ECE 440 **Final Exam** **5/5/22**

Name: Solution

General Instructions:

- **You have 120 minutes to complete the exam.**
- Write your name on every page of the exam.
- Please do not write on the backs of pages.
- The exam is closed book. Calculators are not allowed.
- You are allowed both sides of three 8.5 by 11 inch sheets of paper for your personal notes in addition to the instructor supplied formula sheet.
- Your work must be explained to receive full credit. All plots must be carefully drawn with axes labeled.
- Point values for each problem are as indicated. The exam totals 100 points.
- Please do not leave early as it is disruptive to those working around you.

Do not open the exam until you are told to begin.

Name: ____________________

Problem 1. *Calculations Involving WSS Random Processes.* [15 pts. total]

Let $A(t)$ be a real-valued wide-sense stationary random process with mean $\mu_A = 2$ and auto-correlation $R_A(\tau) = 10e^{-f_c|\tau|}$ and let Θ be a random variable uniformly distributed on $[0, 2\pi)$. Suppose that $\{A(t) : t \in \mathcal{R}\}$ and Θ are statistically independent. Please answer the following questions from first principles, i.e., completely explain your steps.

Define the random process $X(t) = A(t)L[\sin(2\pi f_c t + \Theta)]$ where $L(\cdot)$ is the positive threshold function, which equals $+1$ for positive values of its argument and 0 for negative values.

(a) [5 pts.] Find the mean of $X(t)$.

(b) [10 pts.] Find the autocorrelation function of $X(t)$.

Calculations Involving WSS Random Processes.

Let $A(t)$ be a real-valued wide-sense stationary random process with mean $\mu_A = 2$ and auto-correlation $R_A(\tau) = 10e^{-f_c|\tau|}$ and let Θ be a random variable uniformly distributed on $[0, 2\pi)$. Suppose that $\{A(t) : t \in \mathcal{R}\}$ and Θ are statistically independent. Please answer the following questions from first principles, i.e., completely explain your steps.

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(a) Find the mean of $X(t)$.

$$EX(t) = E\{A(t)L[\sin(2\pi f_c t + \Theta)]\}$$

$$= \underbrace{E\{A(t)\}}_{2 \text{ as given}} \; \underbrace{E\{L[\sin(2\pi f_c t + \Theta)]\}}_{\text{need to work this out}}$$

by indep. of $A(t)$ and Θ

$$= \frac{1}{2\pi}\int_0^{2\pi} L[\sin(2\pi f_c t + \theta)]\, d\theta$$

The integrand is periodic in θ of period 2π. It will be +1 over half the period, and 0 over the other half. Therefore the integral $= \pi$. Putting it all together

$$EX(t) = 2 \cdot \frac{1}{2\pi} \cdot \pi = 1$$

(b) Find the auto-correlation of $X(t)$

The auto-correlation is the expectation of

$$X(t)\,X(t+\tau) =$$

$$A(t)\,A(t+\tau)\, L\left[\sin(2\pi f_c t + \Theta)\right] L\left[\sin(2\pi f_c(t+\tau) + \Theta)\right]$$

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and the independence of $A(t)$ and Θ will show that $E\,X(t)\,X(t+\tau)$ is product of two terms ...

$$E\left\{A(t)\,A(t+\tau)\right\} = R_A(\tau) = 10\,e^{-f_c|\tau|}$$

and

$$E\left\{L\left[\sin(2\pi f_c t + \Theta)\right] L\left[\sin(2\pi f_c(t+\tau) + \Theta)\right]\right\}$$

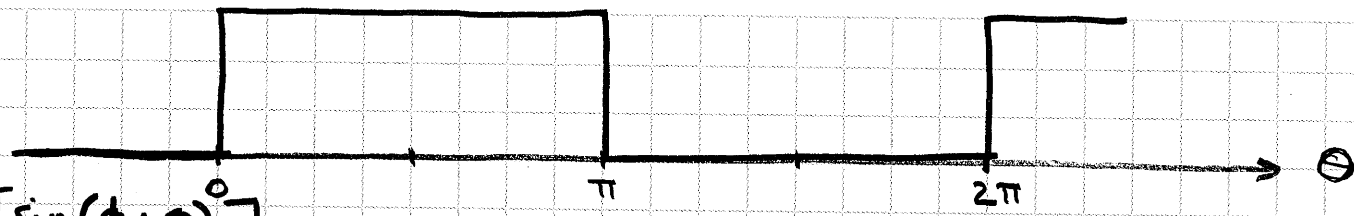
$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{L\left[\sin(2\pi f_c t + \theta)\right] L\left[\sin(2\pi f_c(t+\tau) + \theta)\right]}\, d\theta$$

the integrand is the product of 2 signals that are either 0 or 1 ... hence the product is either 0 or 1 and computing the integral amounts to figuring out the duty cycle of the integrand.

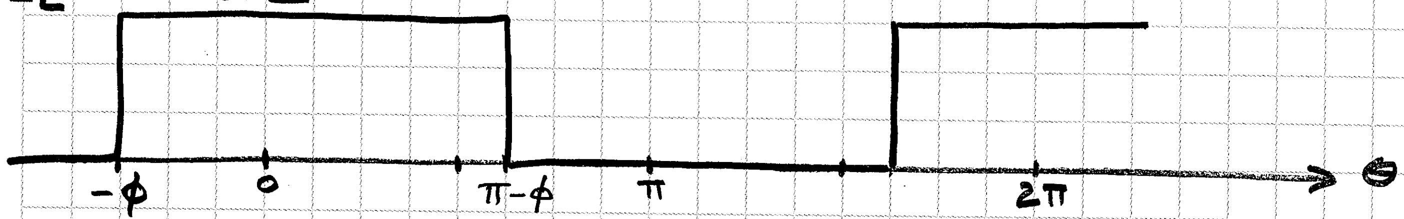
Note that the above integral will not depend on t from periodicity. Therefore let $t = 0$ and define $\phi = 2\pi f_c \tau$ and consider ϕ values over the interval

$$-\pi < \phi < \pi$$

$L[\sin\theta]$



$L[\sin(\phi+\theta)]$



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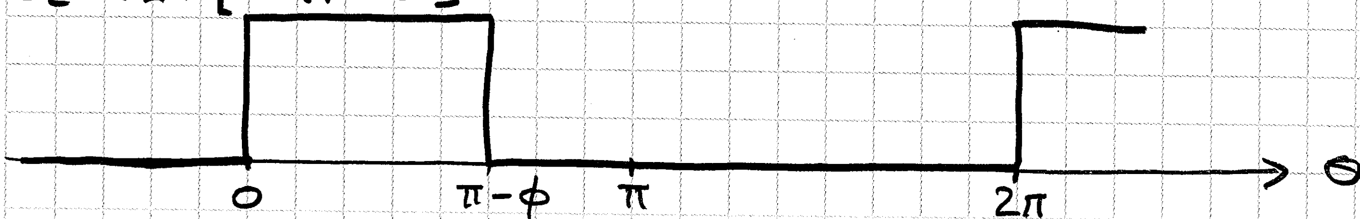
$L[\sin\theta]\,L[\sin(\phi+\theta)]$

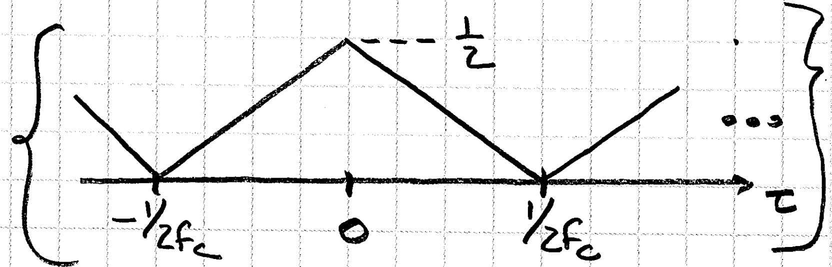


∴ From the drawing shown above for the case where ϕ is positive

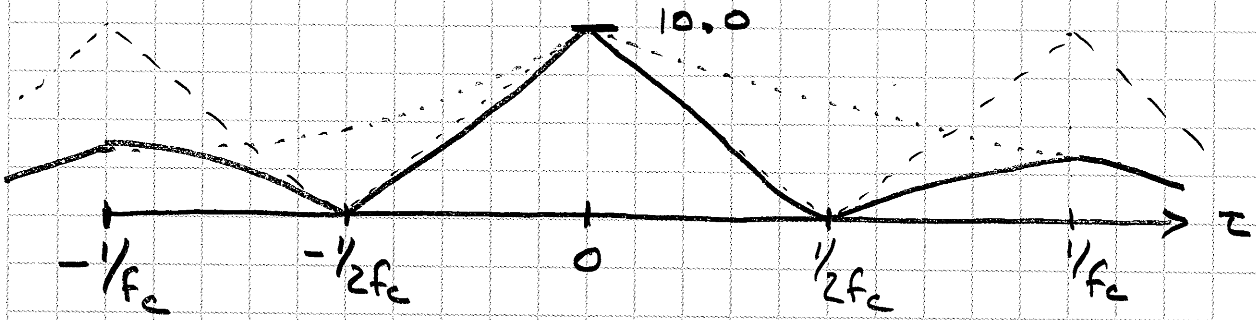
$$\frac{1}{2\pi}\int_0^{2\pi} L[\sin\theta]\,L[\sin(\phi+\theta)]\,d\theta = \frac{\pi-\phi}{2\pi}$$

Using the fact that the auto-correlation is even in ϕ (equiv. in τ) ...

$$R_X(\tau) = 10\,e^{-f_c|\tau|} \cdot$$



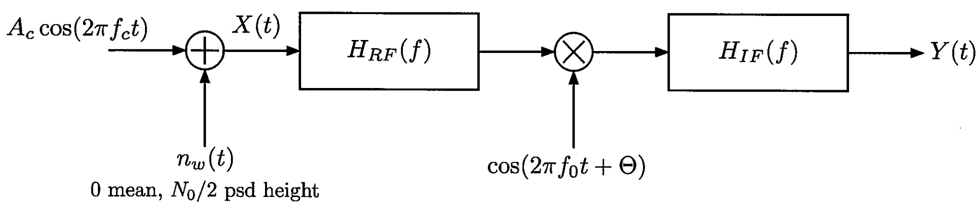
The product of the periodic triangle wave with the decaying exponential will look like ...

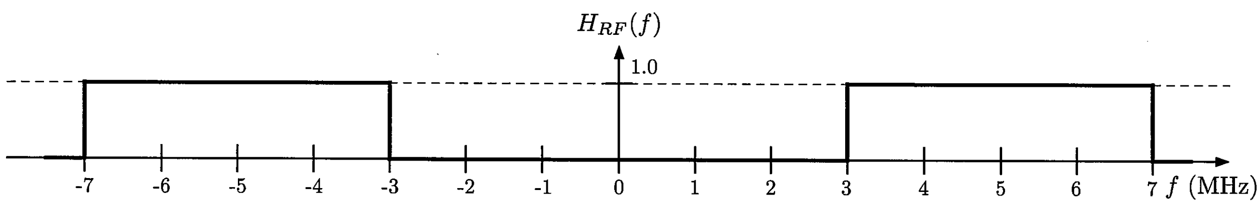


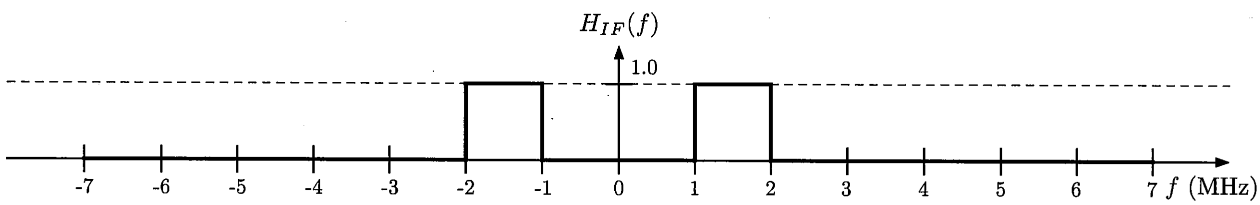
Name: ______________________

Problem 2. *SNR at the Output of a Tunable Receiver.* [15 pts. total]

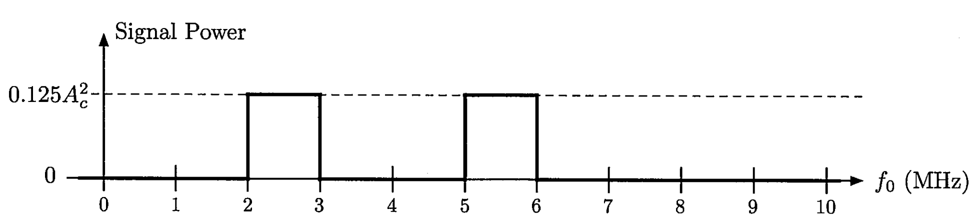
Consider the receiver shown below, which can be tuned by varying the frequency f_0 of the local oscillator. For purposes of modeling, assume that the local oscillator phase Θ is uniformly distributed on $[0, 2\pi)$ and statistically independent of any other random variables or processes that might arise in the course of analysis. As shown the RF filter is bandpass of bandwidth equal to 4 MHz, centered at 5 MHz, and the IF filter is bandpass of bandwidth 1 MHz, centered at 1.5 MHz.





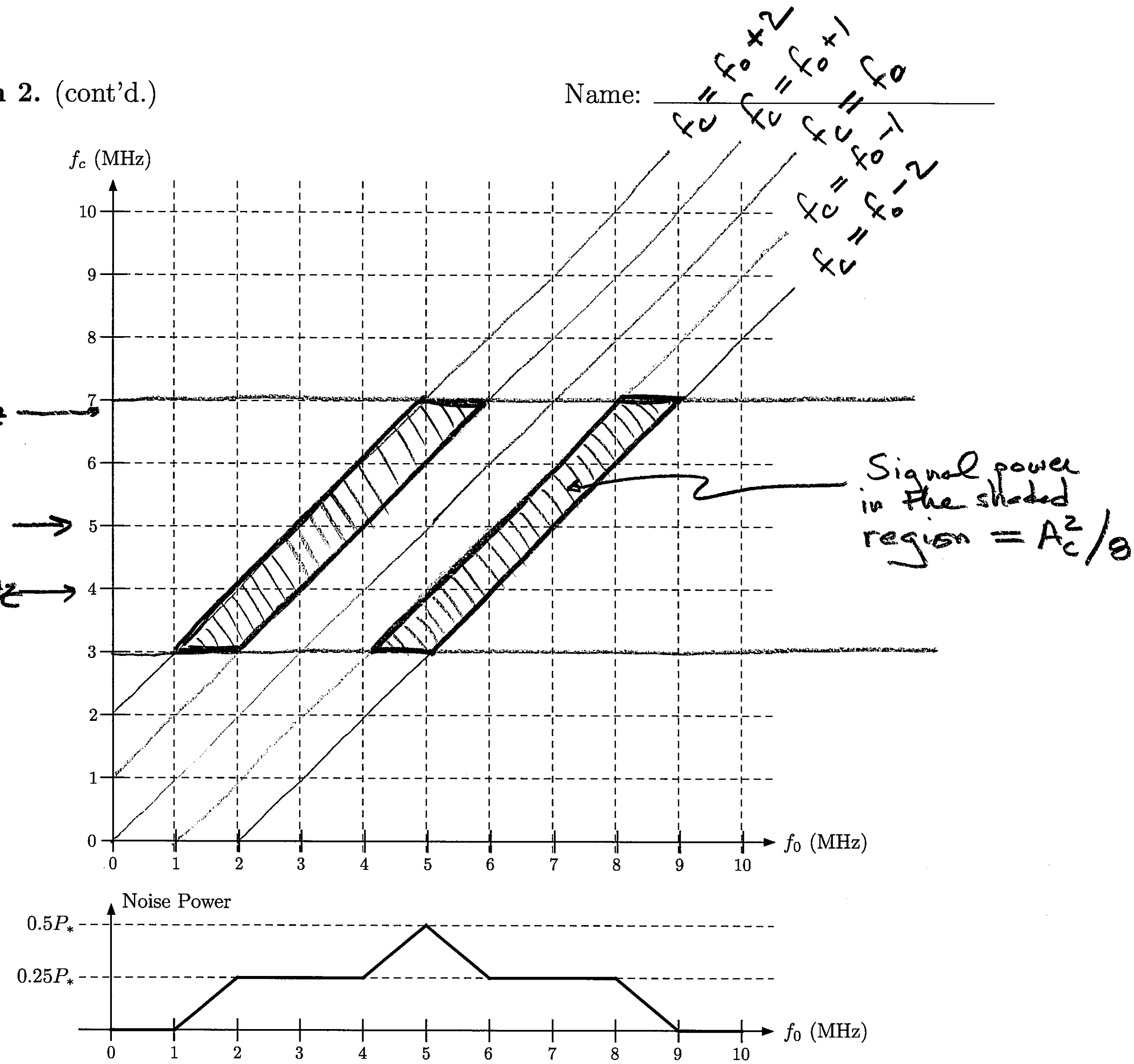


The input signal $X(t)$ is a sinusoid in zero-mean white noise of power spectral density height $N_0/2$. In a previous problem we considered signal and noise separately. For signal moving through the block diagram, two cases were worked out. First, we saw the case $f_c = 1.5$ MHz for which the signal did not make it through the RF filter resulting in zero signal power in the output of the IF filter. Second, we saw the case $f_c = 4$ MHz for which the the signal power in the output of the IF filter as a function of f_0 was shown to be as graphed below ...



In addition, we computed the noise power in the output of the IF filter as function of f_0. Written in terms of the quantity $P_* = 10^6 N_0$ W, the graph is shown on the following page.

Problem 2. (cont'd.) Name: ______________



(a) [6 pts.] Using the axes provided above shade the region in the f_c vs. f_0 plane where the signal power in the output of the IF filter would be greater than zero. What would be the value of the signal power where it is greater than zero?

First, only if $3 < f_c < 7$ (MHz) does any signal pass through the RF filter to have an opportunity to make it to the IF filter output. For $3 < f_c < 7$ the input to the IF filter due to signal is ...

$$V(t) = A_c \cos(2\pi f_c t)\cos(2\pi f_o t + \theta)$$
$$= 0.5 A_c \cos(2\pi(f_c - f_o)t - \theta) + 0.5 A_c \cos(2\pi(f_c + f_o)t + \theta)$$

To appear in the IF filter output must have

$$1 < |f_c - f_o| < 2$$

Since the IF band is below the RF filter band, this term can never appear in IF output for $0 < f_o < \infty$.

There are two cases to consider ...

$0 < f_o < f_c$

Then to satisfy $1 < |f_c - f_o| < 2$ must have

$$1 < f_c - f_o < 2$$

$$\iff 1 - f_c < -f_o < 2 - f_c$$

$$\iff f_c - 2 < f_o < f_c - 1$$

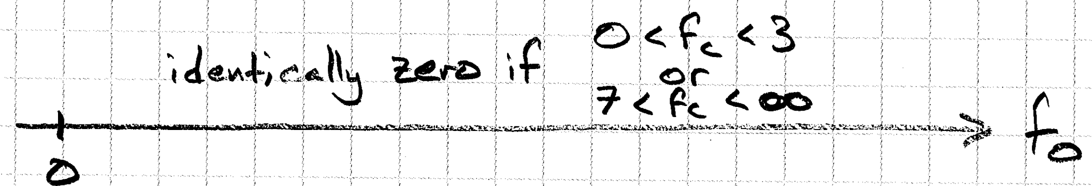
$f_c < f_o < \infty$

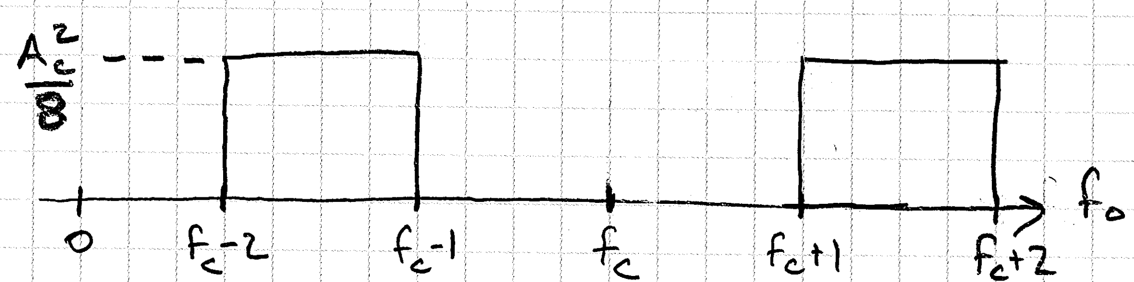
To satisfy $1 < |f_c - f_o| < 2$ must have

$$1 < f_o - f_c < 2$$

$$\iff f_c + 1 < f_o < f_c + 2$$

Therefore the graph of signal power in the IF filter output vs. f_o would look like:





(b1) If $f_c = 4\text{MHz}$ then sig power nonzero for $2 < f_o < 3$, $5 < f_o < 6$

SNR = 0 where sig. power is zero

$2 < f_o < 3$ $\quad SNR = \dfrac{A_c^2/8}{0.25 P_*} = 0.5 A_c^2 / P_*$

$5 < f_o < 6$ $\quad SNR = \dfrac{A_c^2/8}{0.25 P_* (7 - f_o)} = 0.5 \dfrac{A_c^2}{P_*} \left(\dfrac{1}{7 - f_o} \right)$

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(b2) If $f_c = 5$ MHz then sig pow nonzero for $3 < f_o < 4$, $6 < f_o < 7$

SNR = 0 where sig. power is zero

$3 < f_o < 4$ $\quad$ $\text{SNR} = \frac{A_c^2/8}{0.25 P_*} = 0.5 A_c^2 / P_*$

$6 < f_o < 7$ $\quad$ $\text{SNR} = \frac{A_c^2/8}{0.25 P_*} = 0.5 A_c^2 / P_*$

(b3) If $f_c = 7^- $ MHz then sig pow nonzero for $5^- < f_o < 6^-$, $8^- < f_o < 9^-$

SNR = 0 where sig. power is zero

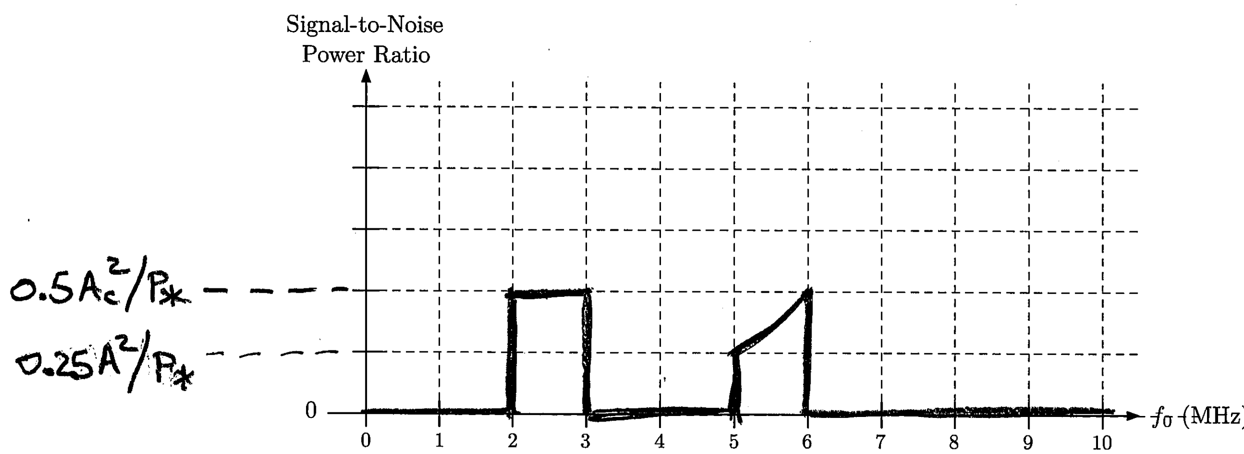
$5^- < f_o < 6^-$ $\quad$ $\text{SNR} = \frac{A_c^2/8}{0.25 P_* (7 - f_o)} = 0.5 \frac{A_c^2}{P_*} \left(\frac{1}{7 - f_o} \right)$

$8^- < f_o < 9^-$ $\quad$ $\text{SNR} = \frac{A_c^2/8}{0.25 P_* (9 - f_o)} = 0.5 \frac{A_c^2}{P_*} \frac{1}{9 - f_o}$

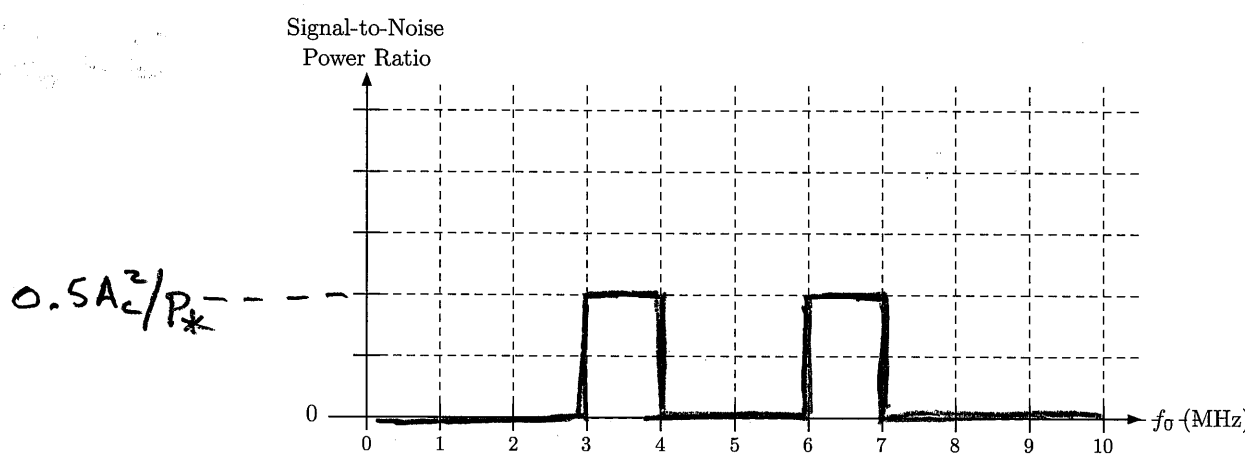
Problem 2. (cont'd.) Name: ____________________

(b) [9 pts.] Using the noise power vs. f_0 plot on the previous page and the answer to part (a), plot the signal to noise ratio at the output of the IF filter as a function of f_0 for the following three values of f_c:

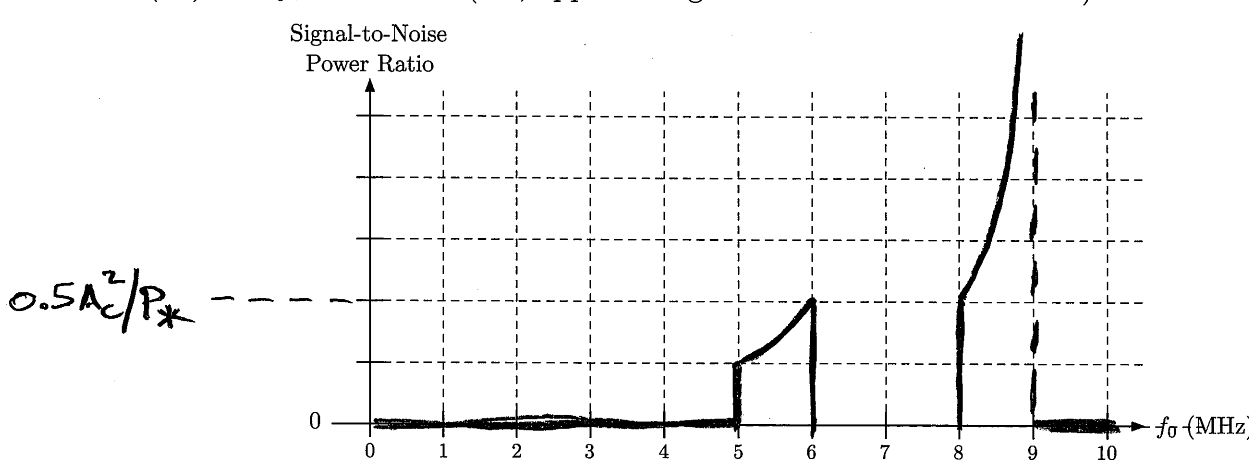
(b1) For $f_c = 4$ MHz ...



(b2) For $f_c = 5$ MHz ...

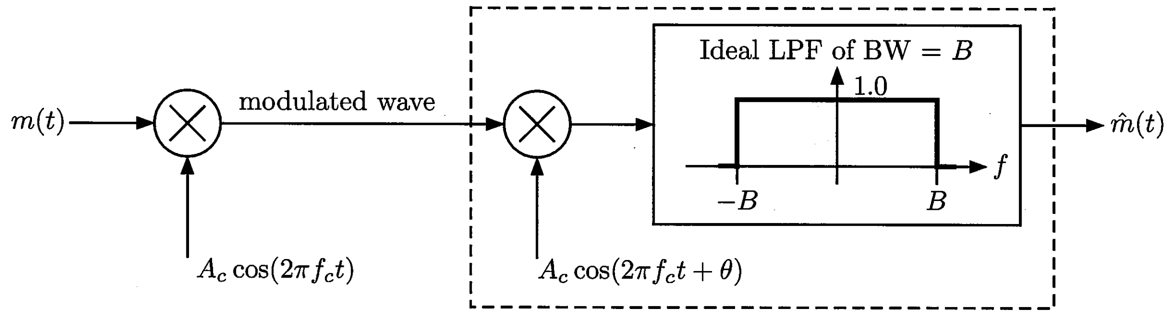


(b3) For $f_c = 7^-$ MHz (i.e., approaching 7 MHz from smaller values) ...

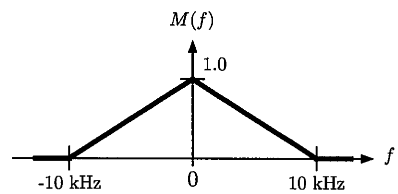


Name: ____________________

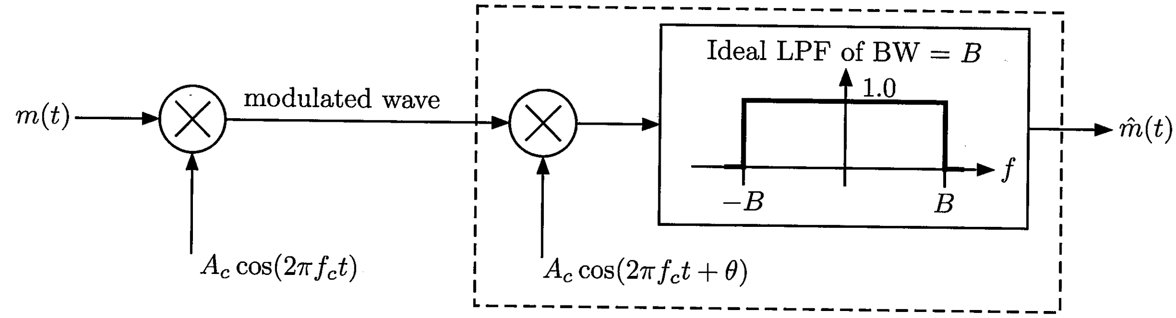
Problem 3. *Related to DSB Modulation and Demodulation.* [5 pts. total]



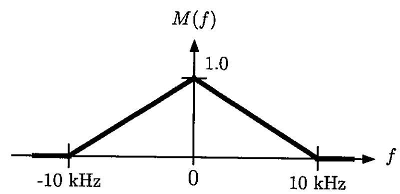
Suppose that a deterministic message $m(t)$ is applied to the block diagram above. Assume that the message has a Fourier transform $M(f)$ with the triangular spectral shape shown below. When plotting $\hat{M}(f)$ plot the real and imaginary parts separately if it is complex-valued. If it is real-valued you may plot only the real part. Note that $M(f)$ is real-valued.



Solve for the general form of $\hat{M}(f)$, i.e., give a formula for $\hat{M}(f)$ that would hold for any choices of the free parameters in the problem: A_c, f_c, θ, and B. Use $H(f)$ to represent the lowpass filter in the block diagram.



Suppose that a deterministic message $m(t)$ is applied to the block diagram above. Assume that the message has a Fourier transform $M(f)$ with the triangular spectral shape shown below. When plotting $\hat{M}(f)$ plot the real and imaginary parts separately if it is complex-valued. If it is real-valued you may plot only the real part. Note that $M(f)$ is real-valued.



Solve for the general form of $\hat{M}(f)$, i.e., give a formula for $\hat{M}(f)$ that would hold for any choices of the free parameters in the problem: A_c, f_c, θ, and B. Use $H(f)$ to represent the lowpass filter in the block diagram.

The input to the LPF is

$$m(t) A_c^2 \cos(2\pi f_c t) \cos(2\pi f_c t + \theta)$$

and so the output $\hat{m}(t)$ is simply ...

$$\hat{m}(t) = h(t) * \left[m(t) A_c^2 \cos(2\pi f_c t) \cos(2\pi f_c t + \theta) \right]$$

To get the desired formula we will want to take the Fourier transform. But first use a trig. identity to write

$$\cos(2\pi f_c t) \cos(2\pi f_c t + \theta)$$
$$= \frac{1}{2} \cos\theta + \frac{1}{2} \cos(2\pi 2 f_c t + \theta)$$

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∴ The LPF input is

$$\frac{A_c^2}{2} m(t) \cos\theta + \frac{A_c^2}{2} m(t) \underbrace{\cos(2\pi 2f_c t + \theta)}$$

(From Euler)

$$\frac{e^{j\theta} e^{j2\pi 2f_c t} + e^{-j\theta} e^{-j2\pi 2f_c t}}{2}$$

Then

$$\hat{M}(f) = H(f) \cdot \left\{ \begin{array}{l} \text{the F.T. of the} \\ \text{LPF input above} \end{array} \right\}$$

$$= H(f) \left\{ \frac{A_c^2}{2} \cos\theta \, M(f) + \frac{A_c^2}{4} e^{j\theta} M(f - 2f_c) + \frac{A_c^2}{4} e^{-j\theta} M(f + 2f_c) \right\}$$

$$= \frac{A_c^2}{2} \cos\theta \, H(f) M(f) + \frac{A_c^2}{4} e^{j\theta} H(f) M(f - 2f_c) + \frac{A_c^2}{4} e^{-j\theta} H(f) M(f + 2f_c)$$

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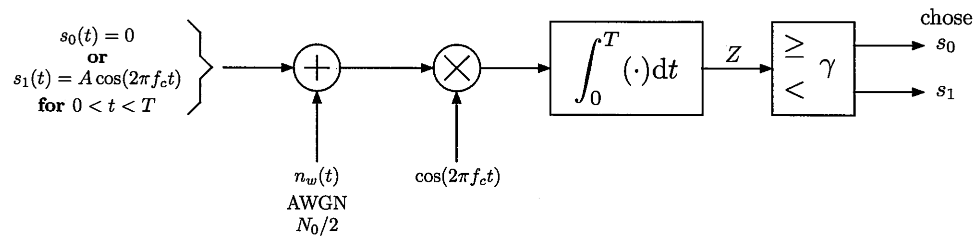
Problem 4. *Amplitude Shift Keying Receiver.* [15 pts. total]



The figure above shows a coherent ASK receiver. The prior probabilities of the two signals $s_0(t) = 0$ and $s_1(t) = A\cos(2\pi f_c t)$ are π_0 and π_1, respectively. The integration time T may be assumed to be an integer multiple of the period of the carrier for simplicity. The threshold has been set to $\gamma = AT/2$.

(a) [6 pts.] Assuming that $s_0(t)$ was transmitted, find the mean and variance of the Gaussian random variable Z. Use this to compute the conditional probability $P_{e|0}$ of an error given that $s_0(t)$ was transmitted. Write your answer in terms of the Gaussian Q-function[1].

(b) [6 pts.] Assuming that $s_1(t)$ was transmitted, find the mean and variance of the Gaussian random variable Z. Use this to compute the conditional probability $P_{e|1}$ of an error given that $s_1(t)$ was transmitted. Write your answer in terms of the Gaussian Q-function

(c) [3 pts.] Use the answers in (a) and (b) to find the expression for the unconditional probability of an error. Would it be possible to lower this error by choosing a different threshold γ? Explain.

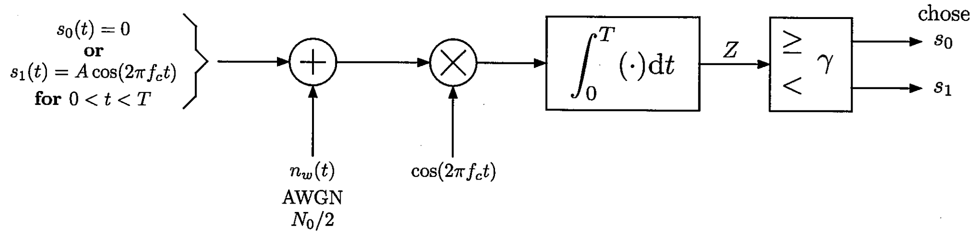
[1] The Gaussian Q-function is the integral

$$Q(\alpha) = \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty e^{-x^2/2} dx$$

Name: ________________________

Problem 4. *Amplitude Shift Keying Receiver.* [15 pts. total]

Typo: Should have been s_1 on top and s_0 on bottom.



The figure above shows a coherent ASK receiver. The prior probabilities of the two signals $s_0(t) = 0$ and $s_1(t) = A\cos(2\pi f_c t)$ are π_0 and π_1, respectively. The integration time T may be assumed to be an integer multiple of the period of the carrier for simplicity. The threshold has been set to $\gamma = AT/2$. → This is also a "typo". It should have been $AT/4$

(a) [6 pts.] Assuming that $s_0(t)$ was transmitted, find the mean and variance of the Gaussian random variable Z. Use this to compute the conditional probability $P_{e|0}$ of an error given that $s_0(t)$ was transmitted. Write your answer in terms of the Gaussian Q-function[1].

The noice part of the integrator output is zero mean Gaussian with variance

$$\int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\cos(2\pi f_c t)\cos(2\pi f_c s)\,dt\,ds = \frac{N_0}{2}\int_0^T \cos^2(2\pi f_c t)\,dt$$

$$= \frac{N_0}{4}\int_0^T \left[1 + \cos(2\pi 2 f_c t)\right] dt = \frac{N_0 T}{4}$$

The variance does not depend on the signal actually transmitted. Since signal $s_0(t) = 0$, the mean of Z will be zero in this case.

$\therefore Z \sim N\left(0, \frac{N_0 T}{4}\right)$. I will provide solution for both typo and non-typo. assuming s_0

As written...

$$\Pr(\text{Error}|s_0) = \Pr(Z < \gamma | s_0)$$

$$= P\left(\frac{Z-0}{\sqrt{N_0T/4}} < \frac{AT/2 - 0}{\sqrt{N_0 T/4}}\right) = 1 - Q\left(\frac{AT/2}{\sqrt{N_0T/4}}\right) = 1 - Q\left(\sqrt{\frac{A^2T}{N_0}}\right)$$

[1] The Gaussian Q-function is the integral

$$Q(\alpha) = \frac{1}{\sqrt{2\pi}}\int_\alpha^\infty e^{-x^2/2}dx$$

Problem 4. (cont'd.) Name: ______________

(b) [6 pts.] Assuming that $s_1(t)$ was transmitted, find the mean and variance of the Gaussian random variable Z. Use this to compute the conditional probability $P_{e|1}$ of an error given that $s_1(t)$ was transmitted. Write your answer in terms of the Gaussian Q-function

The variance will be same as in (a) but the mean will now be non-zero. The mean will be

$$\int_0^T A\cos^2(2\pi f_c t)\,dt = \frac{AT}{2}$$

$$\therefore Z \sim N\left(\frac{AT}{2}, \frac{N_0 T}{4}\right) \quad \text{assuming } s_1$$

Then, solving for the error probability as written...

$$Pr(\text{Error} \mid s_1) = P_r(Z \geq \gamma \mid s_1)$$

$$= P\left(\frac{Z - AT/2}{\sqrt{N_0T/4}} \geq \frac{AT/2 - AT/2}{\sqrt{N_0T/4}}\right) = Q(0) = \frac{1}{2}$$

$\therefore$ The double typos are very unfortunate. But it is what it is.

Problem 4. (cont'd.) Name: ______________

(c) [3 pts.] Use the answers in (a) and (b) to find the expression for the unconditional probability of an error. Would it be possible to lower this error by choosing a different threshold γ? Explain.

As written ...

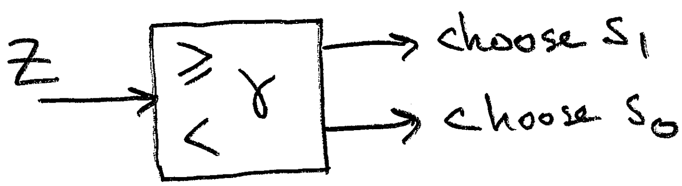
$$P_e = \pi_0 P_{e|0} + \pi_1 P_{e|1}$$

$$= \pi_0 \left(1 - Q\left(\sqrt{\frac{A^2 T}{N_0}}\right)\right) + \pi_1 \frac{1}{2}$$

Yes, because the problem is a "dumpster fire" it is quite possible to fix.

Problem 4. (Extra Work Page) Name: ____________________

The fix:



$$\gamma = \frac{AT}{4}$$

$Z \sim N\left(0, \frac{N_0 T}{4}\right)$ under s_0

$$P(\text{Error}|s_0) = P\left(Z \geq \gamma | s_0\right) = P\left(\frac{Z-0}{\sqrt{N_0T/4}} \geq \frac{AT/4 - 0}{\sqrt{N_0T/4}}\right)$$

$$= Q\left(\frac{AT/4}{\sqrt{N_0T/4}}\right) = Q\left(\sqrt{\frac{A^2T}{4N_0}}\right)$$

$Z \sim N\left(\frac{AT}{2}, \frac{N_0T}{4}\right)$ under s_1

$$P(\text{Error}|s_1) = P\left(Z < \gamma | s_1\right) = P\left(\frac{Z - AT/2}{\sqrt{N_0T/4}} < \frac{\frac{AT}{4} - \frac{AT}{2}}{\sqrt{N_0T/4}}\right)$$

$$= Q\left(\frac{-AT/4}{\sqrt{N_0T/4}}\right) = Q\left(\frac{AT/4}{\sqrt{N_0T/4}}\right)$$

$$= Q\left(\sqrt{\frac{A^2T}{4N_0}}\right)$$

Last part... $P_e = \pi_0 P_{e|0} + \pi_1 P_{e|1} = Q\left(\sqrt{\frac{A^2T}{4N_0}}\right)$

This can be improved in case where $\pi_0 \neq \pi_1$ since γ then should be chosen as

$$= \frac{AT}{4} + \frac{N_0T/4}{AT/2} \ln(\pi_0/\pi_1)$$ ... the Bayes threshold.

Name: ______________________

Problem 5. *Amplitude Shift Keying Performance Calculations.* [30 pts. total]

The optimal performance of a binary communication system in AWGN with equally likely signals $s_0(t)$ and $s_1(t)$ is given by

$$P_e = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

where[2]

$$E_b = \frac{\|s_0\|^2 + \|s_1\|^2}{2} \text{ and } \rho = \langle s_0, s_1 \rangle / E_b.$$

Assume an amplitude shift keying (ASK) system where one bit is sent in each signaling interval of length T_b by choosing one of two possible signals. A zero bit is indicated by using a signal $s_0(t) = 0$. A one bit is transmitted by sending $s_1(t) = A\cos(2\pi f_c t)$ for $0 \le t \le T_b$. For simplicity assume that the bit interval encompasses an integer number of cycles of the carrier, i.e., $T_b = N/f_c$ for some integer N.

(a) [5 pts] For ASK find the signal correlation ρ and the average signal energy per bit E_b in terms of A and T_b.

(b) [3 pts] If the SNR $E_b/N_0 = 16$ (which, incidentally, corresponds to 12.04 dB) find the average probability of a bit error using the below Q-function table.

Table G.1 A Short Table of *Q*-Function Values

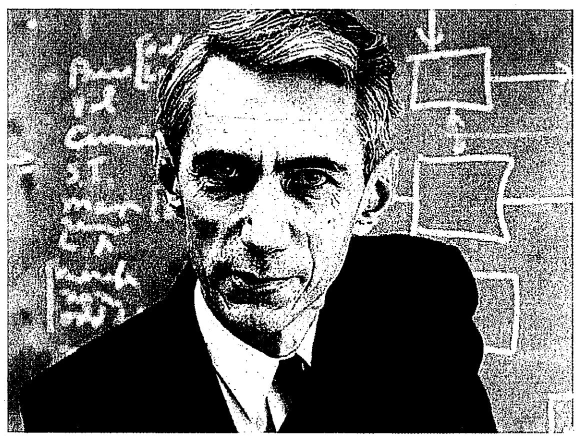
x	*Q(x)*	*x*	*Q(x)*	*x*	*Q(x)*
0	0.5	1.5	0.066807	3.0	0.0013499
0.1	0.46017	1.6	0.054799	3.1	0.00096760
0.2	0.42074	1.7	0.044565	3.2	0.00068714
0.3	0.38209	1.8	0.035930	3.3	0.00048342
0.4	0.34458	1.9	0.028717	3.4	0.00033693
0.5	0.30854	2.0	0.022750	3.5	0.00023263
0.6	0.27425	2.1	0.017864	3.6	0.00015911
0.7	0.24196	2.2	0.013903	3.7	0.00010780
0.8	0.21186	2.3	0.010724	3.8	7.2348×10^{-5}
0.9	0.18406	2.4	0.0081975	3.9	4.8096×10^{-5}
1.0	0.15866	2.5	0.0062097	4.0	3.1671×10^{-5}
1.1	0.13567	2.6	0.0046612	4.1	2.0658×10^{-5}
1.2	0.11507	2.7	0.0034670	4.2	1.3346×10^{-5}
1.3	0.096800	2.8	0.0025551	4.3	8.5399×10^{-6}
1.4	0.080757	2.9	0.0018658	4.4	5.4125×10^{-6}

[2] The inner product of signals is defined as

$$\langle s_0, s_1 \rangle = \int s_0(t) s_1(t) dt$$

where the limits of the integrals are whatever are needed to incorporate all of the non-zero values of the integrand. The norm-squares are defined via the inner product as: $\|s_0\|^2 = \langle s_0, s_0 \rangle$.

(c) [3 pts] The below image of Claude Shannon[3] contains approximately 9 Mbits. If we were to transmit this file using the ASK system specified above, approximately how many bit errors would we expect to incur?



(d) [8 pts] If the power spectral density parameter $N_0 = 2 \times 10^{-21}$ J (approximately corresponding to a noise temperature of 300 K) ...

(d1) compute the bit energy E_b needed to achieve an SNR ratio of $E_b/N_0 = 16$.

(d2) What is the total energy needed to receive the entire Shannon image at this SNR?

(d3) If we send the entire Shannon image in 1 second, what is the signal power needed at the receiver?

(d4) If the above signal power were dissipated in a 50 Ω resistor, what would be the resulting rms voltage?

(e) [3 pts] Now if the power propagation loss from transmitter to receiver were 100 dB, what is the power at the transmitter in order to achieve $E_b/N_0 = 16$ at the receiver?

(f) [5 pts] Now suppose that we increase the distance from transmitter to receiver by a factor of 10 in a scenario where propagation losses are proportional to the distance squared. What would be the new SNR at the receiver and the new resulting average probability of error?

(g) [3 pts] One way to fix the situation of the sudden increase in distance in part (f) is to slow the transmission rate by a certain factor[4] until the required SNR per bit is attained. By what factor should we increase the bit duration T_b?

[3] A luminary of Communication Theory – inventor of Information Theory and the use of Boolean algebra applied to the design of digital logic.

[4] In effect, this is the strategy taken by the wireless, low power, long range networking protocol called LoRaWAN.

Problem 5. (cont'd.) Name: ____________

(b) [3 pts] If the SNR $E_b/N_0 = 16$ (which, incidentally, corresponds to 12.04 dB) find the average probability of a bit error using the below Q-function table.

Table G.1 A Short Table of Q-Function Values

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1.4	0.080757	2.9	0.0018658	4.4	5.4125×10^{-6}

Amplitude Shift Keying Performance Calculations

The optimal performance of a binary coherent communication system is given by

$$P_e = Q\left(\sqrt{\frac{\overline{\mathcal{E}}(1-\rho)}{N_0}}\right)$$

where

$$\overline{\mathcal{E}} = \frac{\|s_0\|^2 + \|s_1\|^2}{2}, \quad \rho = \langle s_0, s_1 \rangle / \overline{\mathcal{E}}$$

In ASK $s_0(t) = 0$ and $s_1(t) = A\cos(2\pi f_c t)$ for $0 \le t \le T_b$ and $= 0$ outside of the bit interval.

For simplicity we assume that the bit interval encompasses an integer number of cycles of the carrier ie

$$T_b = N\frac{1}{f_c} \iff f_c/N = \frac{1}{T_b}$$

(Note that this would be the case if the bit clock were derived from the carrier oscillator upon dividing it down, which would be a convenient way to design the transmitter)

(a) Find sig. correlation + avg. sig. energy per bit.

Then $\|s_0\|^2 = 0$ and

$$\|s_1\|^2 = \int_0^{T_b} A^2 \cos^2 2\pi f_c t \, dt = \frac{1}{2}A^2 T_b$$

Use E_b for $\bar{\mathcal{E}}$ in most formulas.

Then $\bar{\mathcal{E}} = \frac{1}{4} A^2 T_b$ → This is the average energy per bit if 0, 1 bits are equally likely.

Also $\rho = 0$ (correlation coeff.). Therefore, the average probability of a bit error is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

(b) Given $E_b/N_o = 16$ → find P_e

Given that $10 \log_{10}(E_b/N_o) = 10 \log_{10}(16) = 12.04 \text{ dB}$

From the table

$$P_e = Q\left(\sqrt{16}\right) = Q(4) \cong 3.2 \times 10^{-5}$$

(c) If we used this comm. syst. to send the Claude Shannon image (= 1.1 MByte) how many bit errors would we expect ...

$$\#\text{ Bit Errors} = (1.1 \times 10^6)(8 \text{ bits/byte})(3.2 \times 10^{-5})$$

$$\approx (8.8)(3.2) \times 10^1$$

$$\approx 27 \times 10^1 = 270$$

$$\begin{array}{r} 8.8 \\ 3.2 \\ \hline 176 \\ 264 \\ \hline 27.16 \end{array}$$

(d)

$$N_o = kT/2 = (1.38 \times 10^{-23} \text{ J/K})(300 \text{ K})(0.5)$$

$$= (1.38)(150) \times 10^{-23} \text{ J}$$

$$\approx 197 \times 10^{-23} \text{ J} \approx 2 \times 10^{-21} \text{ J}$$

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1
4
150
1.38

1200
450
150
197.00

So needed $\frac{E_b}{N_o} = 16 \Rightarrow E_d = 16 N_o$

$E_b \approx 32 \times 10^{-21}$ J (very small) **(d1)**

Then the energy needed at the receiver to get 8.8×10^6 bits would be

$$E_{total} = (8.8 \times 10^6)(32 \times 10^{-21})$$

$$= 282 \times 10^{-15} \text{ J} \approx 3 \times 10^{-13} \text{ J}$$ **(d2)**

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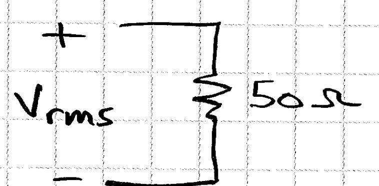
2
8.8
32

176
264
28 1.6

Thus if this were to be accomplished in 1 sec, the needed signal power at the receiver would be

3×10^{-13} W **(d3)**

If this power were dissipated in an equivalent 50Ω resistor, the rms voltage would be



$$\frac{V_{rms}^2}{50} = 3 \times 10^{-13}$$

$$V_{rms} = \sqrt{50 \cdot 3 \times 10^{-13}} \text{ Vrms}$$

$$= \sqrt{15 \times 10^{-12}} = \sqrt{15} \times 10^{-6}$$

$$\approx 4 \mu V \text{ rms}$$ **(d4)**

(e) Now if the power propagation loss from transmitter to receiver were 100 dB, how much is the power at the transmitter?

L = propagation loss as a number

$L_{dB} = 10 \log_{10} L$

$\Longrightarrow 100 = 10 \log_{10} L \Longrightarrow \log_{10} L = 10$

$\Longrightarrow L = 10^{10}$

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$\Longrightarrow P_{RX} = P_{TX}/L \Longrightarrow P_{TX} = L \cdot P_{RX}$

$= 10^{10} \cdot 3 \times 10^{-13}$ W

$= 3 \times 10^{-3}$ W

$= 3$ mW.

(f) Say we increase the distance from transmitter to receiver by a factor of 10 in a scenario where propagation losses are proportional to distance squared.

$\Longrightarrow$ Additional loss factor $10^2 \longrightarrow 20$ dB

Powers, energies are all proportional. New received power $\approx 3 \times 10^{-15}$ W

New received bit energy $\approx 32 \times 10^{-23}$ J

New received SNR

$$\approx \frac{16}{100} = 0.16$$

$Q(\sqrt{.16}) = Q(0.4) = 0.34$

Totally useless now as $\frac{1}{3}$ of the bits are in error.

(g) But if we slow the transmission by a factor of 100 we can get back to the previous bit error rate.

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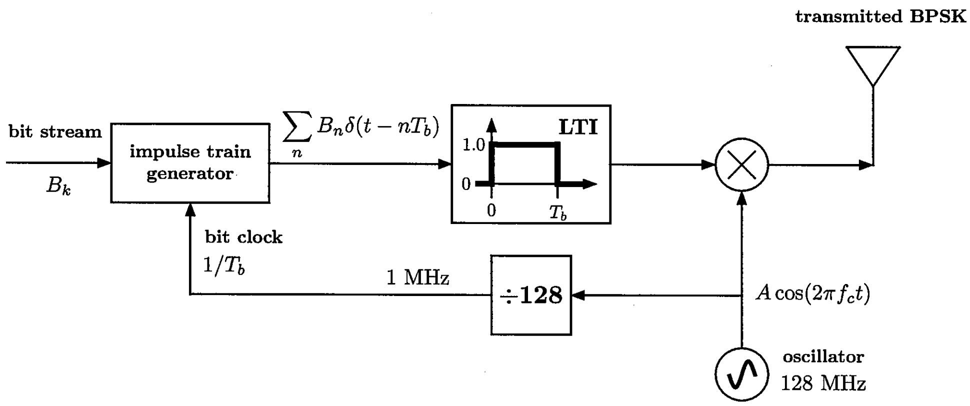
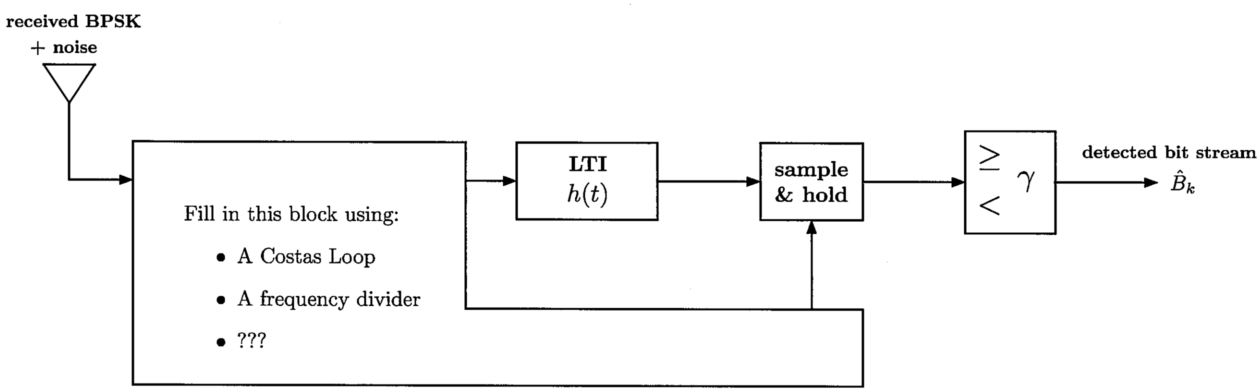
Name: ______________________

Problem 6. *BPSK System and the Costas Loop.* [20 pts. total]



The block diagram above shows a BPSK transmitter operating with a carrier frequency of 128 MHz and a bit rate of 1 MHz. Notice how the carrier frequency and the bit rate clock are automatically locked together since they are related by the frequency divider clock.

The architecture of a receiver for BPSK is partially specified in the block diagram below. In this problem you will show how to complete the design by answering the questions that follow. You don't need to derive anything mathematically but you should give explanations for the design choices made.



(a) [4 pts] What choice should you make for the impulse response $h(t)$ in the LTI filter of the receiver block diagram and why?

h(t) should be matched to the transmitter pulse.
For this problem ...

h(t) =

Why? It maximizes the SNR at the sampler/decision device.

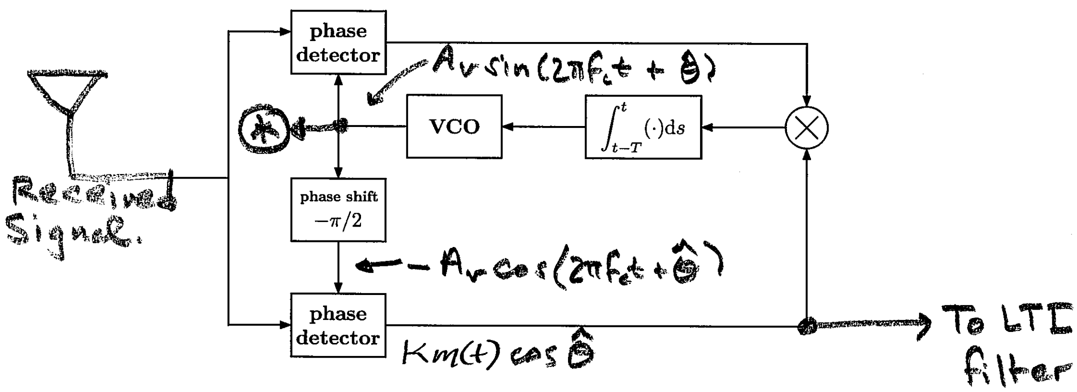
Problem 6. (cont'd.) Name: ______________________

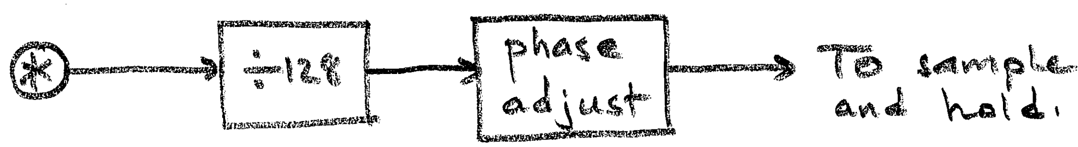
(b) [4 pts] The symbols B_k are i.i.d. and equal to ± 1 with equal probability. What value should be chosen for the threshold γ in the detection device and why?

$\gamma = 0$, which is the average of the two signals.

Note: $+1$ and -1 are equally likely.

(c) [10 pts] The unlabeled block diagram of a Costas PLL is given below. Label the block diagram as needed and show how to place it into the box in the BPSK receiver and how to make the connections to derive both the receiver's estimate of both the carrier wave and the bit clock. What additional block is needed inside the box in order to complete the design? Explain.





(d) [2 pts] Comment on the choice of the parameter T in the sliding window integrator in the Costas PLL.

$\frac{1}{T} << \frac{1}{T_b}$