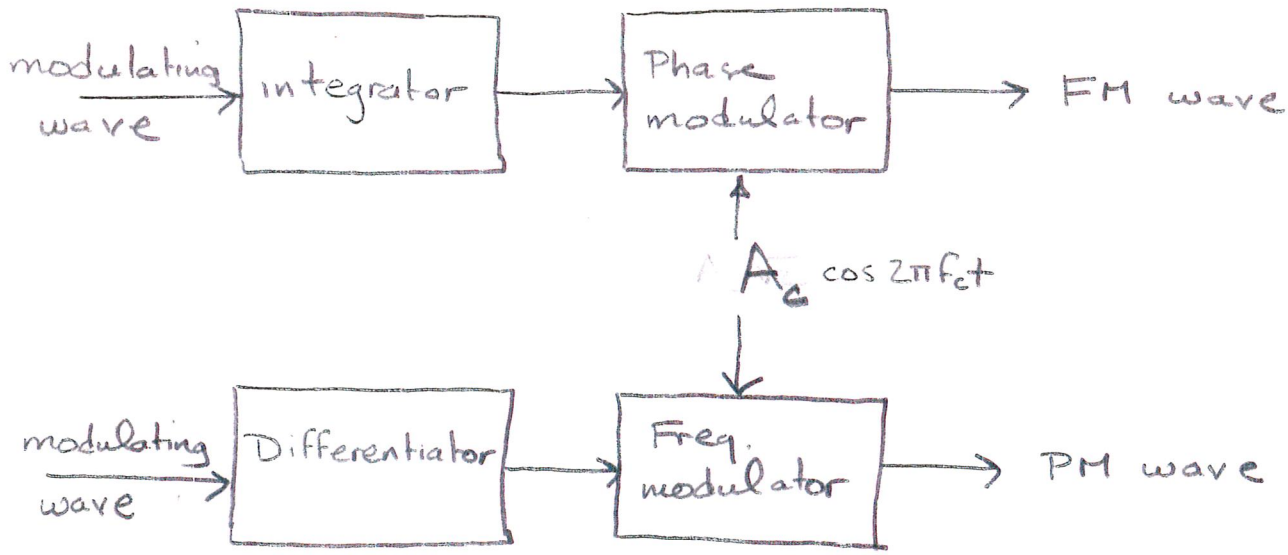
and $k_p = 2\pi f_d$.

Conversely

PM signal is an FM signal where $m(t)$ is replaced by $m'(t)$.

Thus FM & PM are not fundamentally different and we will cover only FM.

Relationship Between FM and PM



Write $\theta_i(t) = 2\pi f_c t + \phi(t)$. Then

$$x(t) = A_c \cos\left[\theta_i(t)\right] = A_c \cos\left[2\pi f_c t + \phi(t)\right]$$

$$= \mathrm{Re}\left\{ A_c e^{j2\pi f_c t} e^{j\phi(t)} \right\}$$

Taylor

$$= 1 + j\phi(t) - \frac{\phi^2(t)}{2!} + \cdots$$

$$\approx 1 + j\phi(t) \quad \text{if } |\phi(t)| \text{ is small.}$$

$$x(t) \cong \mathrm{Re}\left\{ A_c e^{j2\pi f_c t} + A_c j\phi(t) e^{j2\pi f_c t} \right\}$$

$$= A_c \cos 2\pi f_c t - A_c \phi(t) \sin 2\pi f_c t$$

$\Longrightarrow$ This is leading up to the idea of narrowband FM.

<u>Special (though important) Case</u>

$m(t) = A_m \cos(2\pi f_m t)$ input to an FM modulator whence instantaneous freq. is:

$$f_i(t) = f_c + f_d A_m \cos(2\pi f_m t) = f_c + \Delta f \cos 2\pi f_m t$$

where $\Delta f = f_d A_m$ is called the <u>frequency deviation</u> ie max departure of inst. freq. from the carrier freq.

$\Longrightarrow$ Angle of FM signal

$$\theta_i(t) = 2\pi f_c t + 2\pi f_d \int^t A_m \cos(2\pi f_m s)\, ds$$

$$= 2\pi f_c t + 2\pi \Delta f \frac{1}{2\pi f_m} \sin(2\pi f_m t)$$

$$= 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t)$$

$\frac{\Delta f}{f_m} \triangleq \beta$ called the modulation index. Represents max phase deviation. Unit is radian.

$$\boxed{x(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)}$$

Two cases

1) $\beta << 1 \longrightarrow$ narrowband FM

2) $\beta >> 1 \longrightarrow$ wideband FM

Narrowband FM

With a trig. identity

$$x(t) = A_c \cos(2\pi f_c t)\cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t)\sin[\beta \sin(2\pi f_m t)].$$

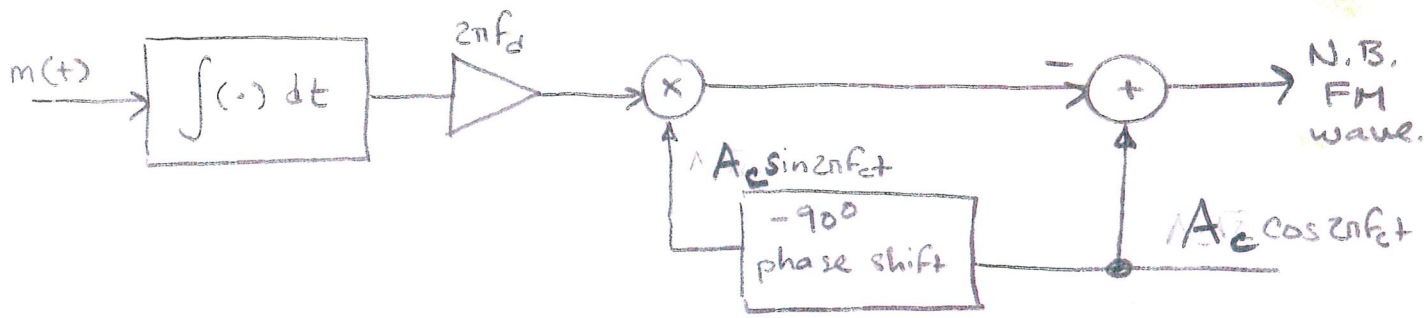
If $\beta << 1$ then

$$\cos[\beta \sin 2\pi f_m t] \approx 1$$

$$\sin[\beta \sin 2\pi f_m t] \approx \beta \sin 2\pi f_m t$$

$$\Longrightarrow \quad x(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_m t)\sin(2\pi f_c t)$$

Approximation suggests following approximate modulator:



Further simp. of $x(t)$

$$x(t) \approx A_c \cos(2\pi f_c t) + \frac{A_c}{2}\left\{\cos\left[2\pi(f_c+f_m)t\right] - \cos\left[2\pi(f_c-f_m)t\right]\right\}$$

(*)

Want to Compare with prev. result for tone modulated AM-LC so recall: $m(t) = A_m \cos(2\pi f_m t)$

$$\Rightarrow \text{AM-LC} \quad x(t) = \left[1 + \underbrace{k_a A_m}_{\mu} \cos(2\pi f_m t)\right] A_c \cos(2\pi f_c t)$$

$$= A_c \cos(2\pi f_c t) + \frac{\mu A_c}{2}\left\{\cos\left[2\pi(f_c+f_m)t\right] + \cos\left[2\pi(f_c-f_m)t\right]\right\}$$

Fund. Diff. Between N.B. FM and AM-LC examples (with Sinusoidal mod.) is in

(*) signs of lower side freq. components.

$\Rightarrow$ N.B. FM requires essentially same transmission BW (ie $2f_m$) as does AM-LC.

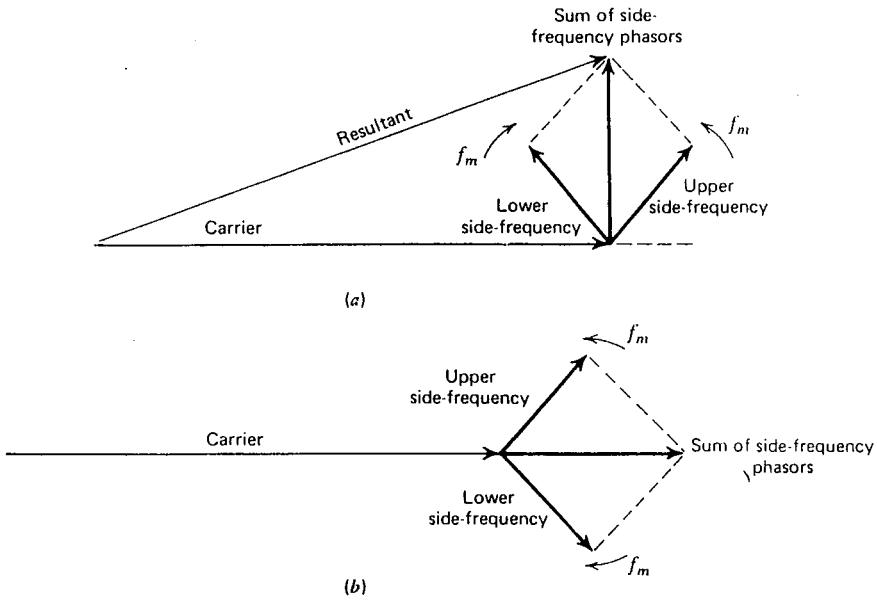


Figure 3.32 A phasor comparison of narrow-band FM and AM waves for sinusoidal modulation. (*a*) Narrow-band FM wave. (*b*) AM wave.

1 Introduction

2 Non-Linear Analog Modulations

Let $m(t) \leftrightarrow M(f)$ be the message. Let W denote its baseband BW. If message is a power signal let P_m denote its power.

- General angle modulated wave:

$$x(t) = A_c \cos(\theta(t))$$

where $\theta(t)$ is some function of the message. Instantaneous frequency is:

$$f(t) = \frac{1}{2\pi}\dot{\theta}(t).$$

- PM: $\theta(t) = 2\pi f_c t + k_p m(t) + \theta_0$ hence

$$x_{pm}(t) = A_c \cos(2\pi f_c t + k_p m(t) + \theta_0).$$

  * k_p is phase sensitivity of modulator in radians/volt.

  * Peak phase deviation from carrier phase is $D = k_p \max_t |m(t)|$.
  * Transmission BW $\approx 2(D+1)W$.
  * power$[x_{pm}(t)] = 0.5A_c^2$.

- FM: $f(t) = f_c + f_d m(t)$ hence[1]

$$x_{fm}(t) = A_c \cos\left(2\pi f_c t + 2\pi f_d \int_{-\infty}^{t} m(s)\mathrm{d}s + \theta_0\right).$$

  * f_d is frequency sensitivity of modulator in Hz/volt.
  * Peak frequency deviation[2] from carrier frequency is $\Delta_f = f_d \max_t |m(t)|$.
  * Transmission BW $\approx 2(\Delta_f + W)$.
  * power$[x_{fm}(t)] = 0.5A_c^2$.

2.1 FM and PM are Closely Related

A phase modulator can be used to create an FM wave and an FM modulator can be used to create a PM wave.

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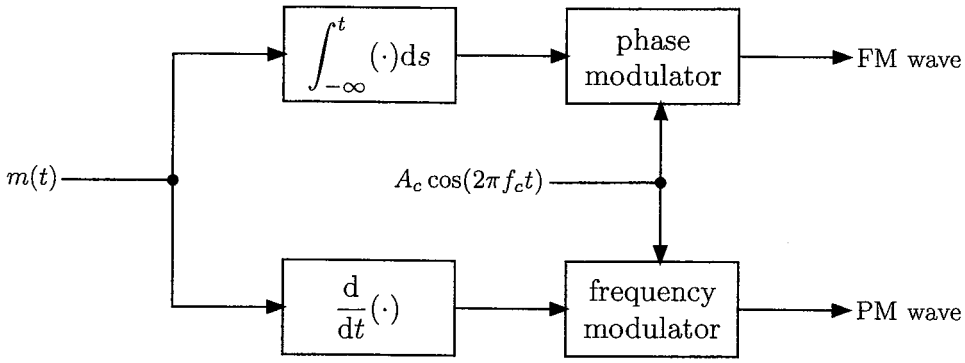
[1] To derive form below for $x_{fm}(t)$ separate variables and integrate $\mathrm{d}\theta(t) = 2\pi f(t)\mathrm{d}t$ from t_0 to t.

[2] The maximum frequency deviation from the carrier, i.e., $\max_t |f(t) - f_c|$ is limited by the available technology. Typically, it is on the order of 5 to 10 percent of the carrier.



2.2 Examples

1. Examples for FM/PM modulation of a sinusoidal message are shown in Figures 1, 2, 3. The following parameters were used:

   - Sinusoidal message frequency = 1 Hz.
   - Carrier frequency = 50 Hz.
   - Peak frequency deviation (FM case) = 20 Hz.
   - Peak phase deviation (PM case) = 20 Hz / 1 Hz = 20 radians (in order to approximately set the transmission bandwidths of FM and PM equal).

2. Examples for FM/PM modulation of a square wave and sawtooth messages are shown in Figures 4 and 5. The following parameters were used:

   - Fundamental message frequency = 1 Hz.
   - Carrier frequency = 50 Hz.
   - Peak frequency deviation (FM case) = 20 Hz.
   - Peak phase deviation (PM case) = 1 radian (here not trying to set the BWs of FM and PM equal).

3. For now on we concentrate on FM.

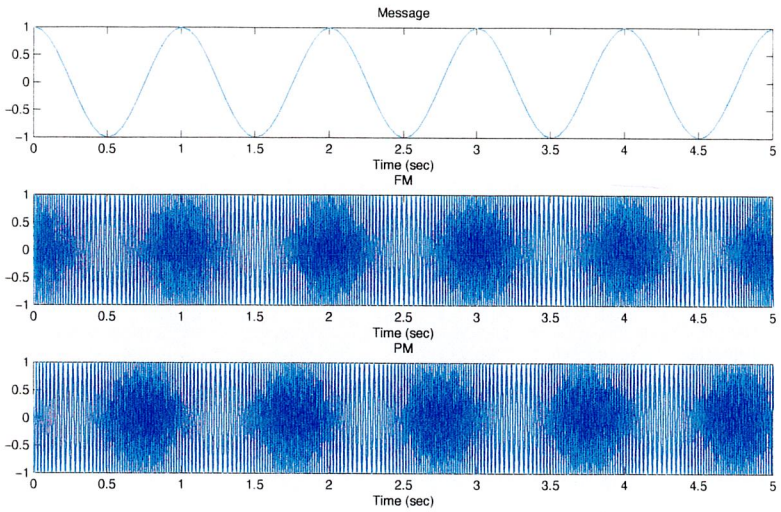


Figure 1: Time domain comparison of FM and PM for sinusoidal message.

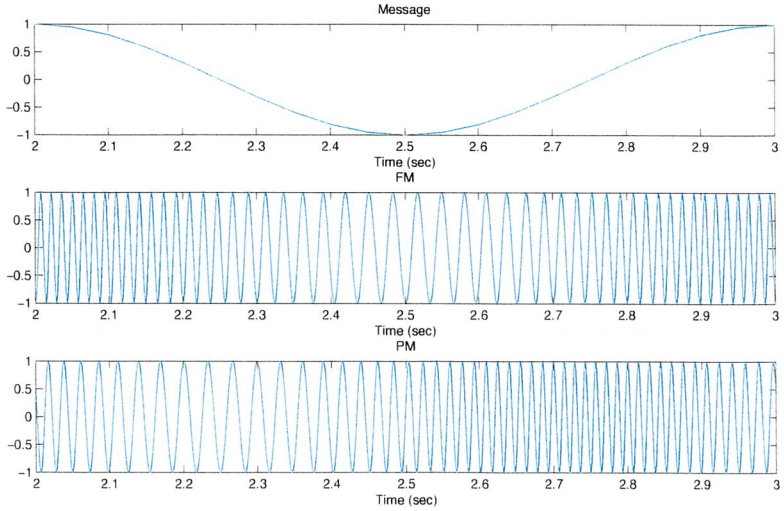


Figure 2: Time domain comparison of FM and PM for sinusoidal message (zoomed).

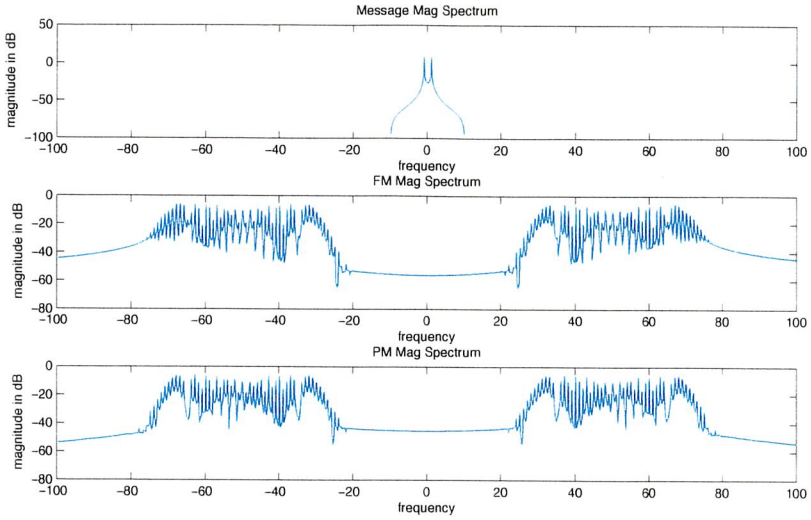
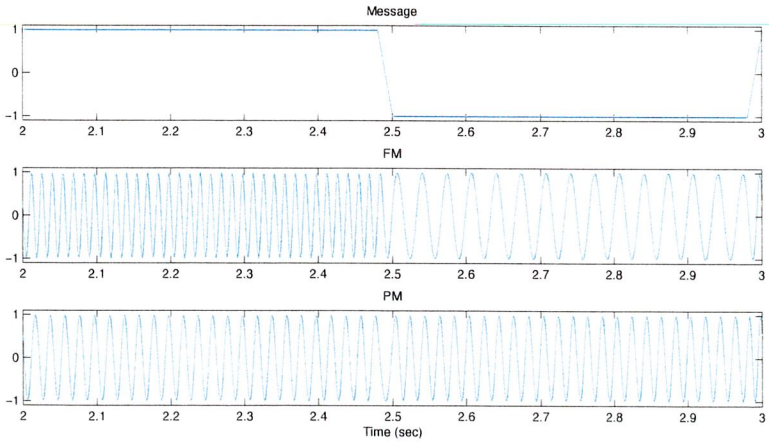
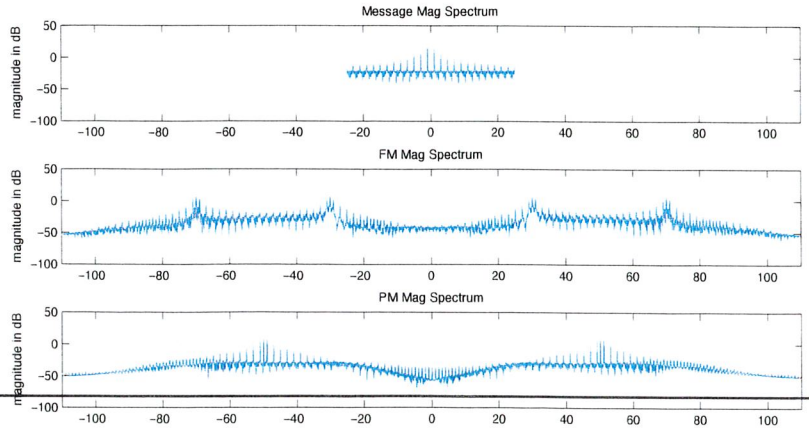


Figure 3: Frequency domain comparison of FM and PM for sinusoidal message.



(a)



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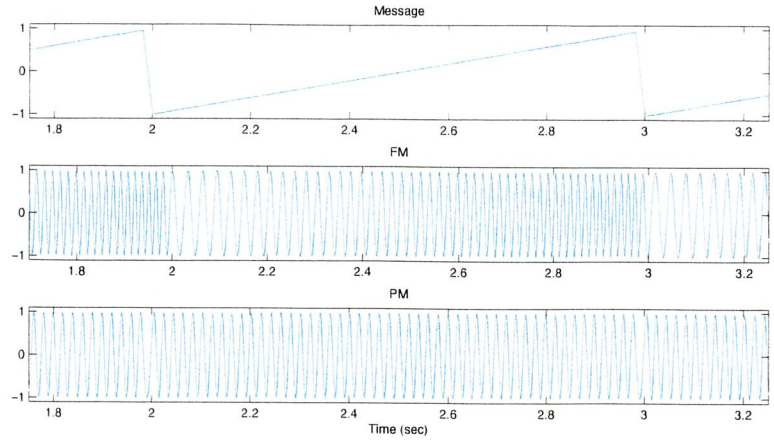
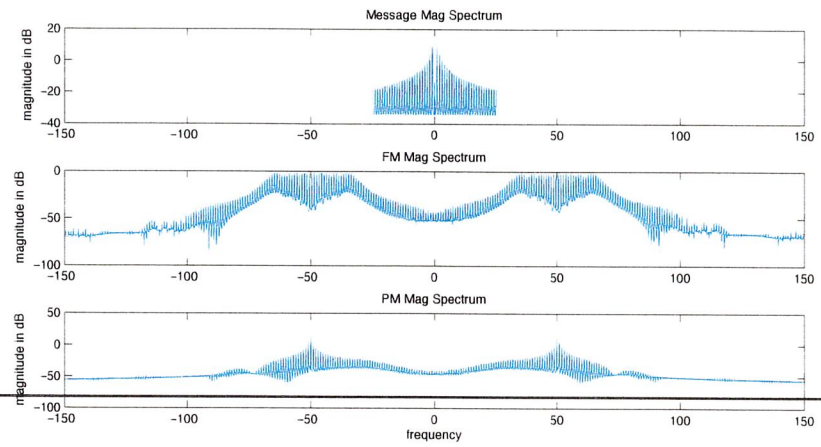


(a)



3 Narrowband FM

1. Consider sinusoidal modulating wave: $m(t) = A_m \cos(2\pi f_m t)$. Instantaneous frequency is

$$f(t) = f_c + f_d A_m \cos(2\pi f_m t) = f_c + \Delta_f \cos(2\pi f_m t),$$

where $\Delta_f = f_d A_m$ is called the peak frequency deviation, i.e., the maximum departure of the instantaneous frequency from the carrier frequency.

2. Compute the angle of the FM signal in this special case (arbitrarily set the phase at $t = 0$ to zero):

$$\begin{aligned}
\theta(t) &= 2\pi f_c t + 2\pi f_d \int_{t_0}^{t} A_m \cos(2\pi f_m s) \mathrm{d}s \\
&= 2\pi f_c t + 2\pi \Delta_f \frac{\sin(2\pi f_m t)}{2\pi f_m} \\
&= 2\pi f_c t + \underbrace{\frac{\Delta_f}{f_m}}_{\beta} \sin(2\pi f_m t).
\end{aligned}$$

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$\beta = \Delta_f / f_m$ (unit is radian) is sometimes called the modulation index or peak phase deviation.

3. The FM wave with sinusoidal $m(t)$ can then be written

$$x_{fm}(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right).$$

4. Two cases are considered

$$\begin{aligned} \beta \ll 1 &\implies \text{narrowband FM} \\ \beta \gg 1 &\implies \text{wideband FM} \end{aligned}$$

5. For remainder of this section consider the narrowband FM case.

6. With a trigonometric identity (i.e., $\cos(u+v) = \cos u \cos v - \sin u \sin v$)

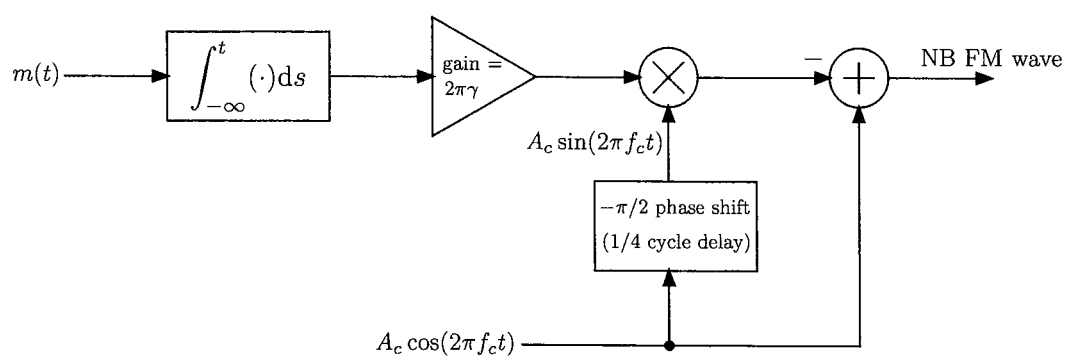
$$\begin{aligned} x_{fm}(t) &= A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right) \\ &= A_c \cos(2\pi f_c t) \cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t) \sin[\beta \sin(2\pi f_m t)] \end{aligned}$$

If $\beta \ll 1$ then:

$$\cos[\beta \sin(2\pi f_m t)] \approx 1 \quad \text{and} \quad \sin[\beta \sin(2\pi f_m t)] \approx \beta \sin(2\pi f_m t)$$

$$x_{fm}(t) \approx A_c \cos(2\pi f_c t) - A_c \beta \sin(2\pi f_m t) \sin(2\pi f_c t)$$

7. The above approximation suggests the following approximate modulator block diagram:



8. Using trigonometric identity $[\sin u \sin v = 0.5\cos(u-v) - 0.5\cos(u+v)]$ we may rewrite the narrowband $x_{fm}(t)$:

$$x_{fm}(t) = A_c \cos(2\pi f_c t) + 0.5 A_c \beta \cos[2\pi(f_c + f_m)t] - 0.5 A_c \beta \cos[2\pi(f_c - f_m)t]. \tag{1}$$

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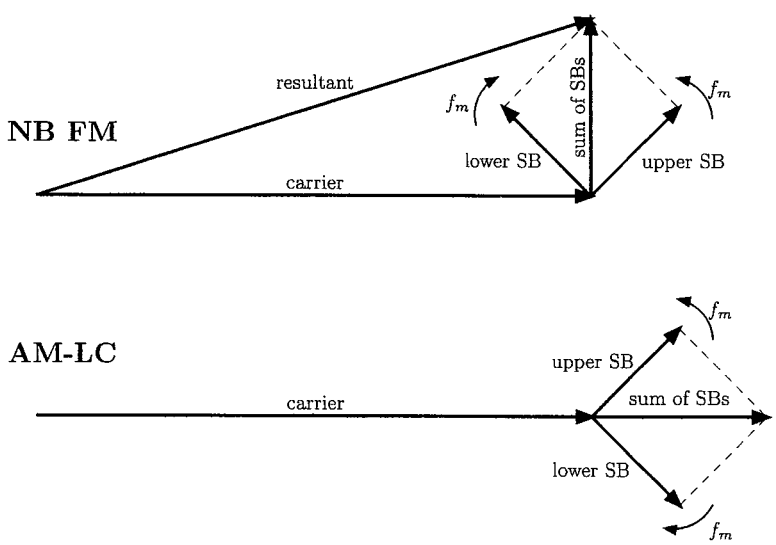
9. Wish to compare the above with the previous result for tone modulated AM-LC. With $m(t) = A_m \cos(2\pi f_m t)$ we had

$$\begin{aligned} x_{am-lc}(t) &= [1 + k_a A_m \cos(2\pi f_m t)] A_c \cos(2\pi f_c t) \qquad (2) \\ &= A_c \cos(2\pi f_c t) + 0.5 A_c \mu \cos[2\pi(f_c + f_m)t] + 0.5 A_c \mu \cos[2\pi(f_c - f_m)t] \end{aligned}$$

with $\mu \stackrel{\text{def}}{=} k_a A_m$.

10. Comparing Equations (1) and (2) we see that the difference between NB FM and AM-LC is entirely in the signs of the lower side frequency components (i.e., the $f_c - f_m$ terms). In particular, we observe that NB FM requires exactly the same transmission BW as does AM-LC (i.e., $2f_m$ for both).

11. Sometimes useful to think of the difference between NB FM and AM-LC by looking at phasor diagrams:



4.2 For Sinusoidal Message

1. As previously noted an FM wave with sinusoidal $m(t)$ can be written

$$\begin{aligned} x_{fm}(t) &= A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right) \\ &= A_c \mathrm{Re}\left\{e^{j\beta \sin(2\pi f_m t)} e^{j2\pi f_c t}\right\}. \end{aligned}$$

Thus, we start by computing the FS of the periodic (complex-valued) waveform $e^{j\beta \sin(2\pi f_m t)}$, which has period $T_m = 1/f_m$.

2. From the FS definition

$$\begin{aligned} C_k &= \frac{1}{T_m}\int_{-T_m/2}^{T_m/2} e^{j\beta \sin(2\pi f_m t)} e^{-jk2\pi f_m t} \mathrm{d}t \\ &= \frac{1}{T_m}\int_{-T_m/2}^{T_m/2} e^{-j(k2\pi f_m t - \beta \sin(2\pi f_m t))} \mathrm{d}t \qquad \text{(c.o.v. } z = 2\pi f_m t) \\ &= \frac{1}{T_m}\int_{-\pi}^{\pi} e^{-j(kz - \beta \sin z)} \frac{\mathrm{d}z}{2\pi f_m} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(kz - \beta \sin z)} \mathrm{d}z \\ &\stackrel{\text{def}}{=} J_k(\beta) \qquad \text{(Bessel funct., first kind, order } k \text{ and argument } \beta) \end{aligned}$$

3. Therefore, have the expansion

$$\begin{aligned} x_{fm}(t) &= A_c \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} J_k(\beta) e^{+jk2\pi f_m t} e^{j2\pi f_c t}\right\} \\ &= A_c \sum_{k=-\infty}^{\infty} \mathrm{Re}\left\{J_k(\beta) e^{+j2\pi(f_c + k f_m)t}\right\} \end{aligned} \tag{3}$$

4. This can be simplified because the $J_k(\beta)$ are real-valued[3]. Therefore, Eq. (3) can be written

$$\begin{aligned} x_{fm}(t) &= A_c \sum_{k=-\infty}^{\infty} J_k(\beta) \mathrm{Re}\left\{e^{+j2\pi(f_c + k f_m)t}\right\} \\ &= A_c \sum_{k=-\infty}^{\infty} J_k(\beta) \cos(2\pi(f_c + k f_m)t). \end{aligned} \tag{4}$$

[3]To see this

$$\begin{aligned} J_k^*(\beta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(kz - \beta \sin z)} \mathrm{d}z \qquad \text{(c.o.v. } \zeta = -z,\ \mathrm{d}\zeta = -\mathrm{d}z) \\ &= \frac{1}{2\pi}\int_{\pi}^{-\pi} e^{j(-k\zeta - \beta \sin(-\zeta))} (-\mathrm{d}\zeta) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(k\zeta - \beta \sin \zeta)} \mathrm{d}\zeta \;=\; J_k(\beta). \end{aligned}$$

5. Taking the CTFT term by term

$$X_{fm}(f) = 0.5A_c \sum_{k=-\infty}^{\infty} J_k(\beta)\left[\delta(f - f_c - kf_m) + \delta(f + f_c + kf_m)\right].$$

6. Summary of useful Bessel function properties:

   (a) $J_n(\beta) = (-1)^n J_{-n}(\beta)$ for all integers n.

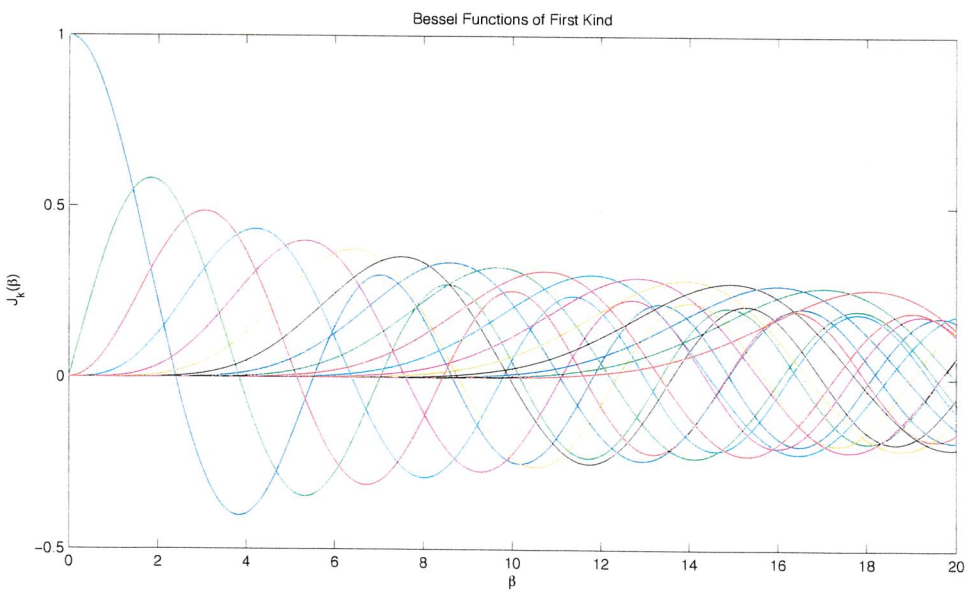
   (b) $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$.

   (c) For small β: $J_0(\beta) \approx 1$, $J_1(\beta) \approx \beta/2$,

   $$J_n(\beta) \approx 0, \quad n \geq 2.$$

   (This corresponds to the NB FM case where have essentially only the carrier and two side frequencies.)

   (d) A plot of Bessel functions is easily created with Matlab:



7. The total power in the sinusoidally modulated FM wave of Eq. (4) can be written

$$\text{Total power} = 0.5A_c^2 \sum_{k=-\infty}^{\infty} J_k^2(\beta) = 0.5A_c^2$$

which does not actually depend on β. While the total power does not depend

Generic FM

$$x_{fm}(t) = A_c \cos\left(2\pi f_c t + 2\pi f_d \int_{-\infty}^{t} m(s)\,ds + \theta_0\right)$$

f_d = freq. sensitivity Hz/Volt

$\Delta_f = f_d \max_t |m(t)|$ = peak freq. deviation

$\text{Power}\left[x_{fm}(t)\right] = A_c^2/2$

Sinusoidal Message

$m(t) = A_m \cos 2\pi f_m t$

$\Delta_f = A_m f_d \qquad \beta \triangleq \Delta_f / f_m$ = modulation index, a.k.a. peak phase deviation

$$\Rightarrow x_{fm}(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)$$

$\beta \ll 1 \Rightarrow$ narrowband FM

$\beta \gg 1 \Rightarrow$ wideband FM

Bandwidth Sinusoidal Message

$$x_{fm}(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)$$

$$= A_c \,\mathrm{Re}\left\{ e^{j\beta \sin(2\pi f_m t)} \, e^{j2\pi f_c t} \right\}$$

$$= A_c \sum_{k=-\infty}^{\infty} J_k(\beta) \cos\left(2\pi (f_c + k f_m) t\right)$$

where

$$J_k(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j(kz - \beta \sin z)}\, dz$$

Bessel, 1st Kind, order k, argument β.

Bessel Properties

$$J_n(\beta) = (-1)^n J_{-n}(\beta)$$

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

For small β: $\quad J_0(\beta) \approx 1 \qquad J_1(\beta) \approx \beta/2$

$$J_n(\beta) \approx 0 \qquad n \geq 2$$

Also easy to create a plot of Bessels using Matlab.

Power in a Band

$$\text{Total Power} = \frac{1}{2}A_c^2 \sum_{k=-\infty}^{\infty} J_k^2(\beta) = \frac{1}{2}A_c^2 \underbrace{\left\{ J_0^2(\beta) + 2\sum_{k=1}^{\infty} J_k^2(\beta) \right\}}_{\triangleq \rho_\infty = 1}$$

$$= \frac{1}{2}A_c^2 \rho_\infty = \frac{1}{2}A_c^2$$

Power in Band: $f_c - Kf_m \leq f \leq f_c + Kf_m$

$$= \frac{1}{2}A_c^2 \rho_K \qquad \left(\rho_K \triangleq J_0^2(\beta) + 2\sum_{k=1}^{K} J_k^2(\beta) \right)$$

ρ_K is monotonically increasing with K

$0 \leq \rho_K < 1$ and $\lim_{K \to \infty} \rho_K = 1$

ρ^* = a target fraction of total power

e.g., $\rho^* = 0.95$ or 0.98

$$K^*(\beta) = \min\left\{K : J_0^2(\beta) + 2\sum_{k=1}^{K} J_k^2(\beta) \geq \rho^*\right\}$$

Transmission BW corresponding to ρ^* power containment:

$$BW_{\rho^*} = 2K^*(\beta) f_m$$

5

Plot ...

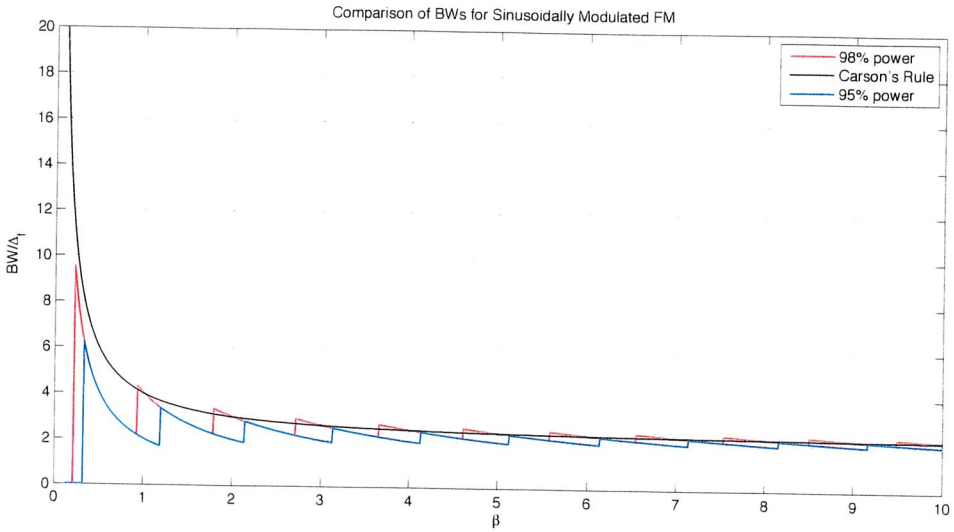
$$BW_{\rho^*}/\Delta_f = \frac{2K^*(\beta) f_m}{\Delta_f} = \frac{2K^*(\beta)}{\beta}$$

Recall Carson's Rule

$$BW_{carson} = 2(\Delta_f + f_m)$$

$$BW_{carson}/\Delta_f = 2\left(1 + \frac{f_m}{\Delta_f}\right) = 2\left(1 + \frac{1}{\beta}\right)$$

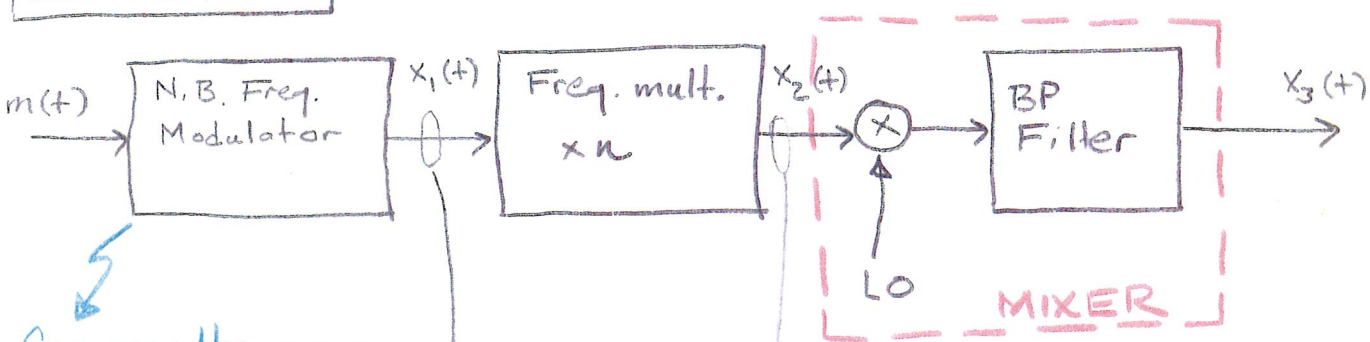
6



Generation of FM Signals

Two methods; indirect FM & direct FM

Indirect FM



Can use the approx. modulator of **NB FM**. **discussion** ...

N.B. FM wave with
- carrier f_{c1}
- peak freq. deviation $f_{\Delta 1}$
- deviation ratio D_1

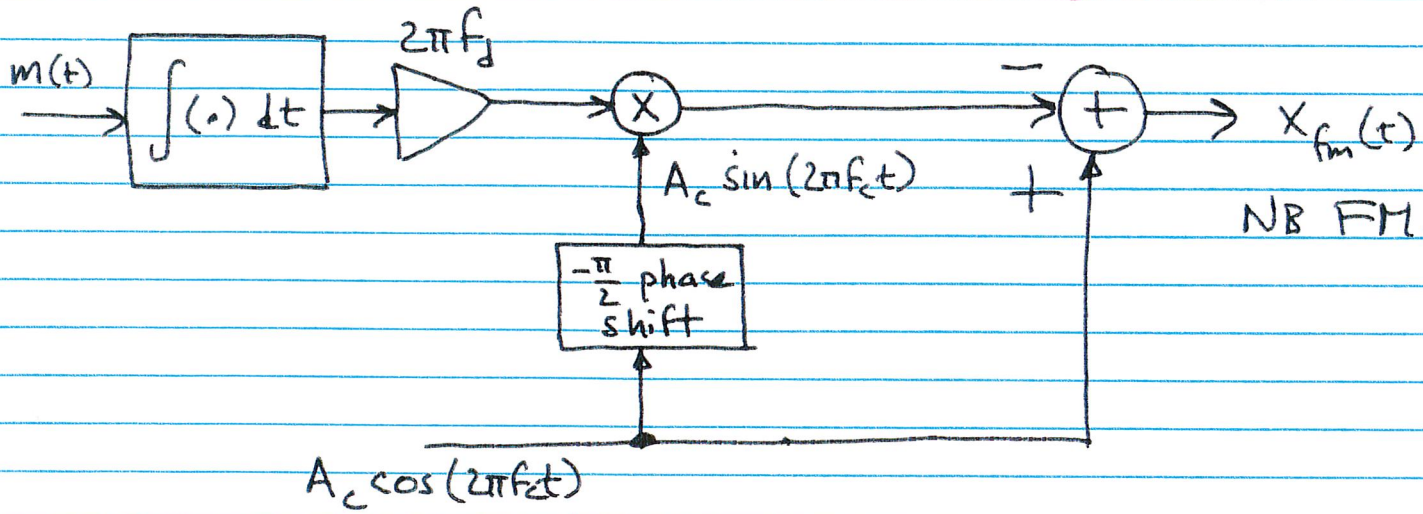
W.B. FM wave with
- carrier $f_{c2} = nf_{c1}$
- peak freq. deviation $f_{\Delta 2} = nf_{\Delta 1}$
- Dev. ratio $D_2 = nD_1$

Approx. FM Modulator

$$x_{fm}(t) \approx A_c \cos(2\pi f_c t) - A_c \beta \sin(2\pi f_m t) \sin(2\pi f_c t)$$

for $\beta \ll 1$

$$\beta \sin(2\pi f_m t) = 2\pi f_d \int^{t} A_m \cos(2\pi f_m s)\, ds$$



9.

Operation of Indirect FM

$$x_1(t) = A_c \cos(2\pi f_{c_1} t + \phi(t))$$

$$x_2(t) = A_c \cos(2\pi n f_{c_1} t + n\phi(t))$$

Say LO is $\cos(2\pi f_{LO} t)$:

$$x_3(t) = \text{BP filter}\left(A_c \cos(2\pi(n f_{c_1} + f_{LO})t + n\phi(t)) + A_c \cos(2\pi(n f_{c_1} - f_{LO})t + n\phi(t)) \right)$$

$$= A_c \cos(2\pi f_c t + n\phi(t)) \quad \text{where } f_c = n f_{c_1} + f_{LO}$$

or

$$f_c = n f_{c_1} - f_{LO}$$

dep. on choice of BPF center freq.

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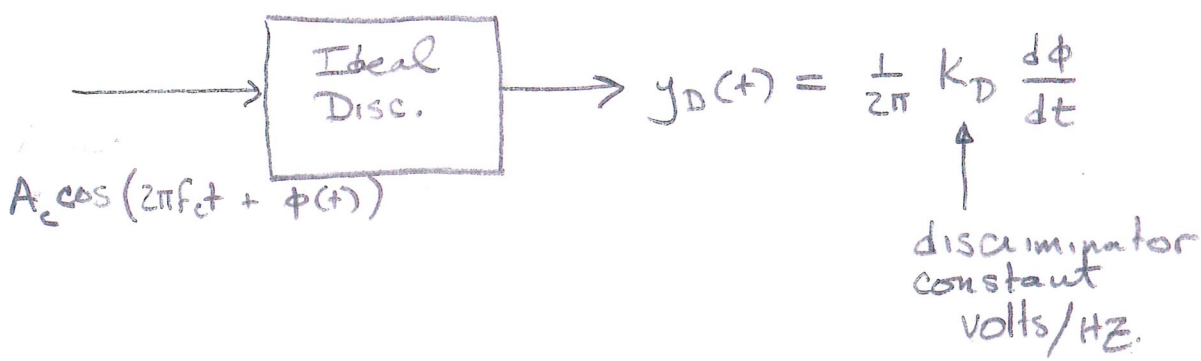
Direct FM

Via a voltage controlled oscillator.

Demodulation of FM Signals

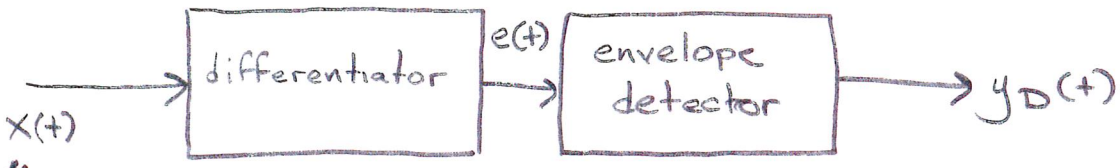
Need to "invert" the FM modulation process (which is a nonlinear system, by the way).

Discriminator = a device whose instantaneous output is proportional to the instantaneous freq. of the input.



For FM: $\phi(t) = 2\pi f_d \int_{t_0}^{t} m(s)\, ds \implies y_D(t) = K_D f_d m(t)$

Approximate Implementation of Discriminator



$$x(t) = A_c \cos(2\pi f_c t + \phi(t))$$

$$\implies e(t) = \frac{d}{dt} x(t) = -A_c \sin(2\pi f_c t + \phi(t)) \left[2\pi f_c + \frac{d\phi}{dt}\right]$$

13.

Let $\frac{d\phi}{dt} \triangleq \dot{\phi}$

$$e(t) = -A_c \left[2\pi f_c + \dot{\phi}(t)\right] \sin(2\pi f_c t + \phi(t))$$

$$\therefore \quad y_D(t) = \text{Envelope}\{e(t)\}$$

$$= A_c \left|2\pi f_c + \dot{\phi}(t)\right|$$

$$= A_c \left[2\pi f_c + \dot{\phi}(t)\right] \quad \text{if } 2\pi f_c > -\dot{\phi}(t) \;\; \forall t$$

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Typically the DC term in $y_D(t)$ is also removed so

$$y_D'(t) = A_c \dot{\phi}(t) = 2\pi f_d m(t) A_c$$

$$= K_D f_d m(t) \quad \text{if} \quad K_D = 2\pi A_c$$

See also Fig. 4.13 Z+T and discussion of BP limiter.

Other practical considerations p178-180 + Figs. 4.16, 4.17

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5 Superheterodyne Receiver

Read Z+T Sec. 3.5 pp. 136–139

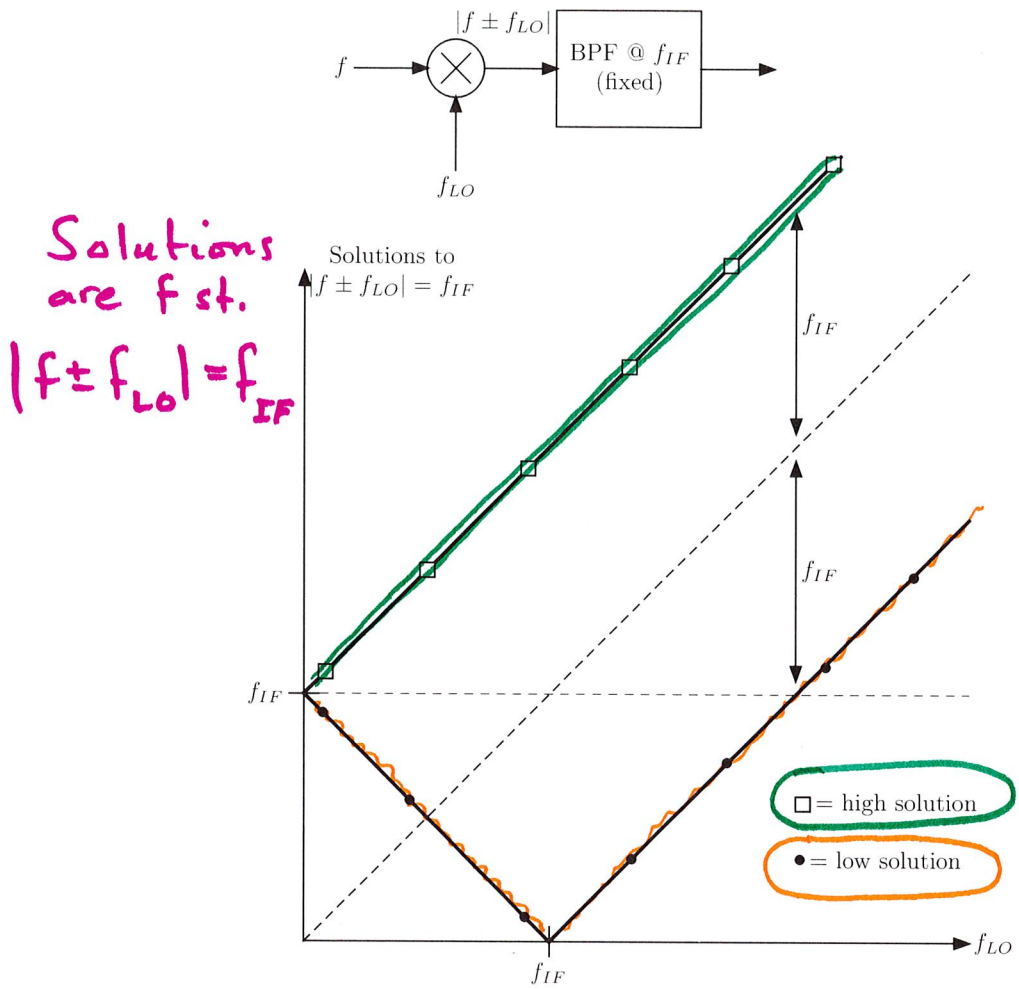
In this section we restrict ourselves to positive frequencies, i.e., f, f_{IF}, and f_{LO} are all > 0.

1. Architecture of SHR is mixer followed by BPF centered at desired IF frequency. Idea is to tune the receiver by varying the frequency f_{LO} of a local oscillator in order to translate desired carrier frequencies to center them on the IF filter.

2. For each choice of f_{LO} there are two frequencies f, which satisfy

$$|f \pm f_{LO}| = f_{IF}$$

   and, therefore, would mix to appear in the output of the BPF. The two solution sets vs. f_{LO} are shown in Figure 4.



Solutions are f st. $|f \pm f_{LO}| = f_{IF}$

Figure 4: For a fixed IF frequency f_{IF} and a local oscillator frequency f_{LO} there are always two input frequencies, which are mixed to the IF.

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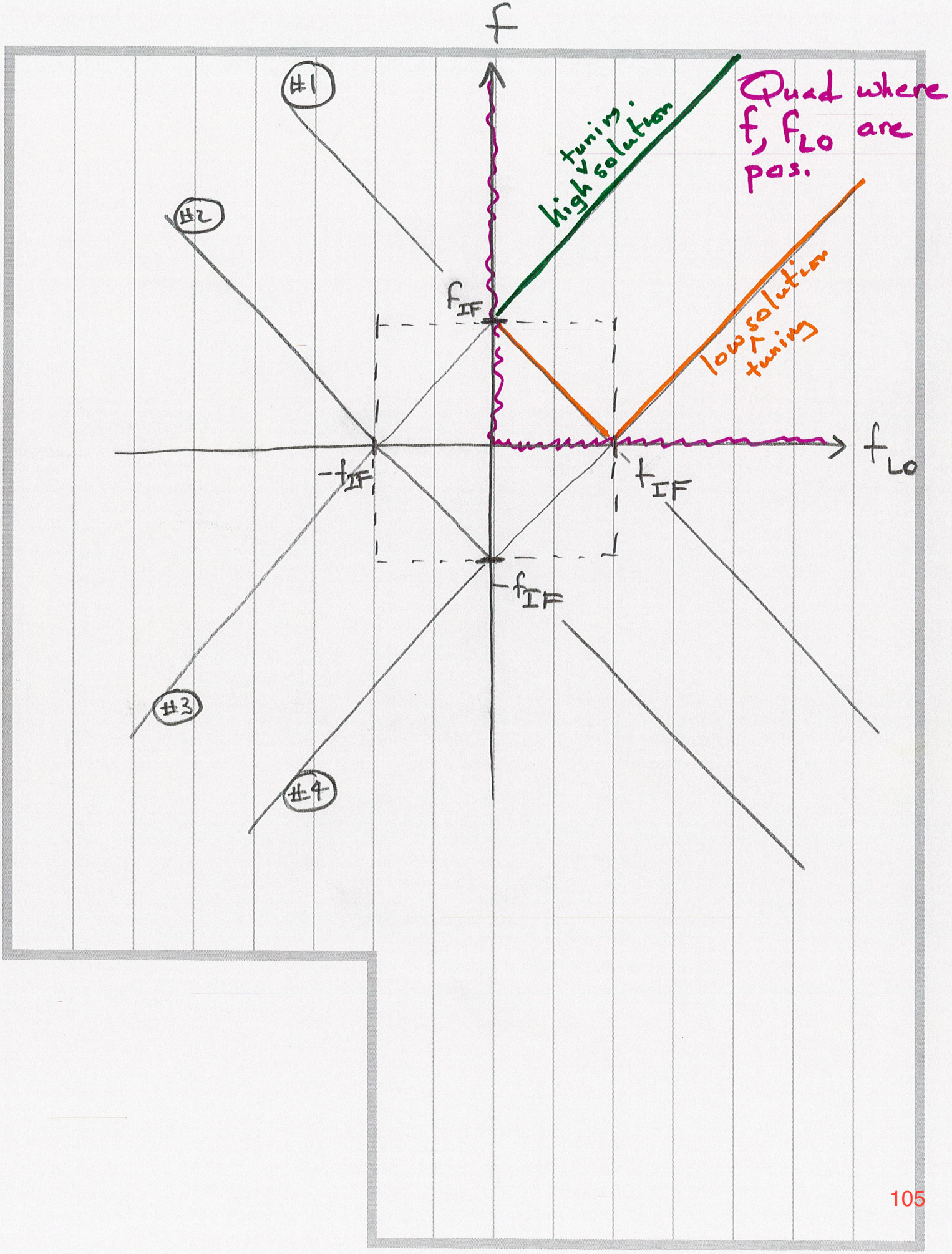
Positive f solutions to $|f \pm f_{LO}| = f_{IF} \iff f \pm f_{LO} = \pm f_{IF}$

$$\iff f = \mp f_{LO} \pm f_{IF}$$

$\therefore$ There are 4 cases ...

$$f = -f_{LO} + f_{IF} \quad \#1$$
$$-f_{LO} - f_{IF} \quad \#2$$
$$+f_{LO} + f_{IF} \quad \#3$$
$$+f_{LO} - f_{IF} \quad \#4$$


f
#1
#2
#3
#4
tuning: high solution
Quad where f, f_LO are pos.
low solution tuning
f_IF
-f_IF
f_IF
-f_IF
f_LO

39a

3. Explanation of operation. Let f_{IF} be a fixed IF frequency and let $f_{desired}$ be the carrier frequency of a signal that we wish to receive in the output of the IF filter. There are two classifications:

   (a) **Upconversion:** $f_{IF} > f_{desired}$. See Figure 5.

   (b) **Downconversion:** $f_{IF} < f_{desired}$. See Figure 6.

4. To prevent both desired and image to appear summed in the output of the BPF a pre-filter must be used to reject the image.

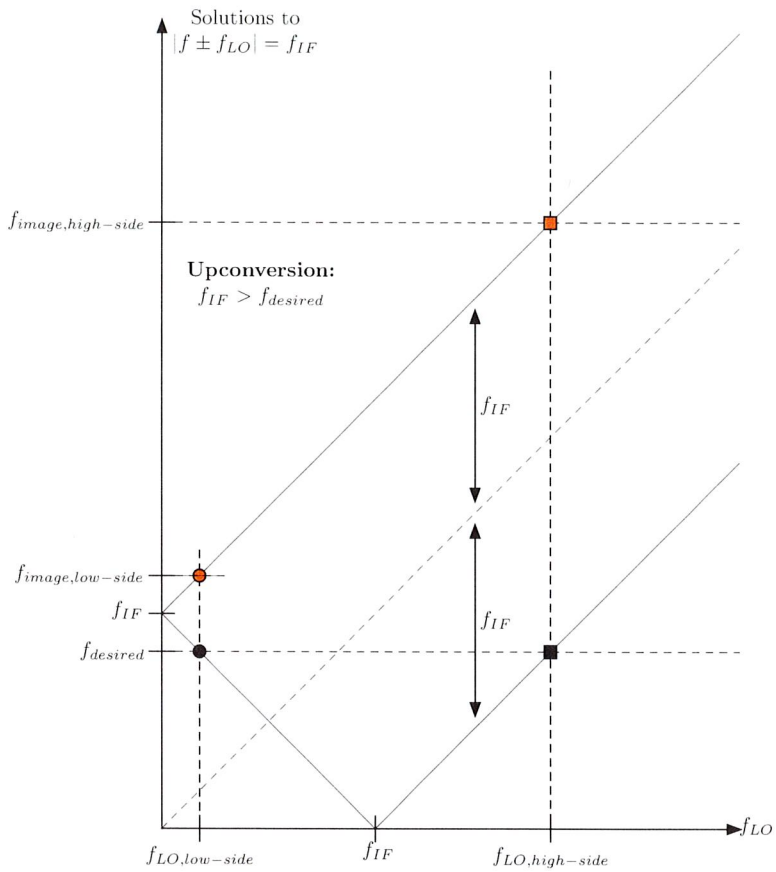


Figure 5: (Upconversion: $f_{IF} > f_{desired}$.) For any $f_{desired}$ there are two choices of LO frequency which will mix it to the IF filter output: $f_{LO,low-side}$ and $f_{LO,high-side}$. Each choice of LO frequency results in an image frequency, which would also appear in the IF filter output.

41

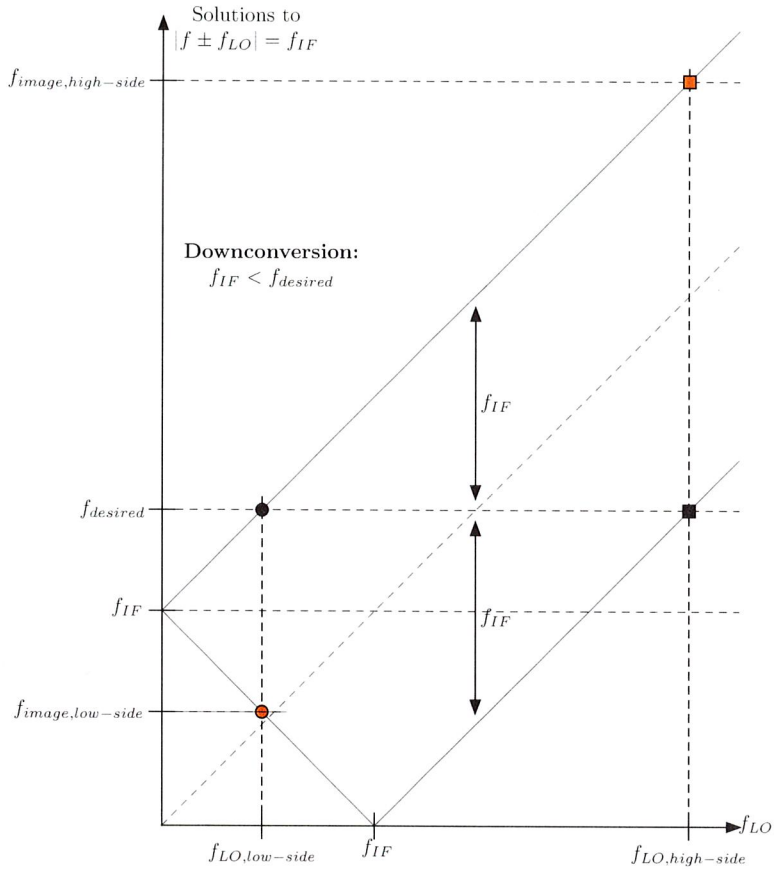
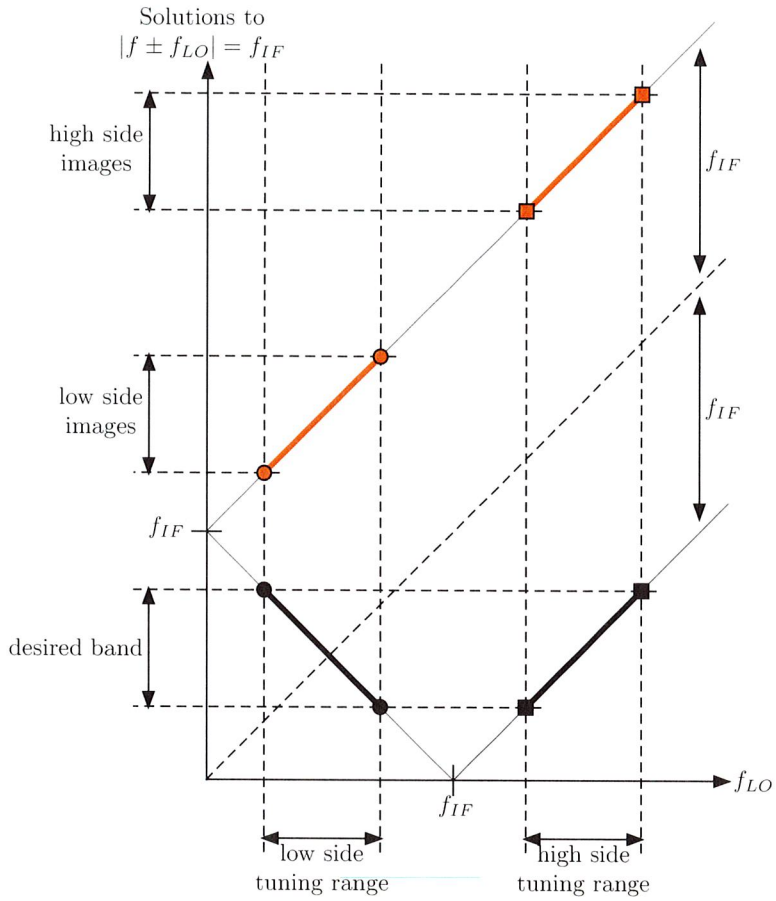


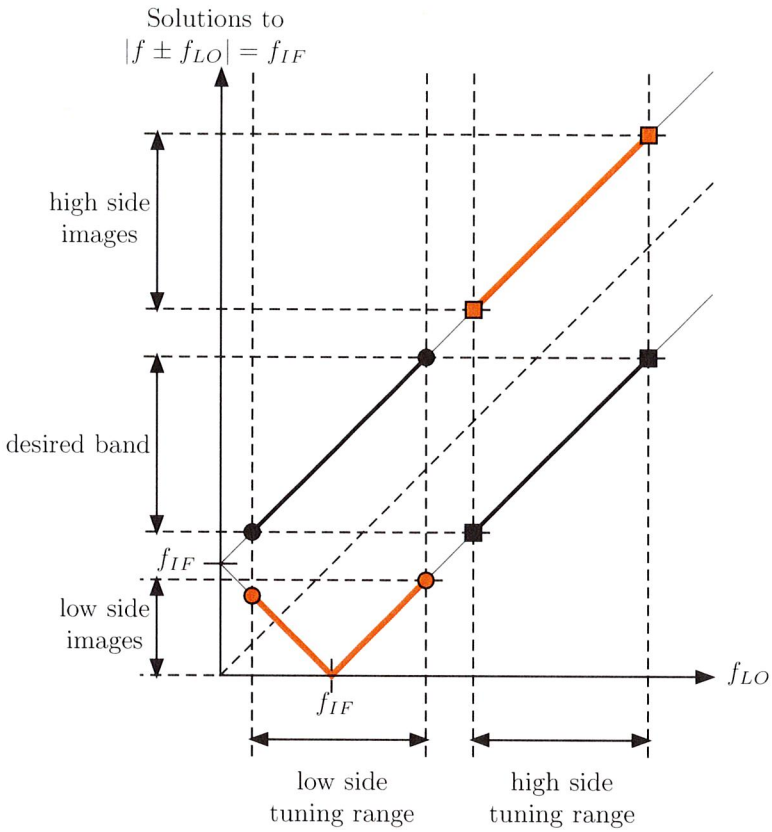
Figure 6: (Downconversion: $f_{IF} < f_{desired}$.) For any $f_{desired}$ there are two choices of LO frequency
42 which will mix it to the IF filter output: $f_{LO,low-side}$ and $f_{LO,high-side}$. Each choice of LO frequency results in an image frequency, which would also appear in the IF filter output.

5. With the SHR architecture one is usually interested in tuning a desired band, e.g., $f_L \leq f_{desired} < f_H$ by varying the local oscillator frequency f_{LO}. This more general case is illustrated in Figures 7 and 8.



44

Figure 7: (Upconversion: $f_{IF} > \text{range}(f_{desired})$.)



45 Figure 8: (Downconversion: $f_{IF} < \text{range}(f_{desired})$.)

6. Sometimes the desired band and the corresponding band of images (either high side or low side images, depending on the tuning scheme) do not overlap. This is the case drawn in Figures 7 and 8. Then a fixed RF prefilter can be used to reject the images.

7. High side tuning has the advantage that is requires the local oscillator to be tunable over a relatively smaller range than does low side tuning. (Relative to the mid point of the required LO frequency range.)

8. The example for AM broadcast band is shown in Figure 9.

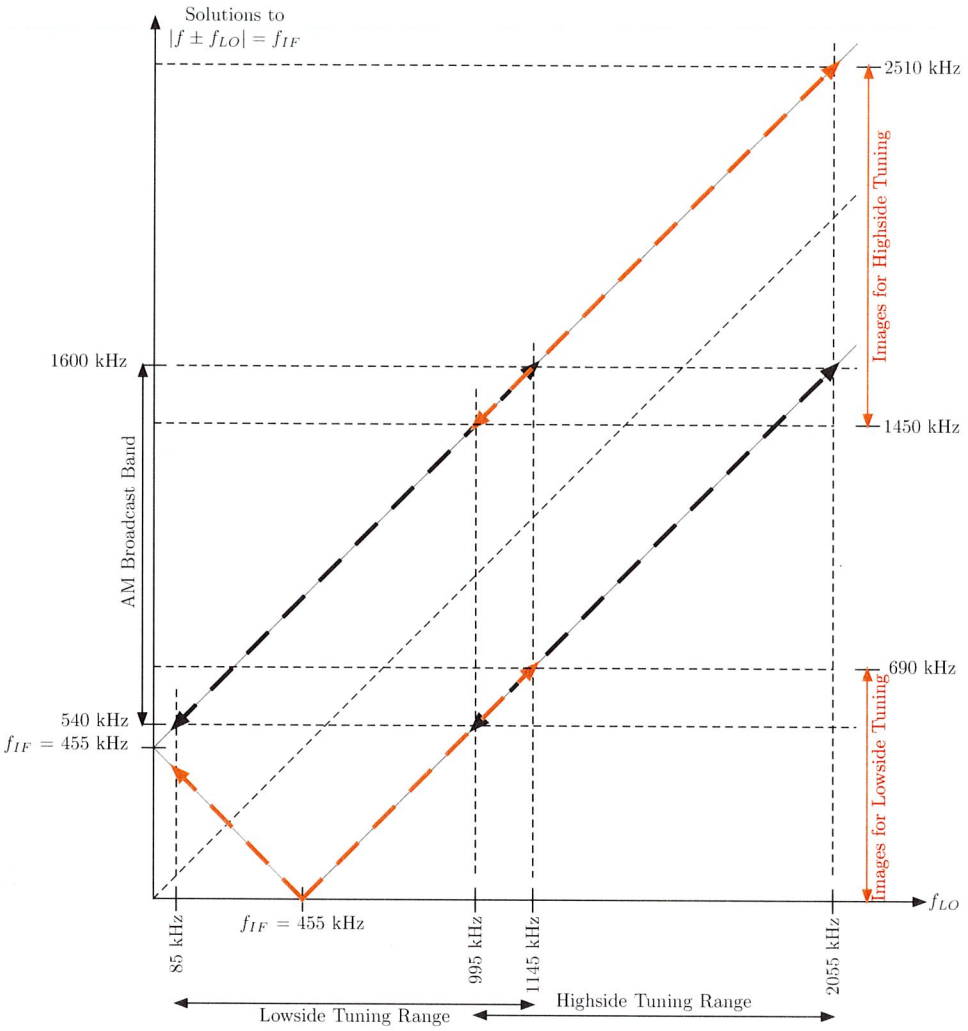


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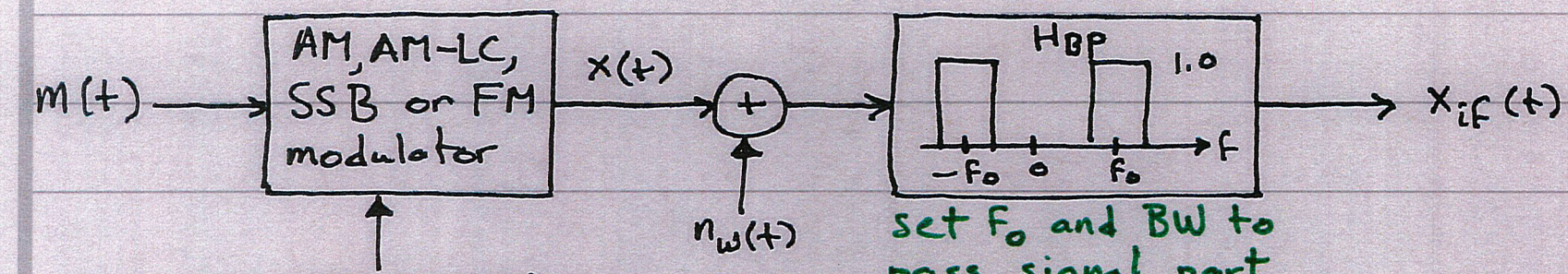
Figure 9: AM broadcast band for IF frequency 455 kHz.

IF Model For All Modulations



set f_0 and BW to pass signal part undistorted and to limit the power in the noise.

(a) $m(t)$ zero-mean WSS
$R_{mm}(\tau) \longleftrightarrow S_{mm}(f)$

(b) Θ unif on $[0, 2\pi)$

(c) $n_w(t)$ AWGN of psd height $N_0/2$

+ (d) signal, phase, noise are stat. indep. of each other.

Since we assume the BPF leaves the signal part alone we have:

$$x_{if}(t) = x(t) + h_{BP} * n_w(t)$$

$$= x(t) + n(t)$$

Band pass filtered white Gaussian noise.

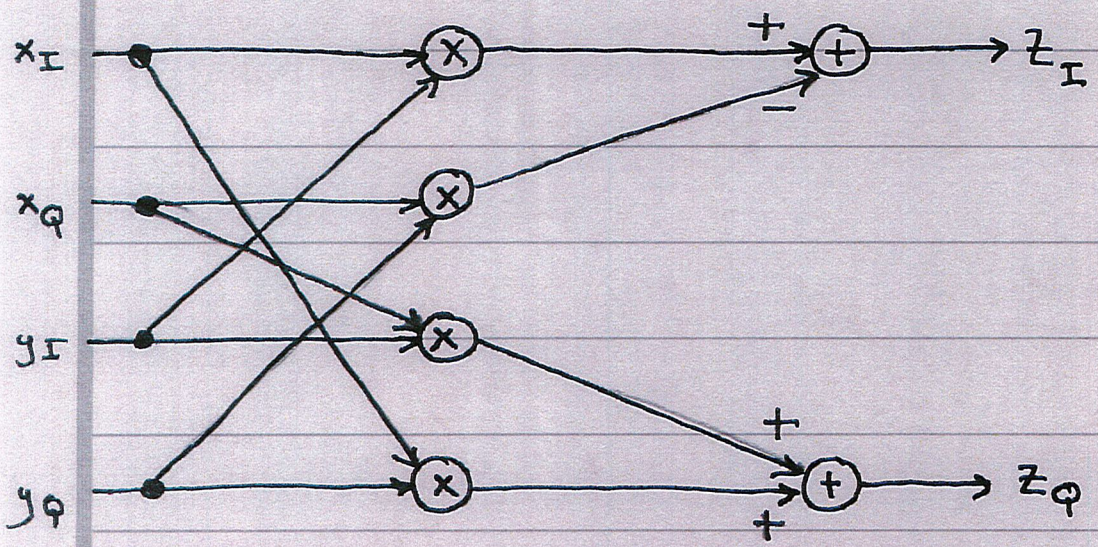
In-Phase/Quadrature & Complex Arithmetic

$$\left.\begin{aligned} x(t) &= x_I(t) + j x_Q(t) \\ y(t) &= y_I(t) + j y_Q(t) \end{aligned}\right\} \quad z(t) = x(t)\,y(t)$$

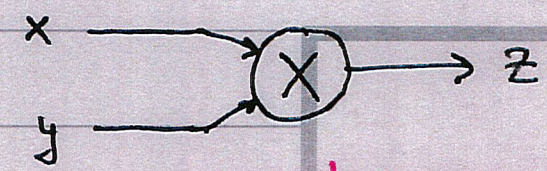
$$= (x_I + j x_Q)(y_I + j y_Q)$$

$$= (x_I y_I - x_Q y_Q) + j(x_I y_Q + x_Q y_I)$$

$$= z_I(t) + j z_Q(t)$$



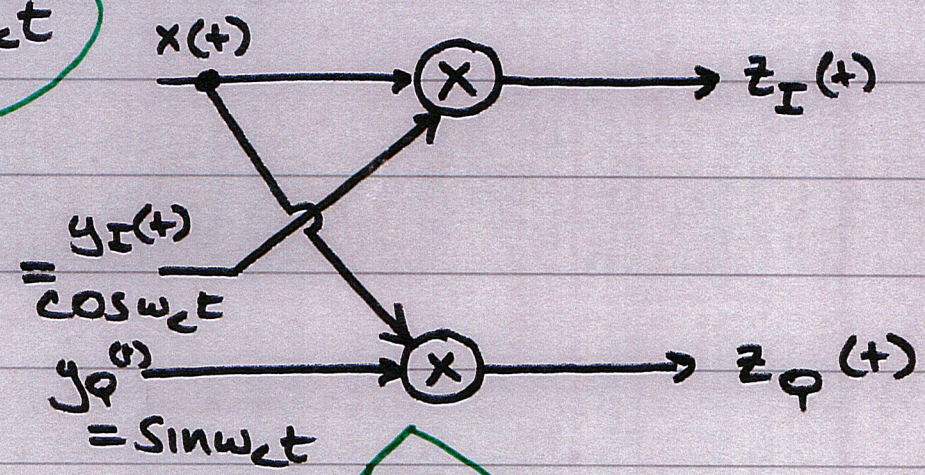
complex number multiplier built of real multipliers and real adders.



complex number multiplier

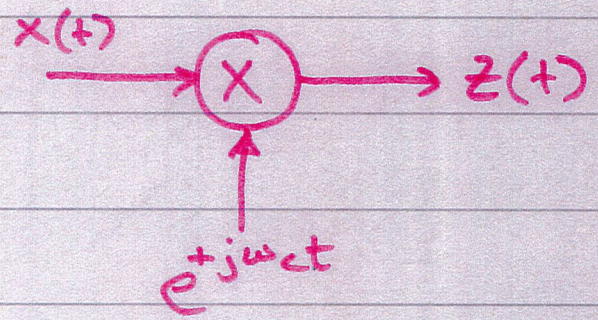
Special Cases

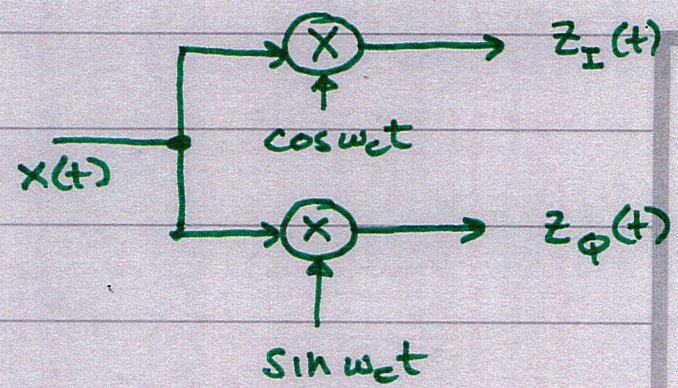
$x(t)$ real, $y(t) = e^{j\omega_c t}$



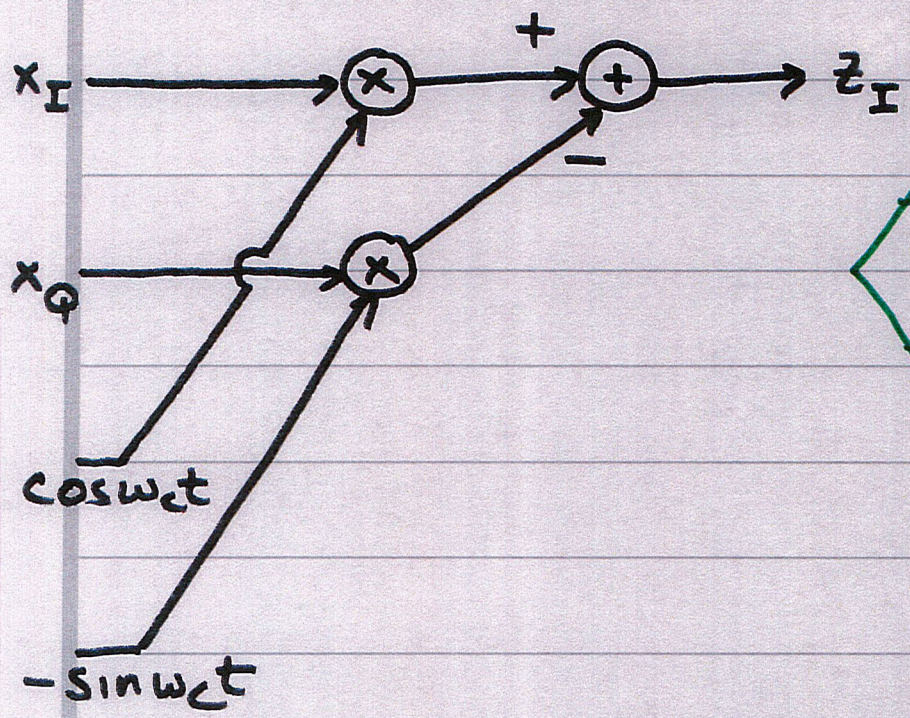
Equivalent to

Complex Form

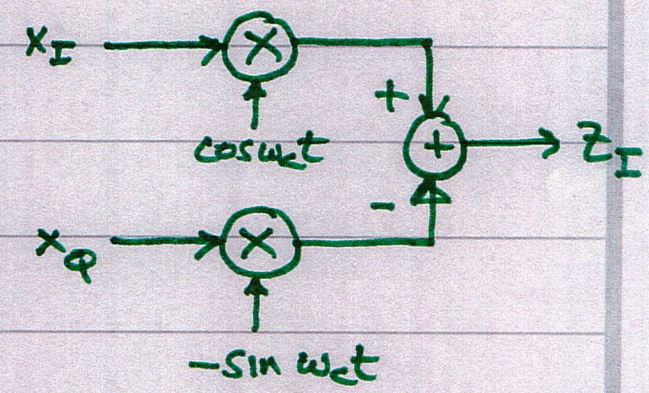


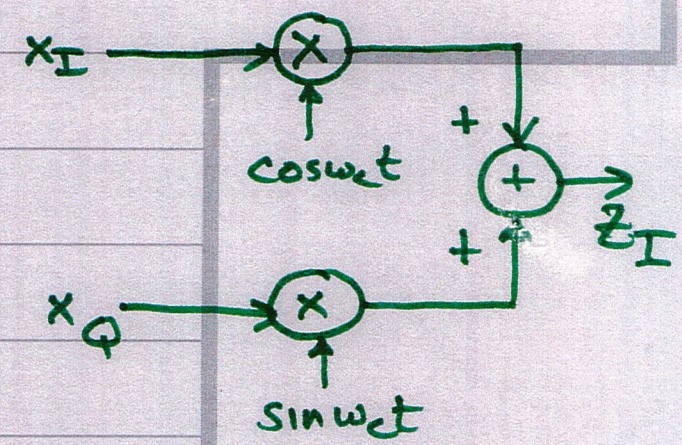


$x(t)$ complex, $y(t) = e^{-j\omega_c t}$,
only keep real part of output

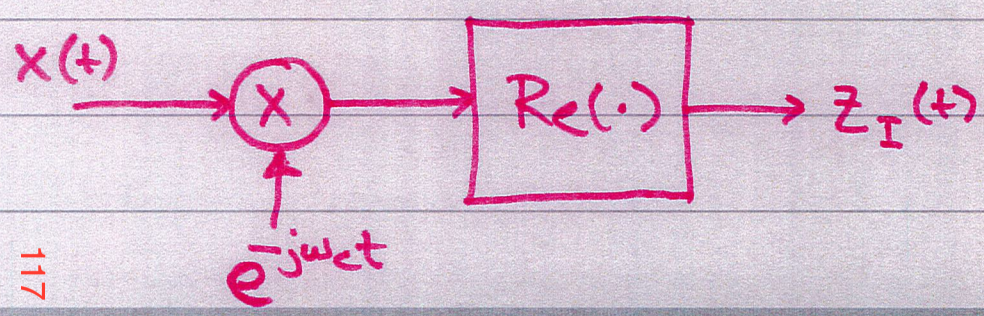


Equivalent to



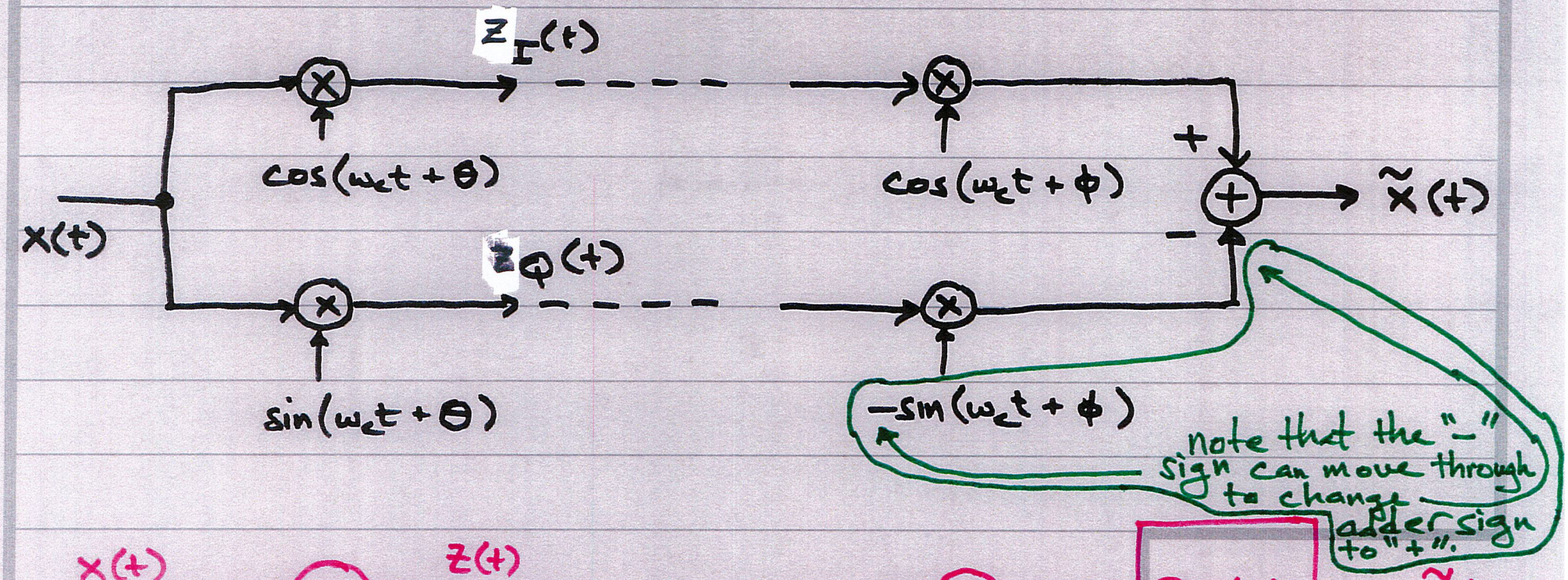


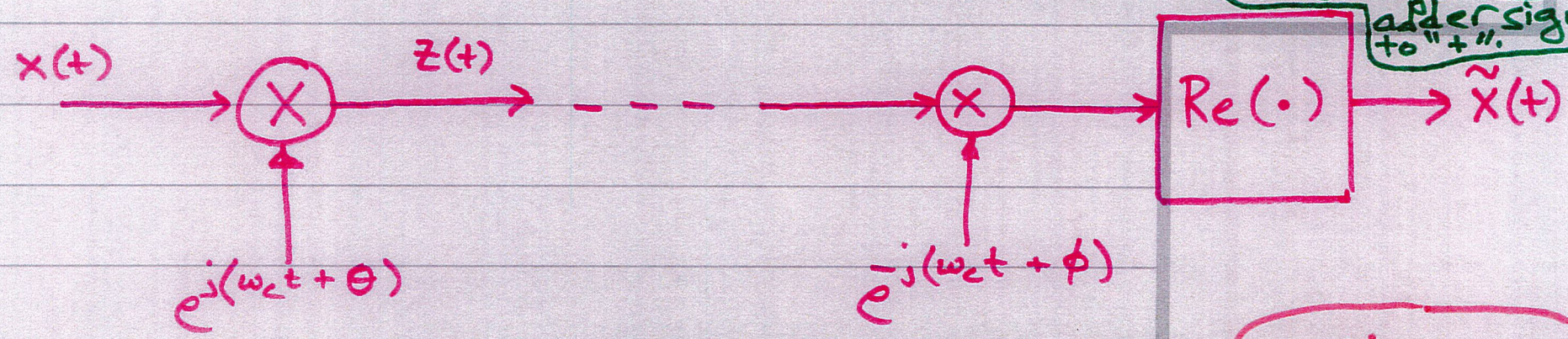
Complex Form



Up Converter – Down Converter

$x(t)$ real-valued





$$\Longrightarrow \tilde{x}(t) = \mathrm{Re}\left\{ x(t)\, e^{+j(\omega_c t + \theta)}\, e^{-j(\omega_c t + \phi)} \right\}$$

$$= x(t)\cos(\theta - \phi)$$

$\therefore \tilde{x}(t) = x(t)$ if $\theta = \phi$

1 Complex Envelope for Deterministic Energy Signals

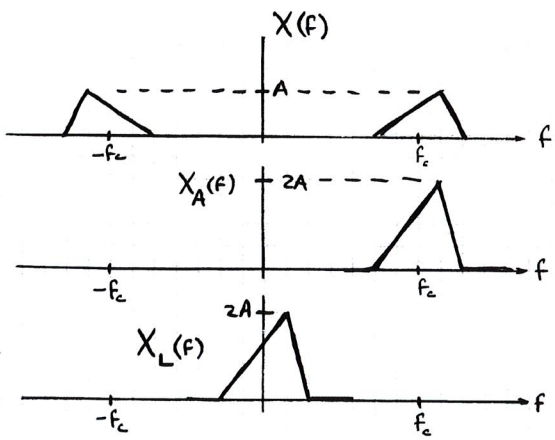
1. Frequency domain definition of complex envelope. Let $x(t)$ be a real-valued deterministic signal and let $x(t) \leftrightarrow X(f)$ denote a Fourier Transform pair[1]. Since the time-domain signal is real-valued, the Fourier Transform has conjugate symmetry about $f = 0$: $X(f) = X^*(-f)$. Define the *complex envelope* with respect to a carrier waveform $\cos(2\pi f_c t)$ of some fixed center frequency $f_c > 0$ by [1]

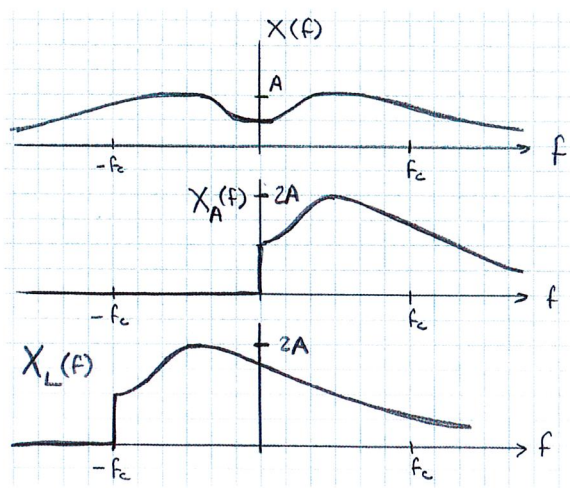
$$\begin{aligned} X_L(f - f_c) &= 2u(f)X(f) \\ &\Updownarrow \\ X_L(f) &= 2u(f + f_c)X(f + f_c). \end{aligned}$$

   (a) In the above definition $u(\cdot)$ denotes the usual unit step function.

   (b) The time domain complex envelope is defined as the inverse Fourier Transform: $x_L(t) \leftrightarrow X_L(f)$.

2. Examples. The complex envelope is usually defined for passband signals as shown below by the situation on the left, but it may also be defined for the general case, as shown on the right.



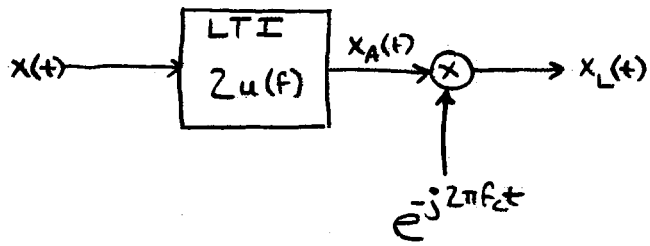


[1]Recall

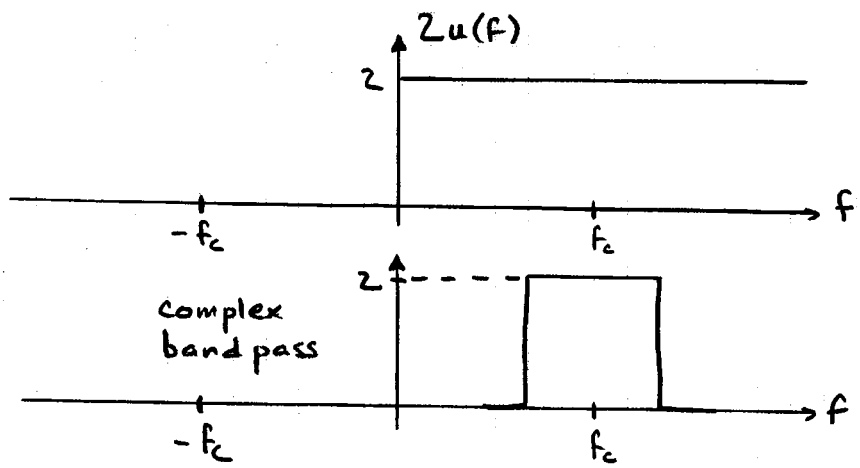
$$\int X(f)e^{j2\pi ft}\mathrm{d}f = x(t) \longleftrightarrow X(f) = \int x(t)e^{-j2\pi ft}\mathrm{d}t.$$

3. The definition suggests two equivalent block diagrams which generate the complex envelope as output when the real-valued signal is input.

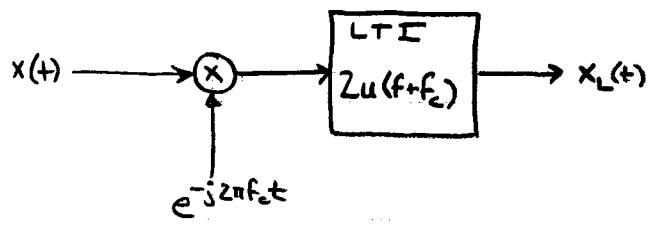
   (a) **Block Diagram 1:** (filtering followed by down-conversion)



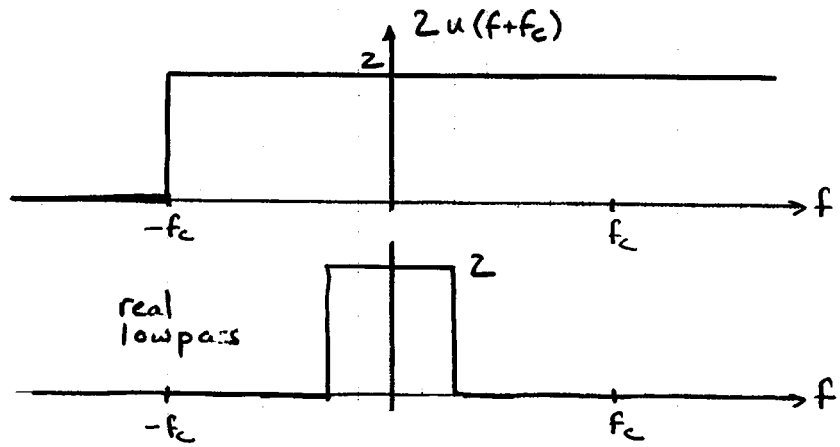
The filter $2u(f)$, called a phase splitter, can be replaced by a complex bandpass filter if $x(t)$ is bandpass as illustrated in the figure below.



(b) **Block Diagram 2:** (down-conversion followed by filtering)



The filter $2u(f + f_c)$ can be replaced by a real lowpass filter if $x(t)$ is bandpass as illustrated in the figure below.



4. Time domain definition of the complex envelope. The impulse response of the filter $2u(f)$ is

$$\delta(t) + j\frac{1}{\pi t}.$$

This can be related to the notion of the *Hilbert transform.* The Hilbert transform $\hat{x}(t)$ of a real-valued signal $x(t)$ is the output of a LTI system driven by $x(t)$. The impulse response and transfer function of the Hilbert transformer are:

$$\frac{1}{\pi t} \leftrightarrow -j\text{sgn}(f).$$

Therefore, the time domain complex envelope can also be given as

$$x_L(t) = [x(t) + j\hat{x}(t)]\, e^{-j2\pi f_c t}$$

where the real-valued signal $\hat{x}(t)$ is the Hilbert transform of $x(t)$ and $x_A(t) \stackrel{\text{def}}{=} (x(t) + j\hat{x}(t))$ is called the *analytic signal.* In general, signals having frequency content containing only positive frequencies are said to be analytic signals.

5. The *in-phase and quadrature components* of a real-valued signal $x(t)$ with respect to $\cos(2\pi f_c t)$ are defined to be the real and imaginary parts, respectively, of the time-domain complex envelope, i.e., $x_L(t) = x_I(t) + jx_Q(t)$.

6. Fact:

$$\begin{aligned} X(f) &= \frac{1}{2}\{X_L(f - f_c) + X_L^*(-f - f_c)\} \\ &\updownarrow \\ x(t) &= \text{Re}\left\{x_L(t)e^{j2\pi f_c t}\right\} \\ &= |x_L(t)|\cos(2\pi f_c t + \angle x_L(t)) \\ &= x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t). \end{aligned}$$

Proof of Fact:

(a) For the relationship among Fourier transforms define

$$\begin{aligned} X_+(f) &= u(f)X(f) \\ X_-(f) &= u(-f)X(f) \end{aligned}$$

(b) Then clear that $X_+(f) = \frac{1}{2}X_L(f - f_c)$ and $X(f) = X_+(f) + X_-(f)$.

(c) From conjugate symmetry have $X_-(f) = X_+^*(-f)$ so

$$\begin{aligned} X(f) &= X_+(f) + X_+^*(-f) \\ &= \frac{1}{2}X_L(f - f_c) + \frac{1}{2}X_L^*(-f - f_c). \end{aligned}$$

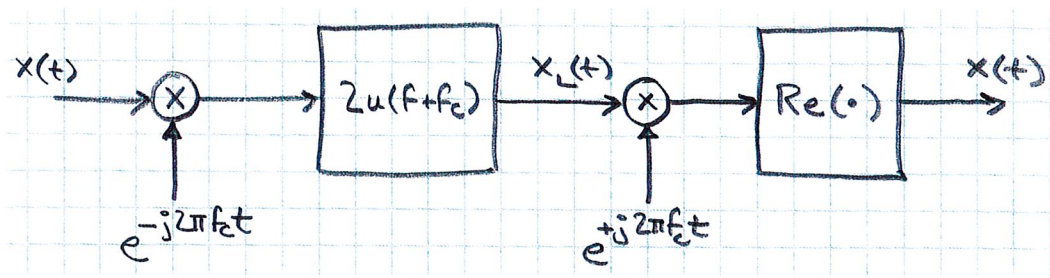
(d) The remainder of the claim follows from basic shift and conjugation properties[2] of the Fourier transform [2]. Thus, note that

$$x_L(t)e^{j2\pi f_c t} \leftrightarrow X_L(f - f_c) \quad \text{and} \quad (x_L(t)e^{j2\pi f_c t})^* \leftrightarrow X_L^*(-f - f_c)$$

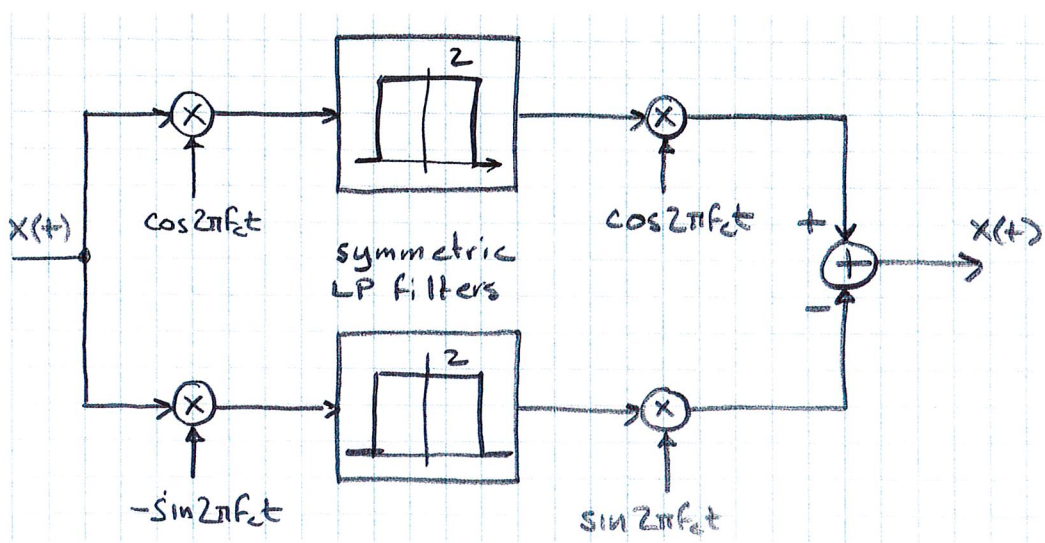
and add terms up, properly scaled.

7. For a real-valued bandpass signal $x(t)$ and its complex envelope $x_L(t)$ the mapping: $x \mapsto x_L$ is implemented by a quadrature down-converter, and the mapping $x_L \mapsto x$ is implemented by a quadrature modulator. This can be seen by converting the complex filtering form of

$$x \mapsto x_L \mapsto x$$



to the in-phase/quadrature form:



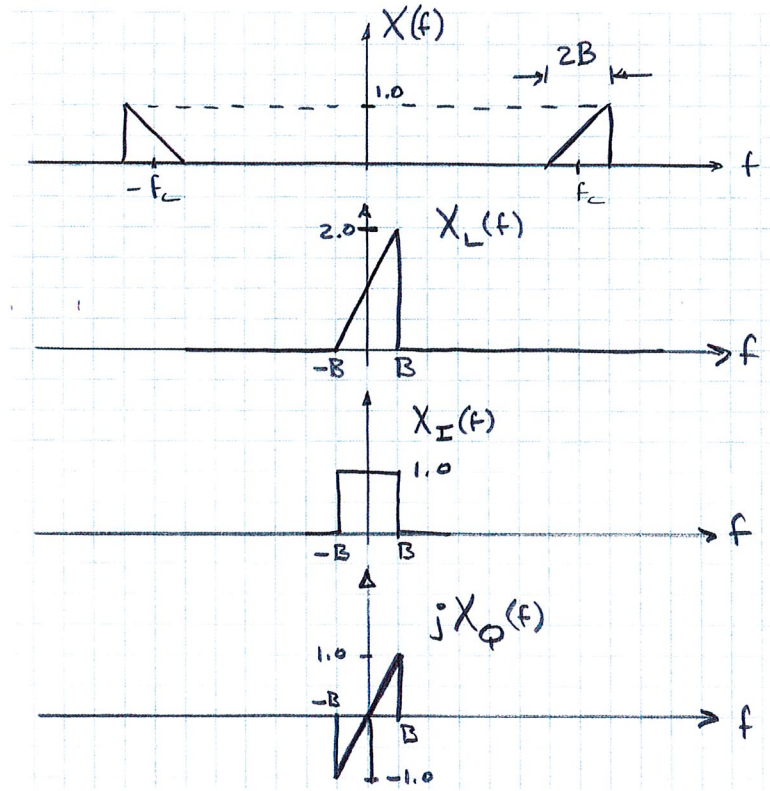
[2]Namely

$$\begin{aligned} g(t) \longleftrightarrow G(f) \quad &\Leftrightarrow \quad g^*(t) \longleftrightarrow G^*(-f) \\ &\Leftrightarrow \quad g(t)e^{j2\pi f_c t} \longleftrightarrow G(f - f_c). \end{aligned}$$

8. Fact. Let $x(t)$ be real-valued with complex envelope $x_L(t)$. Say $x_L(t) \leftrightarrow X_L(f)$, $x_I(t) = \mathrm{Re}\{x_L(t)\}$, $x_Q(t) = \mathrm{Im}\{x_L(t)\}$, $x_I(t) \leftrightarrow X_I(f)$, and $x_Q(t) \leftrightarrow X_Q(f)$. Then [2]:

$$\begin{aligned} X_I(f) &= \left(X_L(f) + X_L^*(-f)\right)/2 \\ X_Q(f) &= \left(X_L(f) - X_L^*(-f)\right)/j2. \end{aligned}$$

Note that $X_I(f) \neq \mathrm{Re}\{X_L(f)\}$ and $X_Q(f) \neq \mathrm{Im}\{X_L(f)\}$, in general, even though it is true that $X_L(f) = X_I(f) + jX_Q(f)$. In pictures, the above relationship is as below.



9. Uniqueness. Say that $x(t)$ is real-valued and $x(t) = \mathrm{Re}\{s(t)e^{j2\pi f_c t}\}$. Then

$$s(t) = x_L(t) \iff S(f) = 0 \text{ for } f < -f_c.$$

2 Complex Envelope and Linear Time Invariant Filtering

10. Would like to relate the complex envelopes of the input and output of a LTI system. Suppose that

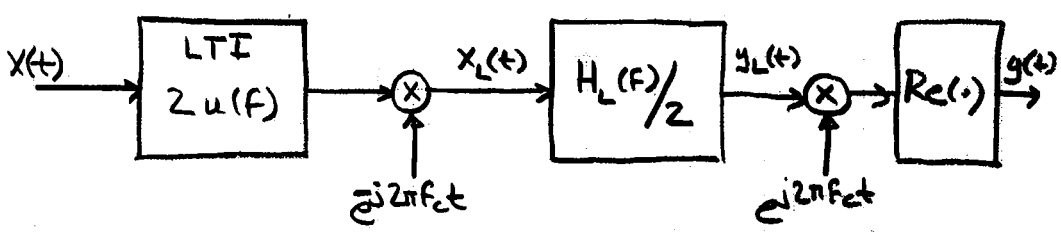
$$y(t) = [h * x](t) \longleftrightarrow Y(f) = H(f)X(f)$$

where the input signal x, the impulse response h, and the output y are real-valued.

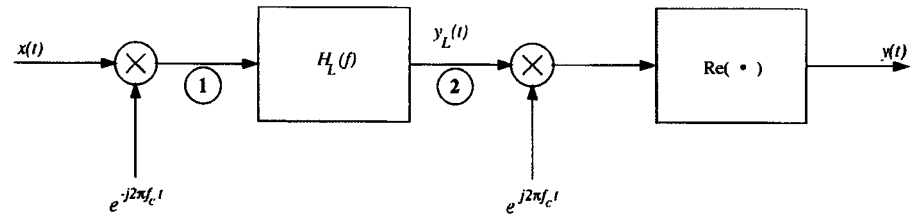
11. From the definition

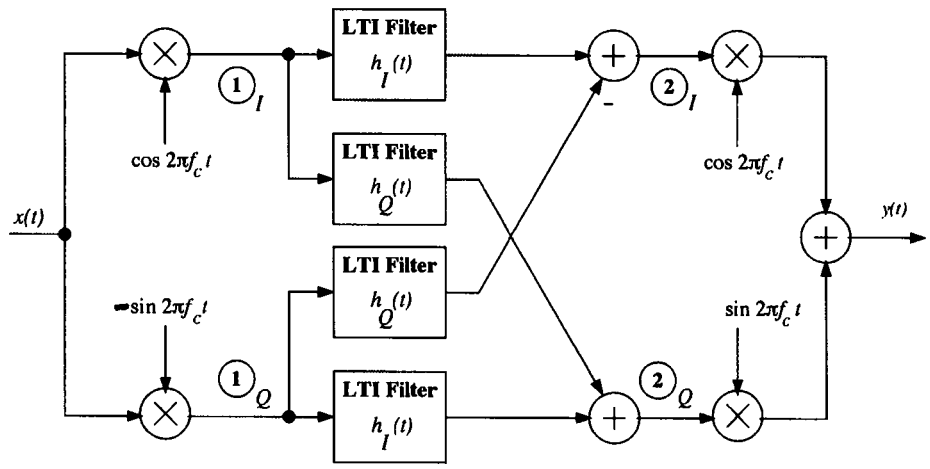
$$\begin{aligned} Y_L(f) &= 2u(f+f_c)Y(f+f_c) \\ &= \frac{1}{2}[2u(f+f_c)H(f+f_c)][2u(f+f_c)X(f+f_c)] \\ &\quad (\text{using } Y = HX \text{ and } u^2 = u) \\ &= \frac{1}{2}H_L(f)X_L(f) \longleftrightarrow y_L(t) = \frac{1}{2}[h_L * x_L](t). \end{aligned}$$

12. Thus the real filtering operation $y(t) = [h * x](t)$ can be implemented via the block diagram below.



13. The previous can be redrawn as shown in the top figure below[3] and then redrawn in I/Q form as shown in the bottom figure.





[3]By replacing Block Diagram 1 in the dashed box with Block Diagram 2 and using the fact that $u(f + f_c)H_L(f) = H_L(f)$.

Complex-Valued Random Processes

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A complex-valued random process is made up of a pair of jointly distributed real-valued random processes

$$\underbrace{Z(t)}_{\mathbb{C}\text{-valued}} = \underbrace{X(t)}_{\text{real-valued real part}} + j\underbrace{Y(t)}_{\text{real valued imag. part}}$$

* Probability of events associated with $Z(t)$ are always defined in terms of events relating to $X(t)$ and $Y(t)$ e.g.

$$P(\{|Z| \le a\}) \triangleq P(\{X^2 + Y^2 \le a^2\})$$

1.

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* Expectations involving Z use linearity and definition of expectations of X and Y...

$$\mu_Z(t) \triangleq E\{Z(t)\} = E\{X(t) + jY(t)\}$$

$$= E\{X(t)\} + jE\{Y(t)\}$$

$$= \mu_X(t) + j\mu_Y(t)$$

* Now need to define correlation / covariance functions for complex-valued random processes

* Power spectral densities also.

2.

What Should Be Correlation Functions Be in Complex Case?

$Z = X + jY$ and $W = U + jV$

be two complex-valued random variables ...

The correlations of interest would be:

$$E\{ZW\} \quad \text{and} \quad E\{Z^* W\}$$

and

$$E\{Z^2\} \quad \text{and} \quad E\{Z^* Z\}$$

3.

For Z Alone

Need correlations of the real-valued rvs X, Y

$$\Longrightarrow E\{XX\} = E\{X^2\}$$

$$E\{YY\} = E\{Y^2\}$$

$$E\{XY\}.$$

Consider:

$$E\{Z \cdot Z\} \quad \text{and} \quad E\{Z^* Z\}$$

4.

$$Z^2 = (X+jY)(X+jY) = (X^2-Y^2)+j(XY+YX)$$

$$|Z|^2 = (X-jY)(X+jY) = (X^2+Y^2)+j(XY-YX)$$

$$\therefore E\{Z^2\} = (E\{X^2\}-E\{Y^2\})+j2E\{XY\}$$

$$E\{|Z|^2\} = (E\{X^2\}+E\{Y^2\})+j0$$

Conclude

Knowing $E\{Z^2\}$ and $E\{|Z|^2\}$ $\Longleftrightarrow$ Knowing $E\{X^2\}$, $E\{Y^2\}$ and $E\{XY\}$.

For Z and W Need correlations of all four real-valued rvs X, Y, U, V

EX^2 EY^2 EU^2 EV^2

EXY EXU EXV

EYU EYV

EUV

This should be equivalent to knowing ...

EZ^2 $E|Z|^2$ EW^2 $E|W|^2$

EZW EZ^*W

Already Know:

$$\begin{matrix} EZ^2 \\ E|Z|^2 \end{matrix} \iff \begin{matrix} EX^2, EY^2 \\ EXY \end{matrix}$$

$$\begin{matrix} EW^2 \\ E|W|^2 \end{matrix} \iff \begin{matrix} EU^2, EV^2 \\ EUV \end{matrix}$$

Thus expand:

$$ZW = (X+jY)(U+jV) = (XU-YV)+j(XV+YU)$$

$$Z^*W = (X-jY)(U+jV) = (XU+YV)+j(XV-YU)$$

7.

Take Expectation:

$$EZW = (EXU - EYV) + j(EXV + EYU)$$

$$EZ^*W = (EXU + EYV) + j(EXV - EYU)$$

Therefore:

$$\begin{matrix} EZW \\ \text{and} \\ EZ^*W \end{matrix} \iff \begin{matrix} EXU, EYV, \\ EXV, EYU \end{matrix}$$

8.

To Summarize

The complete correlation set of two complex-valued rvs Z and W is contained in:

$$EZ^2 \quad E|Z|^2 \quad EW^2 \quad E|W|^2$$

$$EZ^*W \quad EZW$$

9.

Correlation Definitions for $\mathbb{C}$-valued Random Processes $Z(t) = X(t) + j\,Y(t)$, $W(t) = U(t) + j\,V(t)$

Auto Correlation Functions

$$R_{ZZ}(t,s) = E\{Z^*(t)\,Z(s)\}$$

$$\tilde{R}_{ZZ}(t,s) = E\{Z(t)\,Z(s)\}$$

Cross Correlation Functions

$$R_{WZ}(t,s) = E\{W^*(t)\,Z(s)\}$$

$$\tilde{R}_{WZ}(t,s) = E\{W(t)\,Z(s)\}$$

10.

Properties

ⓐ Covariances defined: $C_{WZ}(t,s) = R_{WZ}(t,s) - \mu_W^*(t)\mu_Z(s)$

$\tilde{C}_{WZ}(t,s) = \tilde{R}_{WZ}(t,s) - \mu_W(t)\mu_Z(s)$

ⓑ $$R_{ZZ}(t,t) = C_{ZZ}(t,t) + |\mu_Z(t)|^2$$

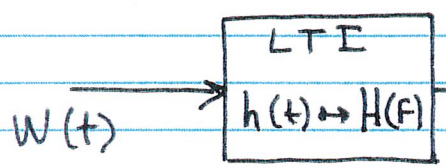
total power at time t = ac power @ t + dc power @ t

ⓒ $$R_{WZ}(t,s) = R_{ZW}^*(s,t)$$

ⓓ $$\left(\mathrm{Re}\{R_{WZ}(t,s)\}\right)^2 \le R_{WW}(t,t)R_{ZZ}(s,s)$$

DEF: Z is WSS if $\mu_Z(t)$, $R_{ZZ}(t,t+\tau)$ and $\tilde{R}_{ZZ}(t,t+\tau)$ do not depend on t.

LTI Filtering



$$Z(t) = \int h(t-\tau)W(\tau)\,d\tau$$

Means $$\mu_Z(t) = \int h(t-\tau)\mu_W(\tau)\,d\tau$$

Correlation $$EW^*(t)Z(s) = E\left\{W^*(t)\int h(s-\tau)W(\tau)\,d\tau\right\}$$

$$= \int h(s-\tau)\,EW^*(t)W(\tau)\,d\tau$$

$$\therefore R_{WZ}(t,s) = \int h(s-\tau)R_{WW}(t,\tau)\,d\tau$$

$$R_{WW}(t,s) \longrightarrow \boxed{\text{LTI } h(s) \text{ operating on } s} \longrightarrow R_{WZ}(t,s)$$

Similarly

$$\tilde{R}_{WZ}(t,s) = \int h(s-\tau)\tilde{R}_{WW}(t,\tau)\,d\tau$$

$$\tilde{R}_{WW}(t,s) \longrightarrow \boxed{\text{LTI } h(s) \text{ operating on } s} \longrightarrow \tilde{R}_{WZ}(t,s)$$

$$\boxed{R_{ZZ}(t,s)}$$

$$E\,Z^*(t)Z(s) = E\int h^*(t-\tau)W^*(\tau)\,d\tau \int h(s-\lambda)W(\lambda)\,d\lambda$$

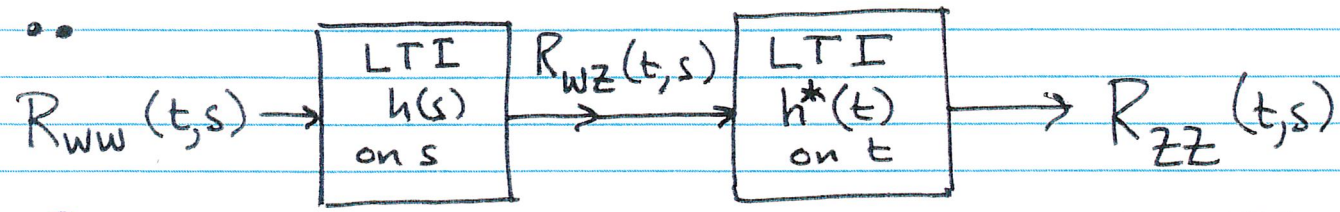
$$= \iint h^*(t-\tau)h(s-\lambda)\,EW^*(\tau)W(\lambda)\,d\tau\,d\lambda$$

$$= \iint h^*(t-\tau)h(s-\lambda)R_{WW}(\tau,\lambda)\,d\tau\,d\lambda$$

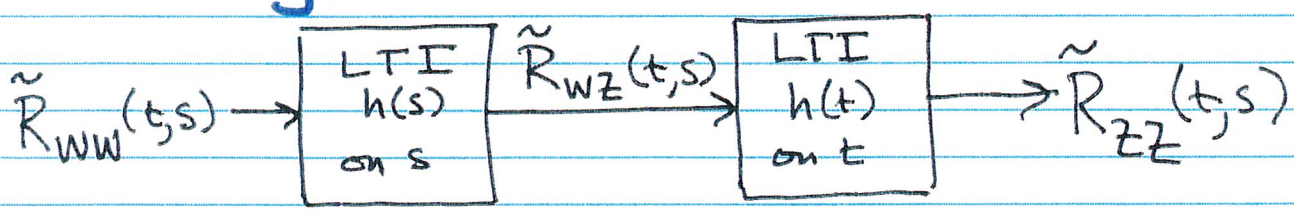
$$= \int h^*(t-\tau)\int h(s-\lambda)R_{WW}(\tau,\lambda)\,d\lambda\;d\tau$$

$$= \int h^*(t-\tau)\, R_{WZ}(\tau, s)\, d\tau$$

$\therefore$



Similarly:



15

$\boxed{\text{If } Z(t) \text{ is WSS}}$

$$\mu_Z = \mu_W \int h(t)\,dt$$

$$\rho_{ZZ}(\tau) = \left[\tilde{h} * h * \rho_{WW}\right](\tau) \qquad \tilde{h}(t) \triangleq h^*(-t)$$

$$\tilde{\rho}_{ZZ}(\tau) = \left[\tilde{\tilde{h}} * h * \tilde{\rho}_{WW}\right](\tau) \qquad \tilde{\tilde{h}}(t) \triangleq h(-t)$$

and in freq. domain

$$\mu_Z = H(0)\,\mu_W$$

$$S_{ZZ}(f) = |H(f)|^2 S_{WW}(f)$$

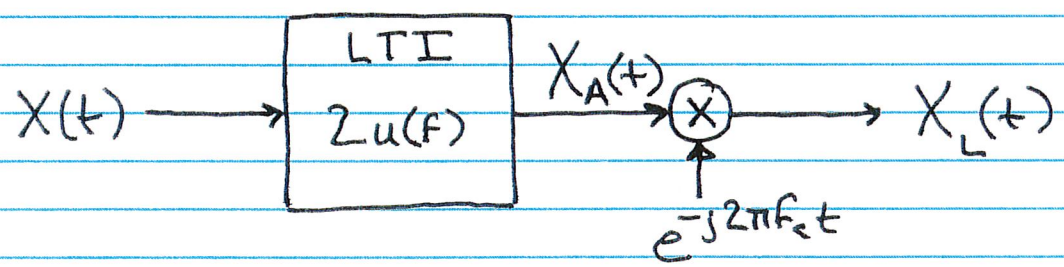
$$\tilde{S}_{ZZ}(f) = H(f)H(-f)\,\tilde{S}_{WW}(f)$$

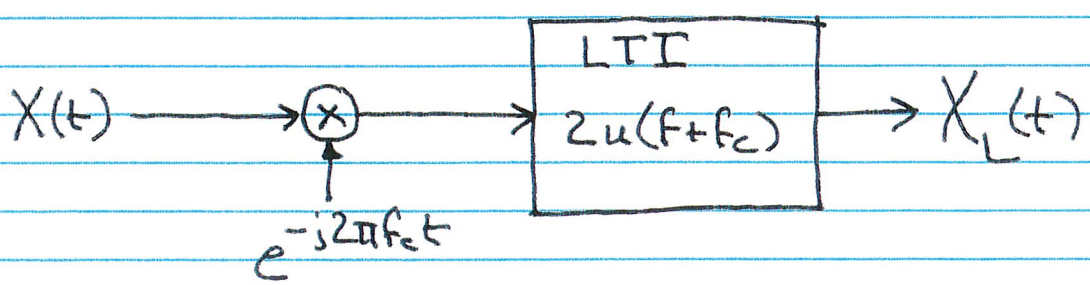
$\rho_{ZZ}(\tau) \triangleq R_{ZZ}(t, t+\tau)$ etc.

16.

Complex Envelope for Real-Valued WSS Random Signals $X(t)$

Equiv. systems that generate $X_L(t)$ from input $X(t)$:





Note: If $S_{XX}(f)$ is non-zero only on $\{f : |f| - f_c| < B\}$ then $2u(f) \to$ complex BP filter, $2u(f+f_c) \to$ real LP filter.

1.

$X_A(t)$ is complex-valued, in general

$X_A(t)$ is WSS in general $\left(\begin{array}{l}\text{it is output of LTI} \\ \text{driven by WSS input}\end{array}\right)$

But $X_L(t) = e^{-j2\pi f_c t} X_A(t)$ is <u>not obviously</u> WSS.

Compute its "moments"

$$\mu_{X_L}(t) = e^{-j2\pi f_c t} \mu_{X_A}$$ ← this is constant wrt $t \iff \mu_{X_A} = 0$

$$R_{X_L X_L}(t, t+\tau) = E\left\{\left(e^{-j2\pi f_c t} X_A(t)\right)^* \left(e^{-j2\pi f_c (t+\tau)} X_A(t+\tau)\right)\right\}$$

$$= e^{-j2\pi f_c \tau} R_{X_A X_A}(t, t+\tau)$$ ← Always a funct. of τ alone

2.

Similarly

$$\tilde{R}_{X_L X_L}(t, t+\tau) = \underbrace{e^{-j2\pi(2f_c)t} e^{-j2\pi f_c \tau} \tilde{R}_{X_A X_A}(t, t+\tau)}_{\text{this will be a function of } \tau \text{ alone} \iff \text{it is zero.}}$$

3.