Name: Solution

General Instructions:

- **You have 50 minutes to complete the exam.**
- Write your name on every page of the exam.
- Please do not write on the backs of pages.
- The exam is closed book. Calculators are not allowed.
- You are allowed both sides of two 8.5 by 11 inch sheets of paper for your personal notes in addition to the instructor supplied formula sheet.
- Your work must be explained to receive full credit. All plots must be carefully drawn with axes labeled.
- Point values for each problem are as indicated. The exam totals 100 points.
- Please do not leave early as it is disruptive to those working around you.

Do not open the exam until you are told to begin.

Name: ____________________

Problem 1. *Calculations Involving WSS Random Processes.* [40 pts. total]

Let $A(t)$ be a real-valued wide-sense stationary random process with mean μ_A and auto-correlation $R_A(\tau)$ and let Θ be a random variable uniformly distributed on $[0, 2\pi)$. Suppose that $\{A(t) : t \in \mathcal{R}\}$ and Θ are statistically independent. Please answer the following questions from first principles, i.e., completely explain your steps.

Define the random process $X(t) = A(t)S(t, \Theta)$ for $t \in \mathcal{R}$.

(a) [5 pts.] If $S(t, \Theta) = \sin(2\pi f_c t + \Theta)$, find the mean of $X(t)$.

(b) [15 pts.] If $S(t, \Theta) = \sin(2\pi f_c t + \Theta)$, find the autocorrelation function of $X(t)$.

(c) [5 pts.] If $S(t, \Theta) = \mathrm{sgn}[\sin(2\pi f_c t + \Theta)]$, find the mean of $X(t)$. Note that $\mathrm{sgn}(\cdot)$ is the *sign* function, which equals +1 for positive values of its argument and -1 for negative values.

(d) [15 pts.] If $S(t, \Theta) = \mathrm{sgn}[\sin(2\pi f_c t + \Theta)]$, find the autocorrelation function of $X(t)$.

Exam 2

P1 $A(t)$ WSS μ_A $R_A(\tau)$ Θ unif $[0,2\pi)$

$A \perp\!\!\!\perp \Theta$

$$X(t) \triangleq A(t)\, S(t,\Theta)$$

$$EX(t) = EA(t)\, ES(t,\Theta)$$

$$= \mu_A \boxed{ES(t,\Theta)} \longrightarrow \text{Figure out}$$

$$R_X(\tau) = E\{X(t)X(t+\tau)\}$$

$$= E\{A(t)A(t+\tau)\, S(t,\Theta)\, S(t+\tau,\Theta)\}$$

$$= R_A(\tau) \boxed{E\{S(t,\Theta)\, S(t+\tau,\Theta)\}} \longrightarrow \text{Figure out.}$$

a $S(t,\Theta) = \sin(2\pi f_c t + \Theta)$

$$ES(t,\Theta) = \frac{1}{2\pi}\int^{2\pi} \sin(2\pi f_c t + \theta)\, d\theta = 0$$

$$E\, S(t,\Theta)\, S(t+\tau,\Theta)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(2\pi f_c t + \theta)\,\sin(2\pi f_c t + 2\pi f_c \tau + \theta)\, d\theta$$

$$\frac{1}{2}\cos(2\pi f_c \tau) - \frac{1}{2}\cos(\cdots + 2\theta)\, d\theta$$

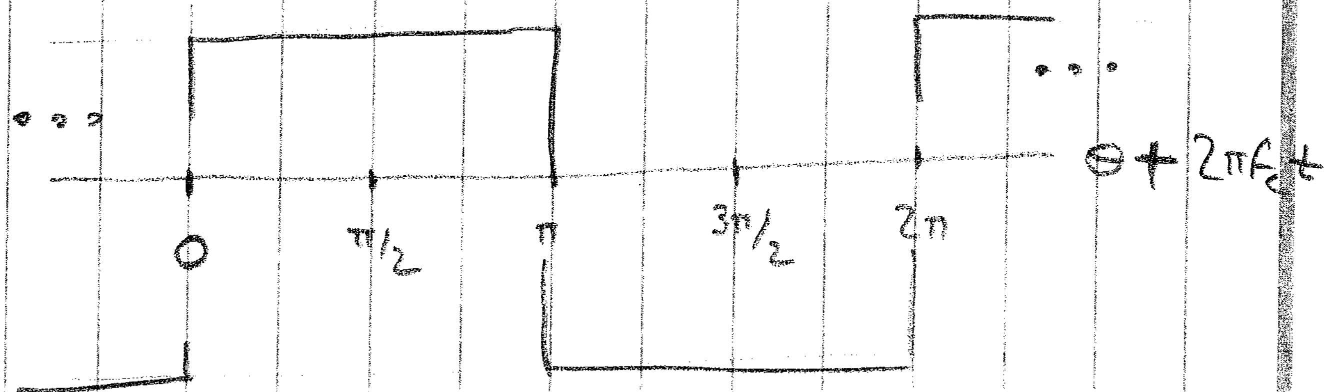
$$= \frac{1}{2}\cos 2\pi f_c \tau$$

(b)

$$R_x(\tau) = \frac{1}{2} R_A(\tau) \cos 2\pi f_c \tau$$

(c) $S(t, \Theta) = \operatorname{sgn} \sin(2\pi f_c t + \Theta)$

$$= \begin{cases} +1 & \text{when } \sin \text{ is } > 0 \\ -1 & \text{" " " } < 0 \end{cases}$$



Clearly $E S(t, \Theta) = 0$

Now for the auto corr.

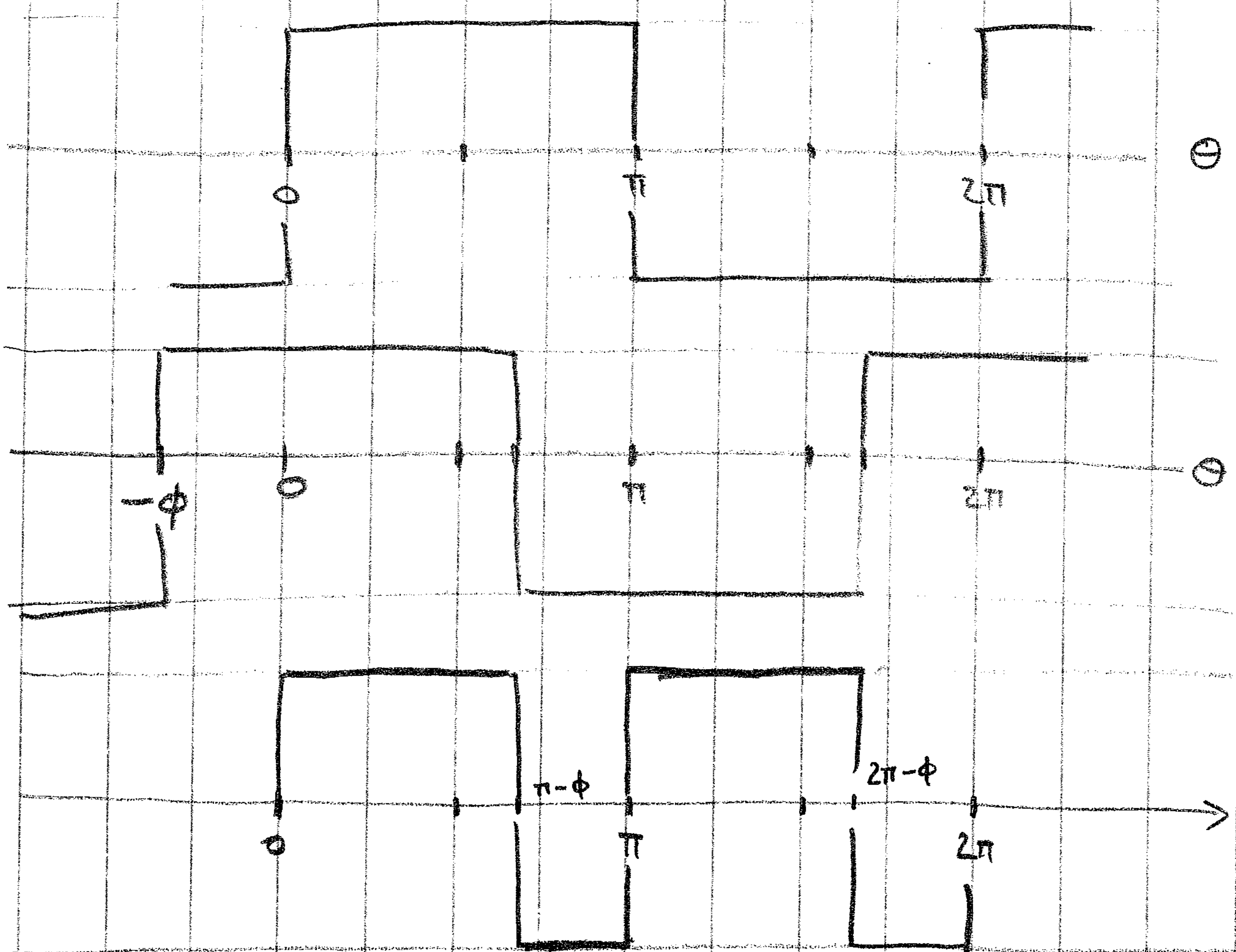
$$\operatorname{sgn} \sin(2\pi f_c t + \Theta) \operatorname{sgn} \sin(2\pi f_c t + 2\pi f_c \tau + \Theta)$$

Easy to see that above is periodic in $2\pi f_c t$ so WLOG set $t = 0$

(d)

It will also be periodic in $2\pi f_c \tau$, though it will depend on it.

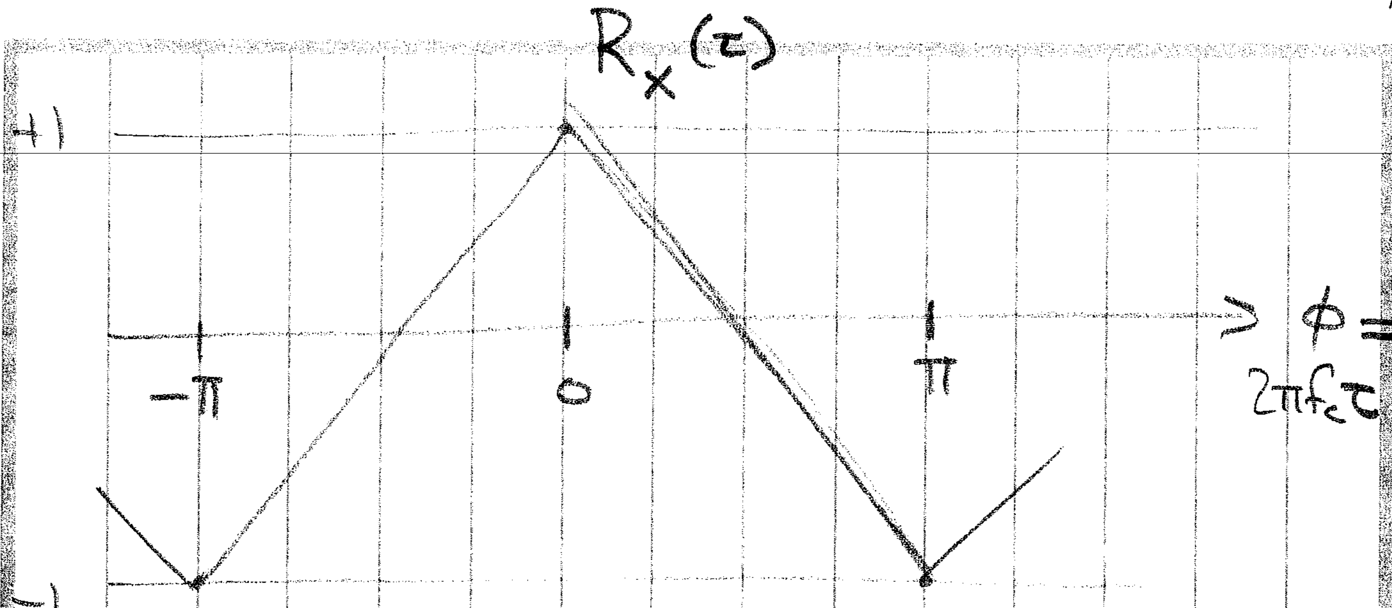
Let $\phi = 2\pi f_c \tau$ and say $-\pi < \phi < \pi$



$$\frac{1}{2\pi}\int_0^{2\pi} \text{above}\; d\theta = \frac{1}{\cancel{2}\pi}\cancel{2}(\pi-\phi) - \frac{1}{\cancel{2}\pi}\cancel{2}(\phi)$$

$$= \frac{\pi-\phi-\phi}{\pi} = \frac{\pi-2\phi}{\pi}$$ linear in ϕ

$0 < \phi < \pi$

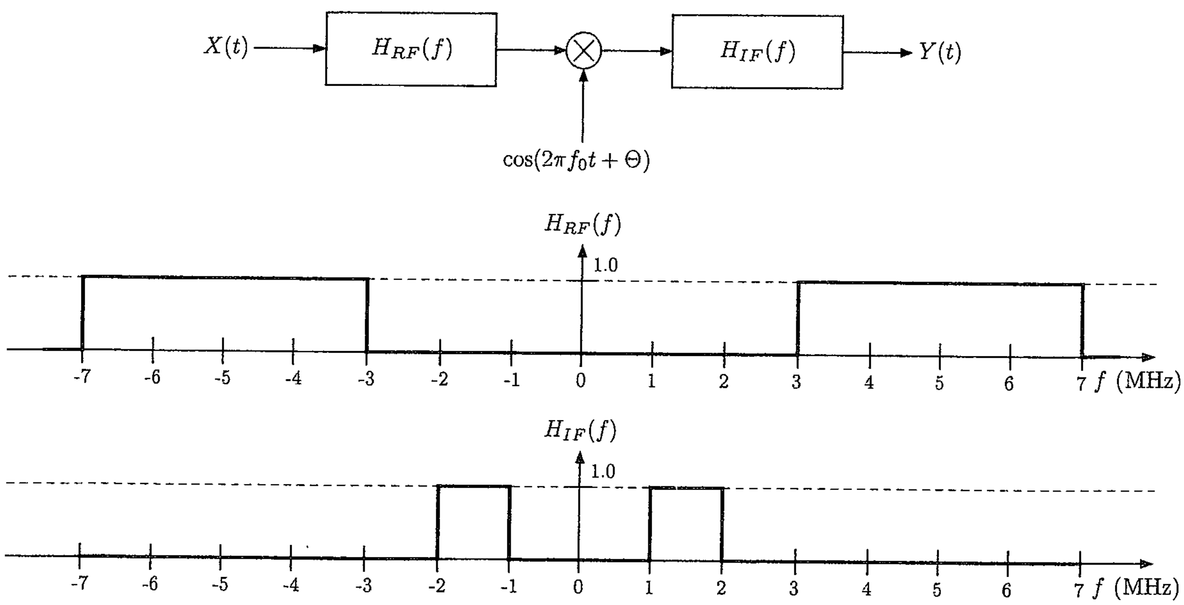


$R_x(\tau)$
+1
-1
$-\pi$
0
π
$\phi = 2\pi f_c \tau$

Name: ____________________

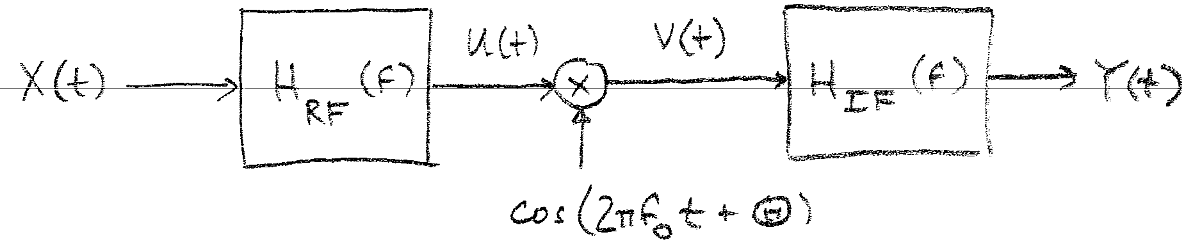
Problem 2. *Signal and Noise at the Input to a Tunable Receiver.* [40 pts. total]

Consider the receiver shown below, which can be tuned by varying the frequency f_0 of the local oscillator. For purposes of modeling, assume that the local oscillator phase Θ is uniformly distributed on $[0,\ 2\pi)$ and statistically independent of any other random variables or processes that might arise in the course of analysis. As shown the RF filter is bandpass of bandwidth equal to 4 MHz, centered at 5 MHz and the IF filter is bandpass of bandwidth 1 MHz, centered at 1.5 MHz.



(a) [25 pts.] Assuming that the input $X(t)$ is a zero-mean white noise of power spectral density height $N_0/2$, compute the power in the output $Y(t)$ as a function of the local oscillator frequency as we vary it over $0 < f_0 < \infty$. Make a plot of the power vs. f_0. Write the power in terms of the quantity $P_* = 10^6 N_0$ W, the nominal power of white noise in a bandwidth of 1 MHz.

(b) [15 pts.] Assuming that the input $X(t)$ is a sinusoid $X(t) = A_c \cos(2\pi f_c t)$ compute and plot the power in the output $Y(t)$ as a function of the local oscillator frequency as we vary it over $0 < f_0 < \infty$. Do this for two cases: $f_c = 1.5$ MHz and $f_c = 4$ MHz. The power should be written in terms of A_c^2.



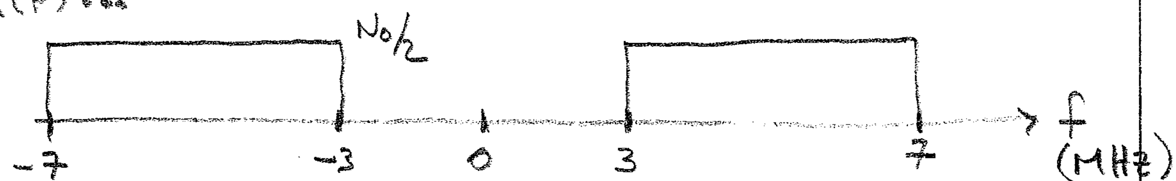
(a)

X(t) AWGN, mean zero, psd height $N_0/2$

From standard results know that U(t) is WSS, zero mean and with psd having the same form as the RF filter ...

$$S_U(f) = |H_{RF}(f)|^2 S_X(f) = \frac{N_0}{2} |H_{RF}(f)|^2$$

$S_U(f)$ ...



From the result of the prev. problem (or from the so-called modulation theorem) ...

$$R_V(\tau) = \frac{1}{2} R_U(\tau) \cos(2\pi f_0 \tau)$$

Taking the Fourier transform and using standard properties ...

$$S_V(f) = \frac{1}{4} S_U(f - f_0) + \frac{1}{4} S_U(f + f_0)$$

Then to determine the power in the IF filter output we would compute

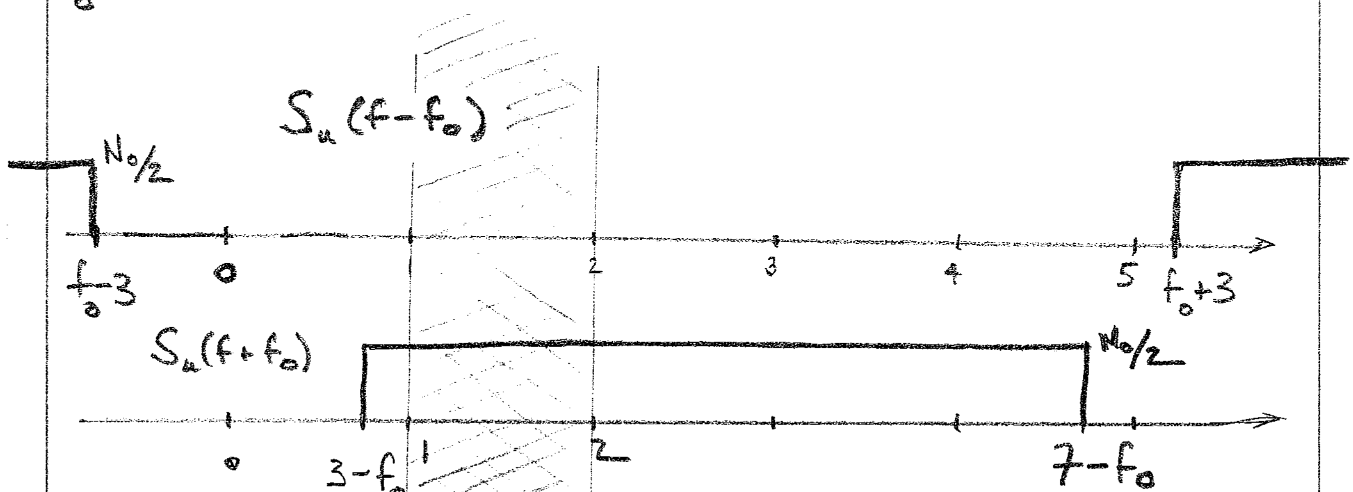
$$P\{Y(t)\} = R_Y(0) = \int S_V(f) |H_{IF}(f)|^2 \, df$$

There is a symmetry in the problem, which allows us to simplify ...

$$R_Y(0) = 2\int_1^2 S_V(f)\, df$$

$$= \frac{2}{4}\int_1^2 S_u(f-f_0)\, df + \frac{2}{4}\int_1^2 S_u(f+f_0)\, df$$

where we then consider varying the local osc. freq. f_0 from 0 to ∞.



Picture above is drawn for $0 < f_0 < 3$. In order for a non-zero power to appear in the output of the IF filter there must be overlap between the sidebands of $S_u(f-f_0)$ and $S_u(f+f_0)$ and the IF filter passband.

By examing the drawing above we conclude: (all frequencies in MHz)

$$S_u(f+f_0) \text{ overlaps } 1<f<2 \iff 3-f_0 < 2 \text{ and } 7-f_0 > 1$$

$$\iff 1 < f_0 < 6$$

Furthermore

$1 < f_0 < 2$ partial overlap
$2 < f_0 < 5$ full overlap
$5 < f_0 < 6$ partial overlap

$S_u(f-f_0)$ overlaps $1<f<2 \iff f_0-3>1$ and $f_0-7<2$
$\iff 4<f_0<9$

Furthermore

$4<f_0<5$ partial overlap
$5<f_0<8$ full overlap
$8<f_0<9$ partial overlap

Putting together for power we compute

$$R_Y(0) = \frac{1}{2}\int_1^2 S_u(f-f_0)\,df + \frac{1}{2}\int_1^2 S_u(f+f_0)\,df$$

Case: $0<f_0<1 \Rightarrow$ no overlap for either $\Rightarrow R_Y(0)=0$

$1<f_0<2 \Rightarrow$ partial overlap for $S_u(f+f_0)$ $\Rightarrow R_Y(0) = \frac{1}{2}\cdot\frac{N_0}{2}\left(2-(3-f_0)\right)$

$$= 0.25N_0(f_0-1)$$

$$= 0.25(f_0-1)P_*$$

$2<f_0<4 \Rightarrow$ full overlap for $S_u(f+f_0)$ $\Rightarrow R_Y(0) = 0.25P_*$

$4<f_0<5 \Rightarrow$ full overlap for $S_u(f+f_0)$, partial overlap for $S_u(f-f_0)$ $\Rightarrow R_Y(0) = 0.25P_* + \frac{1}{2}\frac{N_0}{2}(f_0-3-1)$

$$= 0.25P_* + 0.25P_*(f_0-4)$$

$$= 0.25P_*[f_0-3]$$

$5<f_0<6 \Rightarrow$ partial overlap for $S_u(f+f_0)$, full overlap for $S_u(f-f_0)$ $\Rightarrow R_Y(0) = \frac{1}{2}\frac{N_0}{2}(7-f_0-1) + 0.25P_*$

$$= 0.25P_*[1+6-f_0]$$

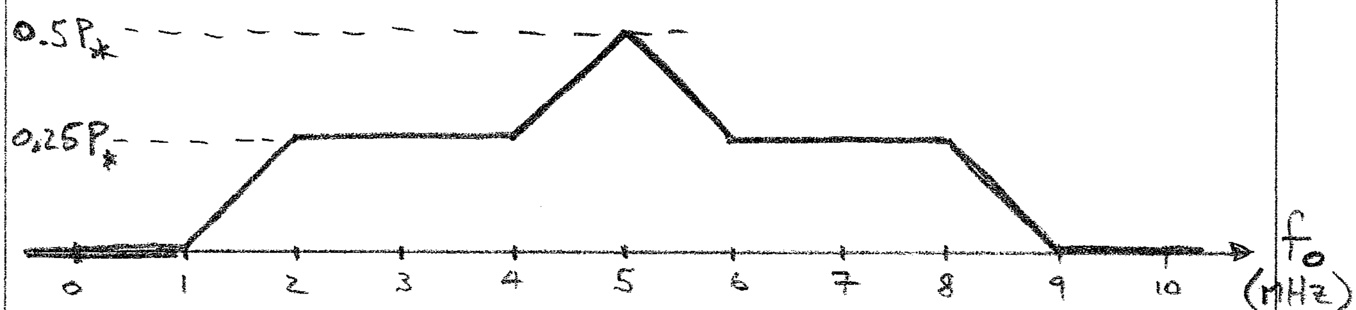
$$= 0.25P_*[7-f_0]$$

$6 < f_0 < 8 \Rightarrow$ full overlap for $S_u(f-f_0)$ $\Rightarrow R_Y(0) = 0.25\, P_*$

$8 < f_0 < 9 \Rightarrow$ partial overlap for $S_u(f-f_0)$ $\Rightarrow R_Y(0) = \frac{1}{2}\frac{N_0}{2}\left[2-(f_0-7)\right]$

$$= 0.25\, P_* \left[9 - f_0\right]$$

$f_0 > 9 \Rightarrow$ no overlap for either $\Rightarrow R_Y(0) = 0$



(b) $X(t) = A_c \cos(2\pi f_c t)$

If $f_c = 1.5$ MHz then the signal at output of the RF filter is zero $\Rightarrow Y(t) = 0$ for any value of $f_0 \Rightarrow R_Y(0) = 0 \;\; \forall f_0$

If $f_c = 4$MHz then the signal makes it through the RF filter

$\Rightarrow U(t) = X(t) = A_c \cos(2\pi f_c t)$

$\Rightarrow V(t) = A_c \cos(2\pi f_c t) \cos(2\pi f_0 t + \Theta)$

$$= \frac{A_c}{2} \cos\left(2\pi(f_c - f_0)t - \Theta\right) + \frac{A_c}{2} \cos\left(2\pi(f_c + f_0)t + \Theta\right)$$

Power from these terms will appear in the output of the IF filter only for those values of f_o st

$$\underbrace{1 < |4 - f_o| < 2}_{\text{Satisfied for}} \quad \text{or} \quad \underbrace{1 < |4 + f_o| < 2}_{\text{This is never satisfied for } 0 < f_o < \infty}$$

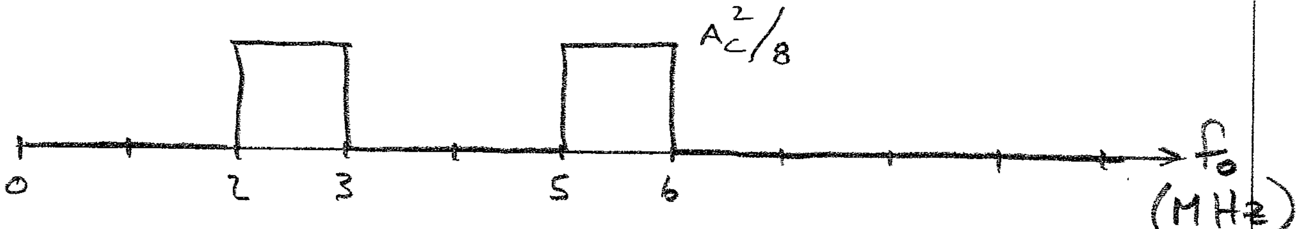
$2 < f_o < 3$

$5 < f_o < 6$

→ For f_o in these ranges

$$Y(t) = \frac{A_c}{2} \cos\left(2\pi(4 - f_o)t - \theta\right)$$

$$\therefore \text{Power } Y(t) = \frac{1}{2}\frac{A_c^2}{4} = A_c^2/8$$



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Problem 3. *SNR at Output of an FM Discriminator.* [20 pts. total]

In the class notes we derived the following expression for the SNR at the output of a discriminator demodulator

$$\mathrm{SNR}_D = K\left(\frac{B_t}{W}\right)^2\left(\frac{P_T}{N_0 W}\right).$$

In the above equation W is the baseband bandwidth of the message used to modulate the FM wave and B_t/W is the bandwidth expansion factor.

Suppose that the FCC decided to allow the maximum frequency deviation of an FM wave to be 125 kHz. The message bandwidth is $W = 15$ kHz. Using Carson's rule find the transmission bandwidth B_t. For these parameters find the factor by which demodulated SNR is improved relative to baseband transmission (express as a ratio). You may assume $K = 1$ for simplicity.

Solution follows that given in homework. For

$$\Delta f_{max} = 125 \text{ kHz}, \quad W = 15 \text{ kHz}$$

the Carson rule bandwidth would be

$$B_t = 2(125 + 15) = 2 \cdot 140 = 280 \text{ kHz}$$

$$\therefore \left(\frac{B_t}{W}\right)^2 = \left(\frac{280}{15}\right)^2 = \left(\frac{56}{3}\right)^2 \approx 248$$

This answer will do.