ECE 440 — Complex Baseband Models for Passband Communications Systems

J. V. Krogmeier

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1 Motivation

1. Certain types of demodulators (envelope detectors, discriminators) are based on the expression of a real-valued modulated signal in the form

$$x(t) = R(t)\cos[2\pi f_c t + \Psi(t)]$$

where $R(t) \geq 0$ is the envelope and $\Psi(t)$ is the phase. In particular, if $x(t)$ is input to:

   (a) an ideal envelope detector the output should be proportional to $R(t)$.

   (b) an ideal phase discriminator the output should be proportional to $\Psi(t)$.

   (c) an ideal frequency discriminator the output should be proportional to $\dot{\Psi}(t)$.

2 Complex Envelope for Deterministic Energy Signals

2. Frequency domain definition of complex envelope. Let $x(t)$ be a real-valued deterministic signal and let $x(t) \leftrightarrow X(f)$ denote a Fourier Transform pair[1]. Since the time-domain signal is real-valued, the Fourier Transform has conjugate symmetry about $f = 0$: $X(f) = X^*(-f)$. Define the *complex envelope* with respect to a carrier waveform[2] $\cos(2\pi f_c t)$ of some fixed center frequency $f_c > 0$ by [1]

$$\begin{array}{rcl} X_L(f - f_c) & = & 2u(f)X(f) \\ & \Updownarrow & \\ X_L(f) & = & 2u(f + f_c)X(f + f_c). \end{array}$$

   (a) In the above definition $u(\cdot)$ denotes the usual unit step function.

   (b) The time domain complex envelope is defined as the inverse Fourier Transform: $x_L(t) \leftrightarrow X_L(f)$.

3. Examples. The complex envelope is usually defined for passband signals as shown below by the situation on the left, but it may also be defined for the general case, as shown on the right. The passband case is the practical case. The general case is considered only for theoretical completeness.

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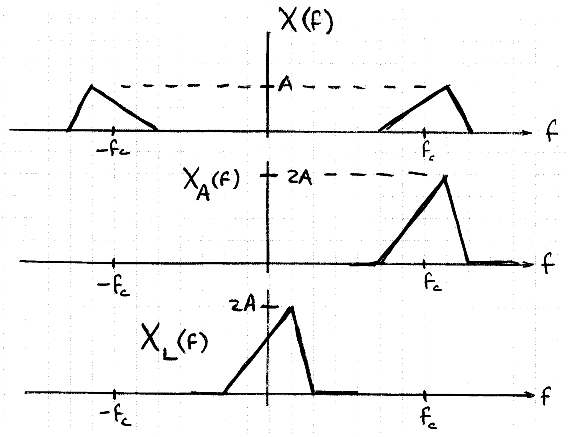
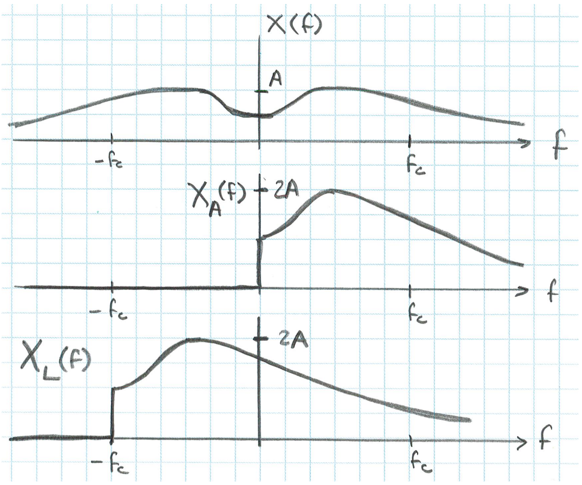
[1] Recall

$$\int X(f)e^{j2\pi ft}\mathrm{d}f = x(t) \longleftrightarrow X(f) = \int x(t)e^{-j2\pi ft}\mathrm{d}t.$$

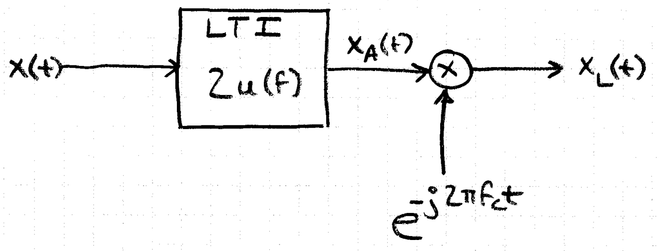
[2] We can also define the complex envelope with respect to a carrier $\cos(2\pi f_c t + \theta)$, i.e., having a phase offset, with a small modification. We will indicate the needed changes later.



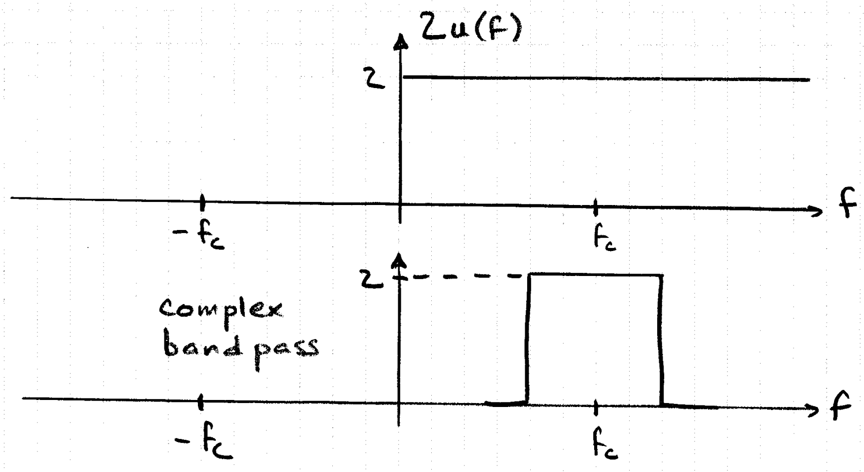


4. The definition suggests two equivalent block diagrams which generate the complex envelope as output when the real-valued signal is input.

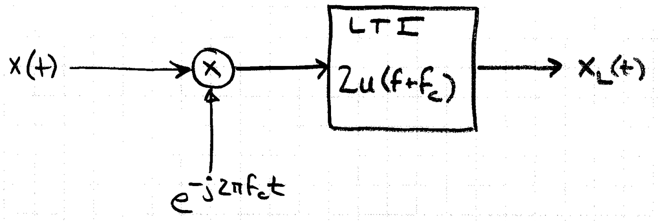
   (a) **Block Diagram 1:** (filtering followed by down-conversion)



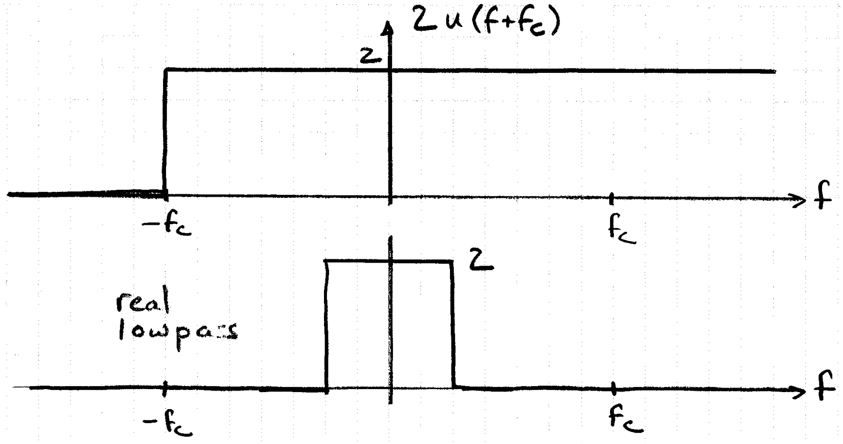
The filter $2u(f)$ can be replaced by a complex bandpass filter if $x(t)$ is bandpass as illustrated in the figure below.



   (b) **Block Diagram 2:** (down-conversion followed by filtering)



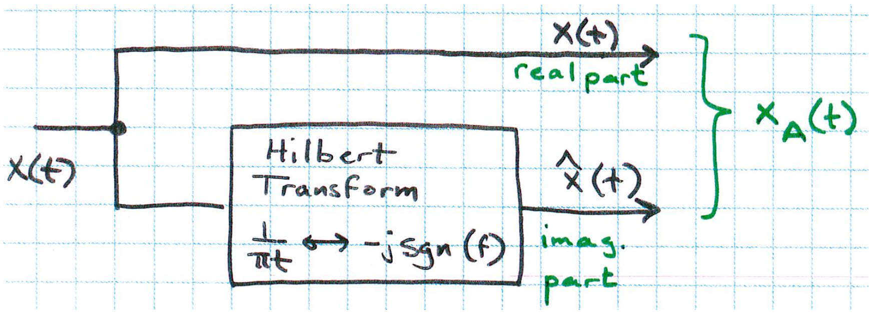
The filter $2u(f+f_c)$ can be replaced by a real lowpass filter if $x(t)$ is bandpass as illustrated in the figure below.



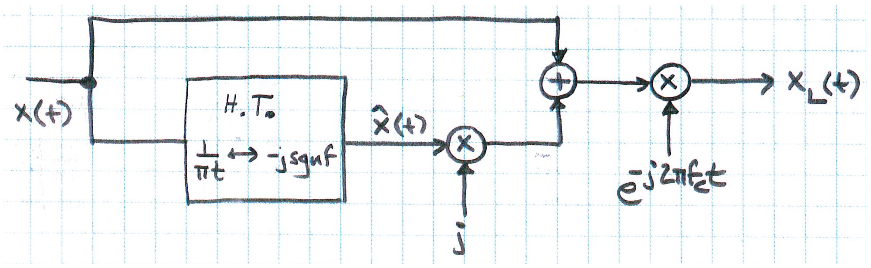
5. Time domain definition of the complex envelope. The impulse response of the filter $2u(f)$ is
$$\delta(t) + j\frac{1}{\pi t}.$$
This can be related to the notion of the *Hilbert transform.* The Hilbert transform $\hat{x}(t)$ of a real-valued signal $x(t)$ is the output of a LTI system driven by $x(t)$. The impulse response and transfer function of the Hilbert transformer are:
$$\frac{1}{\pi t} \leftrightarrow -j\mathrm{sgn}(f).$$

6. Therefore, the time domain complex envelope can also be given as
$$x_L(t) = [x(t) + j\hat{x}(t)]\, e^{-j2\pi f_c t}$$
where the real-valued signal $\hat{x}(t)$ is the Hilbert transform of $x(t)$ and $x_A(t) \overset{\text{def}}{=} (x(t) + j\hat{x}(t))$ is called the *analytic signal.* Signals having frequency content containing only positive frequencies are said to be analytic signals.

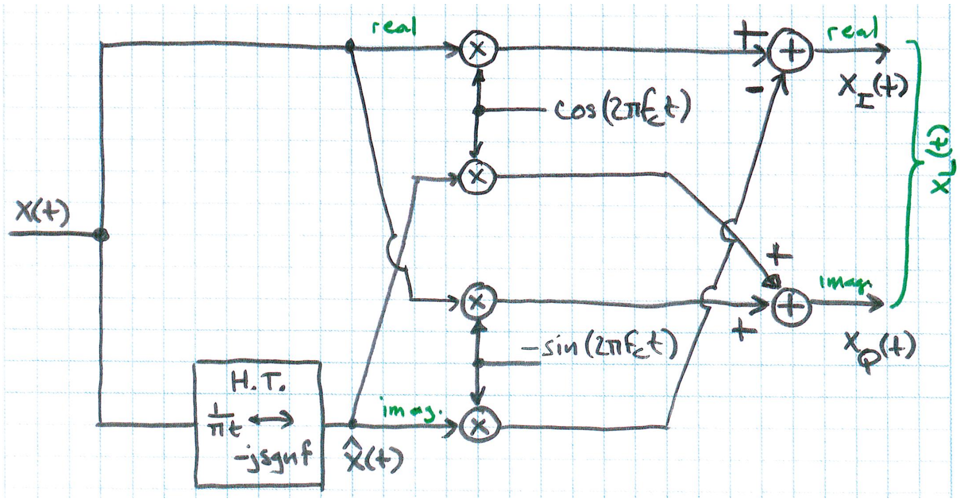
7. A time-domain block diagram generating the analytic signal from a real-valued signal input is ...



The above can be extended to generate the complex envelope signal ...



8. The previous used complex arithmetic. The equivalent real-arithmetic block diagram is ...



9. The *in-phase and quadrature components* of a real-valued signal $x(t)$ with respect to $\cos(2\pi f_c t)$ are defined to be the real and imaginary parts, respectively, of the time-domain complex envelope, i.e., $x_L(t) = x_I(t) + jx_Q(t)$.

10. We can derive a time-domain relationship between the complex envelope and the real-

valued signal using ...

$$\begin{aligned}
X(f) &= \frac{1}{2}\left\{X_L(f-f_c)+X_L^*(-f-f_c)\right\} \\
&\updownarrow \\
x(t) &= \operatorname{Re}\left\{x_L(t)e^{j2\pi f_c t}\right\} \\
&= |x_L(t)|\cos(2\pi f_c t+\angle x_L(t)) \\
&= x_I(t)\cos(2\pi f_c t)-x_Q(t)\sin(2\pi f_c t).
\end{aligned}$$

This relationship is a consequence of the steps below ...

(a) For the relationship among Fourier transforms define

$$\begin{aligned}
X_+(f) &= u(f)X(f) \\
X_-(f) &= u(-f)X(f)
\end{aligned}$$

(b) Then clear that $X_+(f)=\frac{1}{2}X_L(f-f_c)$ and $X(f)=X_+(f)+X_-(f)$.

(c) Since $x(t)$ is real-valued its Fourier Transform has conjugate symmetry and therefore $X_-(f)=X_+^*(-f)$ ...

$$\begin{aligned}
X(f) &= X_+(f)+X_+^*(-f) \\
&= \frac{1}{2}X_L(f-f_c)+\frac{1}{2}X_L^*(-f-f_c).
\end{aligned}$$

(d) The remainder of the claim follows from basic shift and conjugation properties[3] of the Fourier transform [2]. Thus, note that

$$x_L(t)e^{j2\pi f_c t}\leftrightarrow X_L(f-f_c) \quad \text{and} \quad (x_L(t)e^{j2\pi f_c t})^*\leftrightarrow X_L^*(-f-f_c)$$

and add terms up, properly scaled.

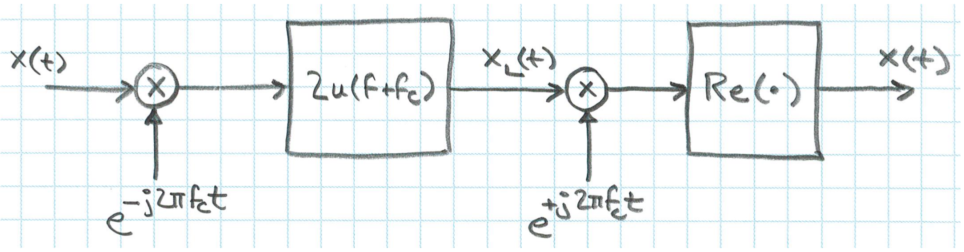
11. For a real-valued bandpass signal $x(t)$ and its complex envelope $x_L(t)$ the mapping: $x\mapsto x_L$ is implemented by a quadrature down-converter, and the mapping $x_L\mapsto x$ is implemented by a quadrature modulator. This can be seen by converting the complex filtering form of

$$x\mapsto x_L\mapsto x$$

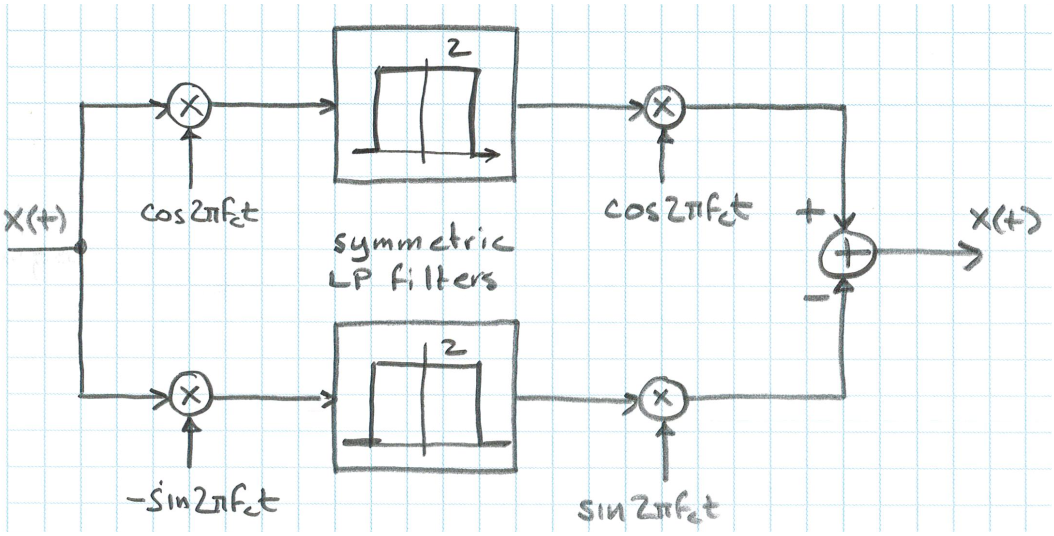
---

[3] Namely

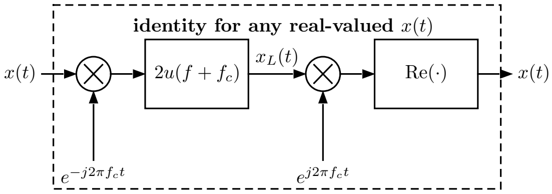
$$\begin{aligned}
g(t)\longleftrightarrow G(f) &\Leftrightarrow g^*(t)\longleftrightarrow G^*(-f) \\
&\Leftrightarrow g(t)e^{j2\pi f_c t}\longleftrightarrow G(f-f_c).
\end{aligned}$$

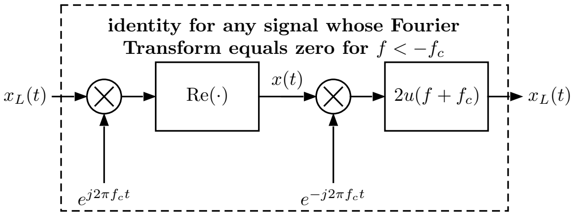


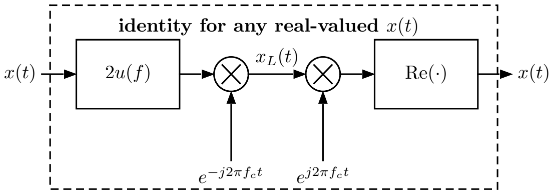
to the in-phase/quadrature form:

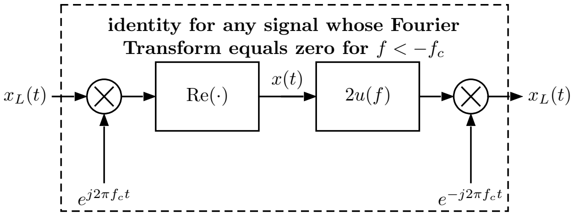


12. In fact, Item 11 leads to the following observation regarding inversion of the $x \mapsto x_L$ and $x_L \mapsto x$ operations ...









13. Let $x(t)$ be real-valued with complex envelope $x_L(t)$. Say $x_L(t) \leftrightarrow X_L(f)$, $x_I(t) = \mathrm{Re}\{x_L(t)\}$, $x_Q(t) = \mathrm{Im}\{x_L(t)\}$, $x_I(t) \leftrightarrow X_I(f)$, and $x_Q(t) \leftrightarrow X_Q(f)$. Then[4] [2]:

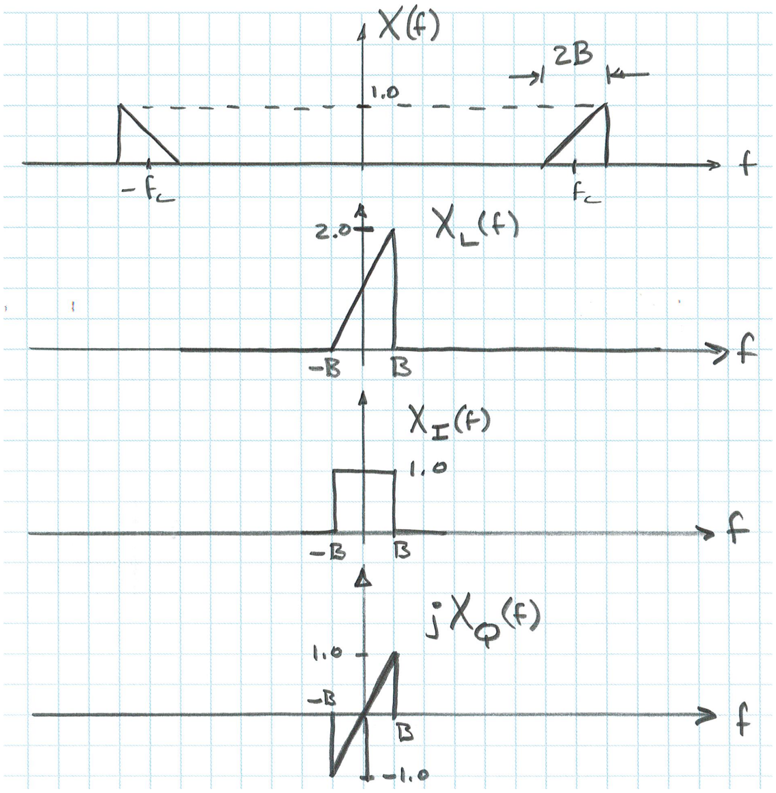
$$\begin{aligned} X_I(f) &= \left(X_L(f) + X_L^*(-f)\right)/2 \\ X_Q(f) &= \left(X_L(f) - X_L^*(-f)\right)/j2, \end{aligned}$$

---

[4] In general, $X_I(f) \neq \mathrm{Re}\{X_L(f)\}$ and $X_Q(f) \neq \mathrm{Im}\{X_L(f)\}$, in spite of the fact that $X_L(f) = X_I(f) + jX_Q(f)$. Of course, $X_I(f)$ and $X_Q(f)$ do not have to be real-valued.

which follows by taking the Fourier Transforms of $x_I(t) = (x_L(t) + x_L^*(t))/2$ and $x_Q(t) = (x_L(t) - x_L^*(t))/j2$.

14. In pictures, the above relationship is as below.



15. Energy
    (a) energy$[x_L(t)] = 2 \cdot$ energy$[x(t)]$.
    (b) energy$[x_L(t)] =$ energy$[x_I(t)] +$ energy$[x_Q(t)]$.
    (c) In general: energy$[x_I(t)] \neq$ energy$[x_Q(t)]$. In fact, possible that either $x_I(t)$ or $x_Q(t)$ equals zero.

16. Uniqueness #1. Say that $x(t)$ is real-valued and $x(t) = \mathrm{Re}\{g(t)e^{j2\pi f_c t}\}$. Then

$$g(t) = x_L(t) \iff G(f) = 0 \text{ for } f < -f_c.$$

17. Uniqueness #2. We can define the complex envelope with respect to other carrier frequencies and then compare. We will need a notation to indicate the carrier frequency used to define the complex envelope. The definition of the complex envelope of a real-valued signal $x(t)$ with respect to $\cos(2\pi f_c t)$ is defined from the analytic signal $x_A(t)$ via

$$x_{L|f_c}(t) = x_A(t)e^{-j2\pi f_c t}.$$

(a) Then easy to show:

$$\begin{aligned} x_{L|f_c}(t) &= x_{L|f_c'}(t)e^{j2\pi(f_c'-f_c)t} \\ |x_{L|f_c}(t)| &= |x_{L|f_c'}(t)| \\ \angle x_{L|f_c}(t) &= \angle x_{L|f_c'}(t) + 2\pi(f_c' - f_c)t \end{aligned}$$

(b) This means that:

  i. Output $R(t)$ of an envelope detector does not depend on the f_c in the definition of complex envelope (this makes sense).

  ii. Outputs of frequency discriminators designed for different values of f_c would differ by a dc term proportional to $2\pi(f_c' - f_c)$.

18. Amplitude scale factor in C.E. definition. The scale factor 2 was used as the gain in the definition of the complex envelope. We can easily change that and still retain the spectral characteristics, etc. So suppose the filter $2u(f)$ is changed to $\alpha u(f)$. Then:

  * $X_{A,new} \mapsto \frac{\alpha}{2}X_A$, $X_{L,new} \mapsto \frac{\alpha}{2}X_L$

  * $x_{A,new} \mapsto \frac{\alpha}{2}x_A$, $x_{L,new} \mapsto \frac{\alpha}{2}x_L$

  * The relative relationship between the C.E. and its in-phase and quadrature parts is unchanged

  * $x(t) = \text{Re}\left\{x_L(t)e^{j2\pi f_c t}\right\} \mapsto x(t) = \text{Re}\left\{\frac{2}{\alpha}x_{L,new}(t)e^{j2\pi f_c t}\right\}$

  * $\text{energy}[x_{L,new}(t)] = \frac{\alpha^2}{4}\text{energy}[x_L(t)]$

  * $\text{energy}[x_{L,new}(t)] = \frac{\alpha^2}{2} \cdot \text{energy}[x(t)]$

A common choice of scale factor is $\alpha = \sqrt{2}$ for then the real signal and its complex envelope have equal energy. The basic relationship would then be written

$$x(t) = \text{Re}\left\{\sqrt{2}x_{L,new}(t)e^{j2\pi f_c t}\right\}$$

meaning, in effect, that the amplitude units of $x_{L,new}$ are RMS units in analogy to the usual treatment of power in sinusoidal steady state circuits.

3 Complex Envelope and Linear Time Invariant Filtering

19. Would like to relate the complex envelopes of the input and output of a LTI system. Suppose that

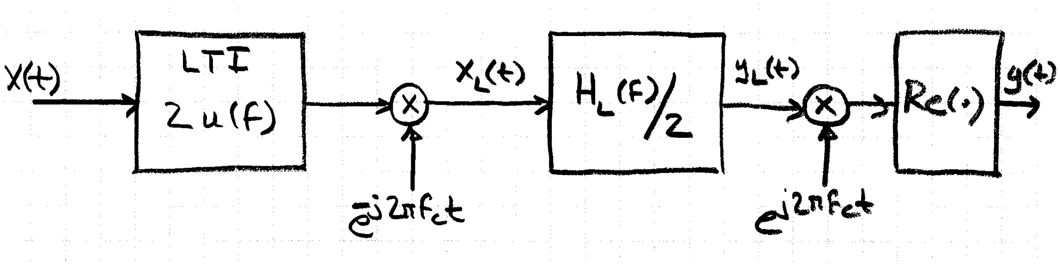
$$y(t) = [h * x](t) \longleftrightarrow Y(f) = H(f)X(f)$$

where the input signal x, the impulse response h, and the output y are real-valued.

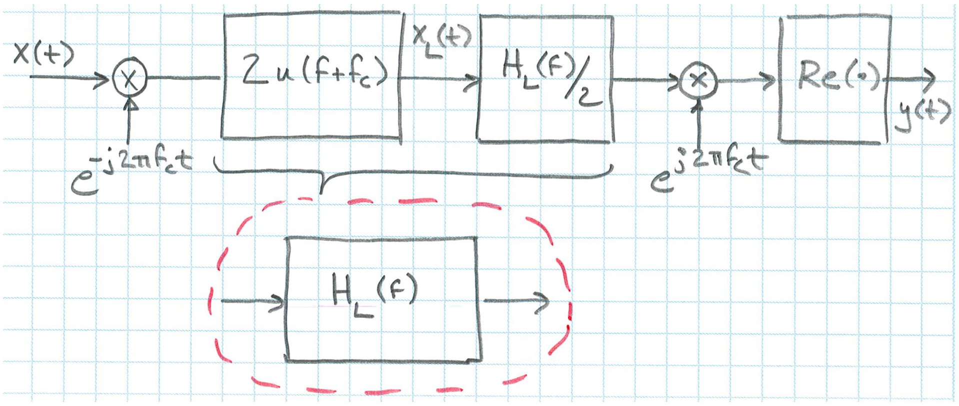
20. From the definition

$$\begin{aligned} Y_L(f) &= 2u(f+f_c)Y(f+f_c) \\ &= \frac{1}{2}[2u(f+f_c)H(f+f_c)][2u(f+f_c)X(f+f_c)] \\ &\text{(using } Y = HX \text{ and } u^2 = u\text{)} \\ &= \frac{1}{2}H_L(f)X_L(f) \longleftrightarrow y_L(t) = \frac{1}{2}[h_L * x_L](t). \end{aligned}$$

21. Thus the real filtering operation $y(t) = [h * x](t)$ can be implemented via the block diagram below. This amounts to the implementation of a bandpass filtering operation by downconversion followed by lowpass filtering and upconversion.



22. The previous can be redrawn[5] as shown ...



and then redrawn in I/Q form as ...

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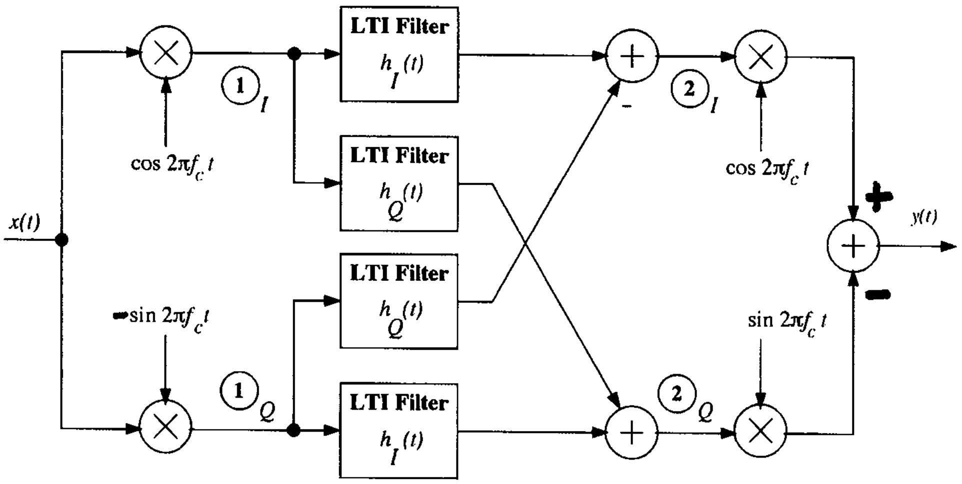
[5] By replacing Block Diagram 4a in the dashed box with Block Diagram 4b and using the fact that $u(f+f_c)H_L(f) = H_L(f)$.



4 Complex Envelope for Wide-Sense Stationary Random Signals

23. Let $X(t)$ be a real-valued WSS random process. Define the complex envelope with respect to a carrier $\cos(2\pi f_c t)$ in the same way as was done for deterministic signals [1]. In otherwords, we let $X(t)$ be input to either Block Diagram 4a or Block Diagram 4b and define the complex envelope $X_L(t)$ to be the random process output.

    Up to now all of our random processes were real-valued and we will have to introduce some extra notation to handle the complex-valued case (see Appendix B).

    (a) When the spectral density $S_X(f)$ is non-zero only for frequencies f in

    $$\{f : ||f| - f_c| < B\}$$

    ($f_c > B$) then the filter in Block Diagram 4a may be replaced by a complex bandpass filter and the filter in Block Diagram 4b may be replaced by a real lowpass filter as was indicated there.

    (b) Since X_A is the output of a LTI system driven by WSS X it must also be WSS.

    (c) But $X_L(t) = X_A(t)e^{-j2\pi f_c t}$ which is not immediately seen to be WSS because it is the product of a time function and a WSS random process. We do have

    $$\begin{aligned} m_{X_L}(t) &= m_{X_A}(t)e^{-j2\pi f_c t} \\ R_{X_L}(t, t+u) &= R_{X_A}(t, t+u)e^{-j2\pi f_c u} \\ \tilde{R}_{X_L}(t, t+u) &= \tilde{R}_{X_A}(t, t+u)e^{-j2\pi(2f_c)t}e^{-j2\pi f_c u}. \end{aligned} \quad (1)$$

    Clearly, $\tilde{R}_{X_L}(t, t+u)$ will be a function of u alone if and only if it is equal to zero. Similarly for the mean.

24. If we look in the frequency domain and use standard LTI filtering formulas

$$\begin{aligned} m_{X_A} &= \left\{ 2u(f)|_{f=0} \right\} \cdot m_X \\ S_{X_A}(f) &= |2u(f)|^2 S_X(f) \;=\; 4u(f)S_X(f) \\ \tilde{S}_{X_A}(f) &= 4u(f)u(-f)S_X(f) \;=\; 0. \end{aligned}$$

Therefore, $\tilde{R}_{X_A}(t, t+u) = 0$ for all t, u, which implies the same for $\tilde{R}_{X_L}(t, t+u)$. Finally, if $X(t)$ is zero mean then so are X_A and X_L. We conclude that the complex envelope X_L is WSS under the assumption of zero mean[6]. At this point we will use WSS notation for the correlation functions and write them as functions of only the lag variable.

(a) In this case, we would write[7]

$$\begin{aligned} R_{X_L}(u) &= R_{X_A}(u)e^{-j2\pi f_c u} \\ &\updownarrow \\ S_{X_L}(f) &= S_{X_A}(f + f_c) \\ &= 4u(f + f_c)S_X(f + f_c). \end{aligned}$$

(b) From integrating psds we have

$$\mathrm{E}\{X^2(t)\} = 0.5\mathrm{E}\{|X_A(t)|^2\} = 0.5\mathrm{E}\{|X_L(t)|^2\},$$

i.e., the power in the complex envelope is twice the power in the original real-valued signal.

25. The in-phase and quadrature components of the complex envelope are defined ...

$$X_I(t) = \mathrm{Re}X_L(t) = \frac{X_L(t) + X_L^*(t)}{2} \tag{2}$$

$$X_Q(t) = \mathrm{Im}X_L(t) = \frac{X_L(t) - X_L^*(t)}{j2}. \tag{3}$$

26. Then $X_L(t) = X_I(t) + jX_Q(t)$, which, when substituted in $X(t) = \mathrm{Re}\{X_L(t)e^{j2\pi f_c t}\}$ yields the familiar expansion ...

$$X(t) = X_I(t)\cos(2\pi f_c t) - X_Q(t)\sin(2\pi f_c t).$$

27. It will follow that X_I and X_Q are zero-mean, WSS, real-valued random processes. We can compute the correlations R_{X_I}, R_{X_Q}, and $R_{X_I X_Q}$ using the expressions in Eqs. (2) and (3) and the definitions of the correlations. In fact,

$$R_{X_I}(u) = R_{X_Q}(u) = 0.5 \cdot \mathrm{Re}\{R_{X_L}(u)\}.$$

---

[6] Note: if the mean of the real-valued process $X(t)$ is zero then both complex processes $X_A(t)$ and $X_L(t)$ have complementary covariances $\tilde{C}_{X_A}(t,s) = \tilde{C}_{X_L}(t,s) = 0$ for all t, s. Complex processes with this property were called *proper* complex processes in [3].

[7] By the way $S_{X_L}(f) = 2 \cdot 2u(f + f_c)S_X(f + f_c)$ shows that $R_{X_L}(u)$ is 2 times the complex envelope of $R_X(u)$, considered as a deterministic real-valued signal.

For the cross-correlation we would start with

$$\left(\frac{X_L(t)+X_L^*(t)}{2}\right)\left(\frac{X_L(t+u)-X_L^*(t+u)}{j2}\right)$$

expand, take the expectation, and simplify. The resulting expression is ...

$$R_{X_I X_Q}(u) = 0.5 \cdot \mathrm{Im}\{R_{X_L}(u)\}.$$

28. Summarizing:

    $$R_{X_L}(u) = 2R_{X_I}(u) + j2R_{X_I X_Q}(u).$$

    Thus, the auto-correlation of X_L will be complex-valued in the general case.

29. However, the cross-correlation is purely odd, i.e.,

    $$R_{X_I X_Q}(u) = -R_{X_I X_Q}(-u)$$

    $\forall u \in \mathcal{R}$. As a consequence $R_{X_I X_Q}(0) = 0$, which is $\Leftrightarrow$ $X_I(t)$ and $X_Q(t)$ are uncorrelated when sampled at the same time instant.

30. Define Fourier Transforms associated with the correlation functions above

    $$\begin{array}{ccc} R_{X_I}(u) & \leftrightarrow & S_{X_I}(f) \\ \shortparallel & & \shortparallel \\ R_{X_Q}(u) & \leftrightarrow & S_{X_Q}(f) \end{array}$$

    and

    $$R_{X_I X_Q}(u) \leftrightarrow S_{X_I X_Q}(f).$$

    (a) We can easily show that $R_{X_I} = R_{X_Q}$ are real and even. Since they are auto-correlation functions, they are positive semi-definite[8]. Equivalently, $S_{X_I} = S_{X_Q}$ are real, positive, and even.

    (b) In a similar way we can show that $R_{X_I X_Q}$ is real and odd, which is equivalent to $S_{X_I X_Q}$ being purely imaginary and odd.

    (c) Therefore:

    $$\begin{array}{rcccc} 0.5 \cdot \mathrm{Re}\{R_{X_L}(u)\} & = & R_{X_I}(u) & = & R_{X_Q}(u) \\ & \updownarrow & & & \\ \dfrac{S_{X_L}(f)+S_{X_L}(-f)}{4} & = & S_{X_I}(f) & = & S_{X_Q}(f). \end{array}$$

    (d) For the cross-correlation:

    $$\begin{array}{rcccc} 0.5 \cdot \mathrm{Im}\{R_{X_L}(u)\} & = & R_{X_I X_Q}(u) & = & -R_{X_I X_Q}(-u) \\ & \updownarrow & & & \\ \dfrac{S_{X_L}(f)-S_{X_L}(-f)}{j4} & = & S_{X_I X_Q}(f) & = & -S^*_{X_I X_Q}(f). \end{array}$$

---

[8] For our purposes, a positive semi-definite function of a time variable is one whose Fourier Transform is real-valued and positive (or zero) for all frequencies.

(e) A statement equivalent to the above is ...

$$\begin{aligned} \mathrm{Even}\{S_{X_L}(f)\} &= \frac{S_{X_L}(f)+S_{X_L}(-f)}{2} &= 2S_{X_I}(f) \\ \mathrm{Odd}\{S_{X_L}(f)\} &= \frac{S_{X_L}(f)-S_{X_L}(-f)}{2} &= j2S_{X_IX_Q}(f). \end{aligned}$$

(f) In terms of the original spectrum:

$$\begin{aligned} S_{X_I}(f) &= u(f+f_c)S_X(f+f_c)+u(-f+f_c)S_X(-f+f_c) \\ S_{X_IX_Q}(f) &= j\left[u(-f+f_c)S_X(-f+f_c)-u(f+f_c)S_X(f+f_c)\right] \end{aligned}$$

31. The representation $X(t)=\mathrm{Re}\{X_L(t)e^{j2\pi f_c t}\}=X_I(t)\cos(2\pi f_c t)-X_Q(t)\sin(2\pi f_c t)$ has an analog for the correlation functions.

    (a) Upon inserting the expression above into the definition for $R_X(u)$ and using the properties $R_{X_I}(u)=R_{X_Q}(u)$ and $R_{X_IX_Q}(u)=-R_{X_QX_I}(u)$ we get:

    $$R_X(u)=R_{X_I}(u)\cos(2\pi f_c u)-R_{X_IX_Q}(u)\sin(2\pi f_c u).$$

    (b) Also have

    $$R_X(u)=\mathrm{Re}\{(R_{X_I}(u)+jR_{X_IX_Q}(u))\exp(j2\pi f_c u)\}=\mathrm{Re}\{0.5\cdot R_{X_L}(u)\exp(j2\pi f_c u)\}.$$

    Therefore $0.5\cdot R_{X_L}(u)$ is the complex envelope of $R_X(u)$ w.r.t. $\cos(2\pi f_c u)$.

32. Important Special Case: X is WSS of zero mean and symmetric bandpass.

    (a) Suppose, in addition to the standing hypotheses, that X is symmetric bandpass in the sense that

    $$u(f_c+f)S_X(f_c+f)=u(f_c-f)S_X(f_c-f).$$

    (b) Then:

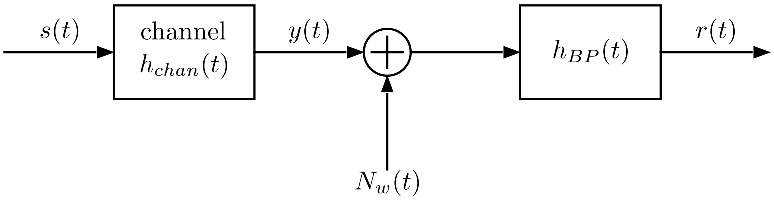
    - $S_{X_L}(f)$ is even whence $R_{X_L}(u)$ is real-valued.
    - $S_{X_I}(f)=S_{X_Q}(f)=S_{X_L}(f)/2=u(f+f_c)S_X(f+f_c)$.
    - $S_{X_IX_Q}(f)=0$ and consequently $R_{X_IX_Q}(u)=0$. That is, $X_I(t)$ and $X_Q(t+u)$ are uncorrelated for any value of u.

5 Passband Communications

5.1 Channel Models

33. The vast majority of communication systems are *passband systems.* This means that the transmitted information bearing signal $s(t)$ has its energy restricted to a band of frequencies located around some nominal carrier frequency and above and relatively far away from dc (baseband).

34. A simple channel model is as shown below:



35. Here:

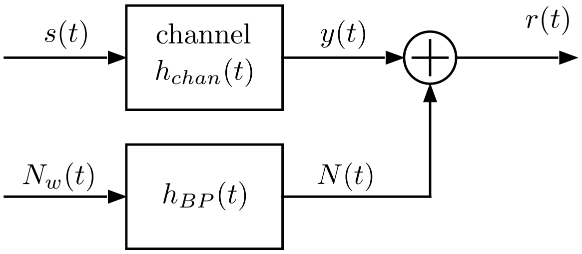
    (a) The information bearing signal is $s(t)$ and the received signal is $r(t)$, the signal that the receiver must use to estimate the desired message information. Typically, $s(t)$ has energy restricted to a known frequency band.

    (b) The receiver channel filter is represented by $h_{BP}(t) \leftrightarrow H_{BP}(f)$ which is assumed to have a bandpass characteristic just sufficient to pass $s(t)$ without distortion.

    (c) The communication channel itself is represented by some filtering $h_{chan}(t) \leftrightarrow H_{chan}(f)$ and the addition of a white Gaussian noise denoted by $N_w(t)$.

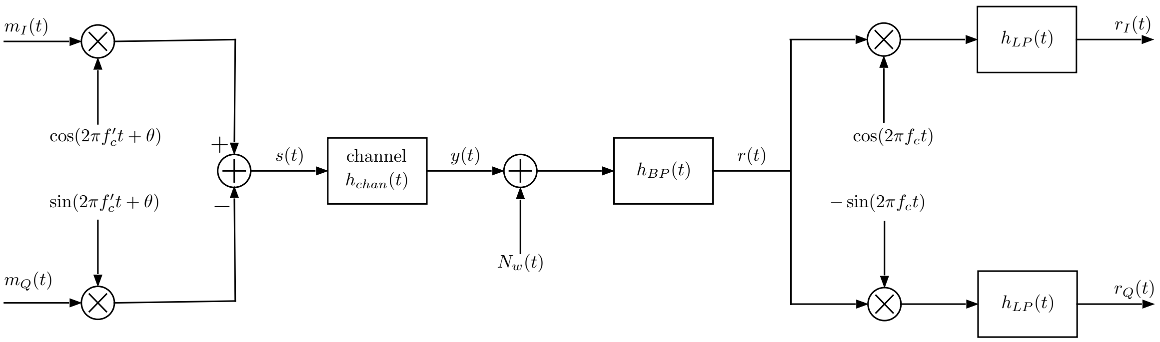
    (d) The channel filter is here indicated as LTI but it could be generalized. Note that if it is LTI then it is no loss of generality to consider its passband to be fully contained in that of $h_{BP}(t) \leftrightarrow H_{BP}(f)$. In such cases the BP filter will have no effect on the signal part so it could be dropped from the signal path.

    (e) Hence, if we may assume $h_{chan} * h_{BP} = h_{chan}$, then the simple channel model above is equivalent to:

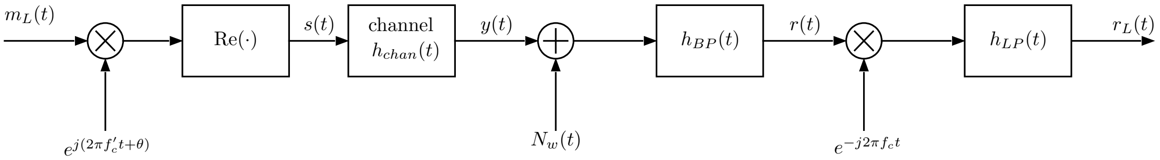


36. If we assume that the information bearing signal is created using a so-called quadrature amplitude modulator (QAM) and that a QAM demodulator is used to recover it, then the standard passband modulator-channel-demodulator is as shown in the figure below. All signals and impulse responses are real-valued.

37. Note that the model allows for the possibility of frequency and phase offsets between transmitter and receiver, i.e., it is possible that $f_c' \neq f_c$ and/or $\theta \neq 0$.



38. The above passband modulator-channel-demodulator can be drawn more compactly by using complex notation as shown in the figure below where $m_L(t) = m_I(t) + jm_Q(t)$ and $r_L(t) = r_I(t) + jr_Q(t)$. Note that the inputs and outputs of $h_{chan}(t)$ and $h_{BP}(t)$ are still real-valued although the input to $h_{LP}(t)$ is now complex. However, $h_{LP}(t)$ is a real-valued impulse response so it operates on the real and imaginary parts of its input without "crosstalk."



39. A complex baseband model for the passband system is a way to directly compute the output $r_L(t)$ from the input $m_L(t)$ using baseband operations, i.e., baseband LTI filtering and/or frequency conversions with small (relative to baseband) center frequencies. This has several benefits:

    (a) A baseband model is a simpler model.

    (b) A baseband model can be numerically simulated with much lower computation than can a passband model because the sampling rate is much lower.

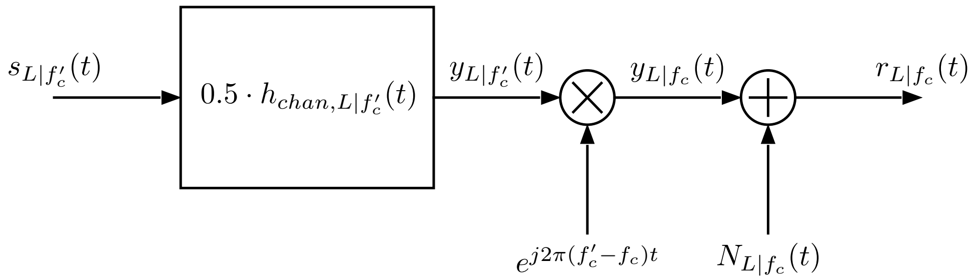
    (c) A baseband model can form the basis for a discrete-time implementation of a bandpass communications system.

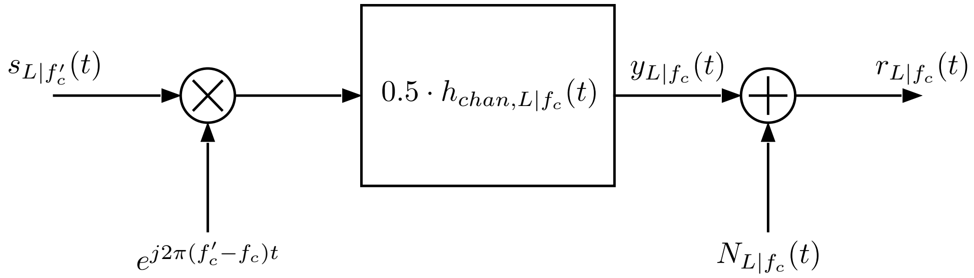
40. Assuming that $m_I(t)$ and $m_Q(t)$ are baseband waveforms with narrow bandwidths in comparision to either f_c' of f_c (which are close to equal) then

$$\begin{aligned} s(t) &= \mathrm{Re}\left\{(m_I(t) + jm_Q(t))e^{j\theta}e^{j2\pi f_c' t}\right\} \\ r(t) &= \mathrm{Re}\left\{(r_I(t) + jr_Q(t))e^{j2\pi f_c t}\right\} \end{aligned}$$

which means that $s_{L|f_c'}(t) = m_L(t)e^{j\theta}$ is the complex envelope of $s(t)$ with respect to the carrier $e^{j2\pi f_c' t}$ and $r_{L|f_c}(t)$ is the complex envelope of $r(t)$ with respect to the carrier $e^{j2\pi f_c t}$.

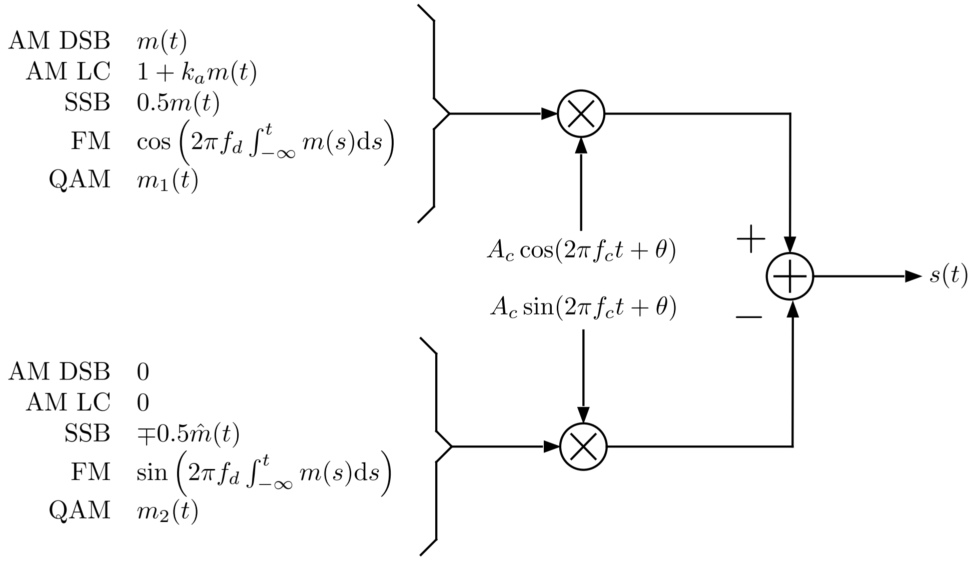
41. We can show that the equivalent complex baseband model can be written in two equivalent ways based on the assumptions above (see Appendix C) ...





5.2 Information Bearing Signals

42. We have studied five modulations to this point. When put in the QAM modulator format, they can be summarized in the figure below ...



43. The message waveforms are $m(t)$, $m_1(t)$, and $m_2(t)$, which are modeled as real-valued WSS random processes statistically independent of each other. We assume correlation functions and power spectral densities

$$R_m(\tau) \leftrightarrow S_m(f)$$

$$R_{m_1}(\tau) \leftrightarrow S_{m_1}(f) \text{ and } R_{m_2}(\tau) \leftrightarrow S_{m_2}(f).$$

In order to have a WSS model for the transmitted bandpass signal $s(t)$ we include a random phase Θ in the model, which is assumed uniform over the interval $[0, 2\pi)$ and statistically independent of all other random variables and processes.

44. Summary of the information bearing signals:

    (a) <u>AM DSB.</u>

$$\begin{aligned}
s(t) &= A_c m(t)\cos(2\pi f_c t + \Theta) \\
s_L(t) &= A_c m(t) e^{j\Theta} \\
R_s(\tau) &= 0.5A_c^2 R_m(\tau)\cos 2\pi f_c \tau \\
&\updownarrow \\
S_s(f) &= 0.25A_c^2\,[S_m(f-f_c) + S_m(f+f_c)] \\
R_{s_L}(\tau) &= A_c^2 R_m(\tau) \\
&\updownarrow \\
S_{s_L}(f) &= A_c^2 S_m(f)
\end{aligned}$$

    (b) <u>AM LC.</u>

$$\begin{aligned}
s(t) &= A_c[1 + k_a m(t)]\cos(2\pi f_c t + \Theta) \\
s_L(t) &= A_c[1 + k_a m(t)] e^{j\Theta} \\
R_s(\tau) &= 0.5A_c^2[1 + k_a^2 R_m(\tau)]\cos 2\pi f_c \tau \\
&\updownarrow \\
S_s(f) &= 0.25A_c^2\,[\delta(f-f_c) + \delta(f+f_c)] \\
&\quad + 0.25k_a^2 A_c^2\,[S_m(f-f_c) + S_m(f+f_c)] \\
R_{s_L}(\tau) &= A_c^2[1 + k_a^2 R_m(\tau)] \\
&\updownarrow \\
S_{s_L}(f) &= A_c^2[\delta(f) + k_a^2 S_m(f)]
\end{aligned}$$

    (c) <u>AM SSB.</u> (top choice of sign is SSB-lower, bottom choice of sign is SSB-upper)

$$\begin{aligned}
s(t) &= 0.5A_c m(t)\cos(2\pi f_c t + \Theta) \pm 0.5A_c \hat{m}(t)\sin(2\pi f_c t + \Theta) \\
s_L(t) &= 0.5A_c[m(t) \mp j\hat{m}(t)] e^{j\Theta} \\
R_s(\tau) &= 0.25A_c^2[R_m(\tau)\cos 2\pi f_c \tau \pm \hat{R}_m(\tau)\sin 2\pi f_c \tau] \\
&\updownarrow \\
S_s(f) &= 0.125A_c^2\,[(1 \mp \mathrm{sgn}(f-f_c))S_m(f-f_c) + (1 \pm \mathrm{sgn}(f+f_c))S_m(f+f_c)] \\
R_{s_L}(\tau) &= 0.5A_c^2[R_m(\tau) \mp j\hat{R}_m(\tau)] \\
&\updownarrow \\
S_{s_L}(f) &= 0.5A_c^2 S_m(f)[1 \mp \mathrm{sgn}(f)]
\end{aligned}$$

(d) FM. Let $\phi(t) = 2\pi f_d \int_{-\infty}^{t} m(s)\mathrm{d}s$. Then

$$\begin{aligned} s(t) &= A_c \cos\left(2\pi f_c t + \phi(t) + \Theta\right) \\ s_L(t) &= A_c e^{j\phi(t)} e^{j\Theta} \end{aligned}$$

The power in $s(t)$ is $0.5A_c^2$ and the transmission BW is $2(\Delta_f + W)$ even though we don't have formulas for correlations or spectra except in the case of sinusoidal modulations.

(e) QAM.

$$\begin{aligned} s(t) &= A_c m_1(t)\cos(2\pi f_c t + \Theta) + A_c m_2(t)\sin(2\pi f_c t + \Theta) \\ s_L(t) &= A_c[m_1(t) + jm_2(t)]e^{j\Theta} \\ R_s(\tau) &= 0.5A_c^2[R_{m_1}(\tau) + R_{m_2}(\tau)]\cos 2\pi f_c \tau \\ &\updownarrow \\ S_s(f) &= 0.25A_c^2\left[(S_{m_1}(f-f_c) + S_{m_1}(f+f_c) + S_{m_2}(f-f_c) + S_{m_2}(f+f_c)\right] \\ R_{s_L}(\tau) &= A_c^2[R_{m_1}(\tau) + R_{m_2}(\tau)] \\ &\updownarrow \\ S_{s_L}(f) &= A_c^2[S_{m_1}(f) + S_{m_2}(f)] \end{aligned}$$

5.3 Complex White Gaussian Noise

45. The complex baseband model shown in Item 41 uses $N_{L|f_c}(t)$ as its noise input. This is the complex envelope of the real-valued bandpass noise $N(t)$ from the real-valued bandpass model. On the other hand, the noise input to the final block diagram in Figure 9 is the complex envelope of a white noise.

5.3.1 Complex Envelope of White Gaussian Noise

46. However, the complex envelope of real-valued white noise is not white. This is demonstrated now.

(a) Let $W(t)$ be real-valued additive white Gaussian noise (AWGN). Thus $\mathrm{E}W(t) = 0$ and

$$R_W(\tau) = \frac{N_0}{2}\delta(\tau) \leftrightarrow S_W(f) = \frac{N_0}{2} \quad \forall f \in \mathcal{R}.$$

(b) Following the previous development we can easily show that $S_{W_L}(f) = 4u(f + f_c)N_0/2$ which is equal to

$$S_{W_L}(f) = \begin{cases} 2N_0 & \text{for } f > -f_c \\ 0 & \text{for } f < -f_c \end{cases}$$

which is not constant for all f and by definition, not white.

(c) Solving for the auto-correlation we find

$$R_{W_L}(\tau) = \left[N_0\delta(\tau) + \frac{N_0}{\pi\tau}\sin(2\pi f_c\tau)\right] + j\frac{N_0}{\pi\tau}\cos(2\pi f_c\tau).$$

(d) Following the standard development we can also find the correlations of the in-phase and quadrature components $W_I(t)$ and $W_Q(t)$ ...

$$\begin{aligned} R_{W_I}(\tau) &= R_{W_Q}(\tau) = \frac{N_0}{2}\delta(\tau) + \frac{N_0}{2}\frac{\sin(2\pi f_c\tau)}{\pi\tau} \\ R_{W_IW_Q}(\tau) &= \frac{N_0}{2}\frac{\cos(2\pi f_c\tau)}{\pi\tau} \end{aligned}$$

meaning the in-phase and quadrature components are also non-white and statistically dependent.

5.3.2 Complex Envelope of Bandpass-Filtered White Gaussian Noise

47. Instead of starting with white noise, use $N(t)$ the bandpass filtered noise with an ideal bandpass filter of gain one, centered at f_c, and symmetric about f_c, i.e., $N(t)$ has psd

$$S_N(f) = \begin{cases} N_0/2 & \text{for } ||f| - f_c| < B \\ 0 & \text{else} \end{cases} .$$

Here we presume the bandwidth B is much wider than the minimum needed. Receivers will likely use filters to further reduce the noise power.

48. By following the standard development we can compute N_L, N_I, N_Q and all correlations and spectra. We would have $S_{N_L}(f) = 4u(f+f_c)S_N(f+f_c)$, ...

$$S_{N_L}(f) = \begin{cases} 2N_0 & \text{for } |f| < B \\ 0 & \text{else} \end{cases}$$

which is symmetric about $f = 0$. Because of the symmetry it follows that $R_{N_IN_Q}(\tau) = 0$ for all τ. In addition, $S_{N_I}(f) = S_{N_Q}(f) = 0.5 \cdot S_{N_L}(f)$. From a table of Fourier Transforms

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = 0.5 \cdot R_{N_L}(\tau) = 2N_0B\frac{\sin(2\pi B\tau)}{2\pi B\tau}.$$

49. Then we could use this complex noise as the input to the block diagrams in Item 41.

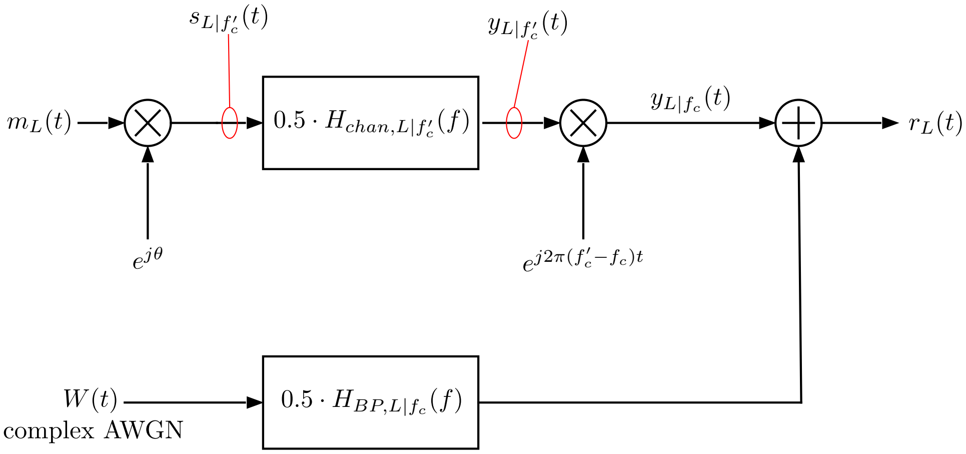
50. However, this lowpass complex noise can be replaced by a complex white noise process found by taking the limit as $B \to \infty$ in the previous model and all calculations of probability or power would be the same. Since, this last model is so simple, it is preferred. This noise is defined by $W(t) = W_I(t) + jW_Q(t)$ where[9]

$$R_W(\tau) = 2N_0\delta(\tau) \quad \text{and} \quad \tilde{R}_W(\tau) = 0.$$

---

[9] Also have $R_{W_I}(\tau) = R_{W_Q}(\tau) = N_0\delta(\tau)$ and $R_{W_IW_Q}(\tau) = 0$.

This gives the final and simplest model ...



A Important Fourier Transform Properties

51. Symmetries associated with the continuous-time Fourier Transform are at the heart of the complex envelope representation. Therefore, we review and summarize these here. See Figures 2 and 3 to review more generally. Let $g(t) \leftrightarrow G(f)$ denote a generic FT pair, where either time-domain or frequency-domain signal can be complex-valued ...

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} \mathrm{d}f \\ &\updownarrow \\ G(f) &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} \mathrm{d}t. \end{aligned}$$

52. The properties of interest for signals in either domain are:

    - Purely real. Time domain: $g(t) = g^*(t)$ for all $t \in \mathcal{R}$. Frequency domain: $G(f) = G^*(f)$ for all $f \in \mathcal{R}$.
    - Purely imaginary. Time domain: $g(t) = -g^*(t)$ for all $t \in \mathcal{R}$. Frequency domain: $G(f) = -G^*(f)$ for all $f \in \mathcal{R}$.
    - Even (possibly complex). Time domain: $g(t) = g(-t)$ for all $t \in \mathcal{R}$. Frequency domain: $G(f) = G(-f)$ for all $f \in \mathcal{R}$.
    - Odd (possibly complex). Time domain: $g(t) = -g(-t)$ for all $t \in \mathcal{R}$. Frequency domain: $G(f) = -G(-f)$ for all $f \in \mathcal{R}$.
    - Even-conjugate symmetric about the origin. Time domain: $g(t) = g^*(-t)$ for all $t \in \mathcal{R}$. Frequency domain: $G(f) = G^*(-f)$ for all $f \in \mathcal{R}$.
    - Odd-conjugate symmetric about the origin. Time domain: $g(t) = -g^*(-t)$ for all $t \in \mathcal{R}$. Frequency domain: $G(f) = -G^*(-f)$ for all $f \in \mathcal{R}$.

53. Of interest is how a particular property in the time domain is manifest in the frequency domain, and vice-versa. Note also that these are not necessarily independent. For example, if a signal is purely real and even, then it must also be even-conjugate symmetric about the origin. The symmetry properties are summarized in Figure 1.

54. The proofs are simple and left to the reader. But a few related observations are given below. Note also that the duality evident in the definition of time and frequency domain Fourier Transforms ensures that the properties checked in Figure 1 have to appear in a symmetric fashion.

    (a) $x(t)$ is purely real $\Leftrightarrow X(f) = X^*(-f)$, $\forall f \in \mathcal{R}$. This is well known, of course. But it might be useful to recall that the even-conjugate symmetry of the Fourier Transform is equivalent to the following ...

$$\Leftrightarrow \left\{ \begin{array}{rcl} |X(f)| &=& |X(-f)| \\ \text{and} && \\ \angle X(f) &=& -\angle X(-f) \end{array} \quad \forall f \in \mathcal{R} \right\} \Leftrightarrow \left\{ \begin{array}{rcl} \mathrm{Re}X(f) &=& \mathrm{Re}X(-f) \\ \text{and} && \\ \mathrm{Im}X(f) &=& -\mathrm{Im}X(-f) \end{array} \quad \forall f \in \mathcal{R} \right\}$$

Freq. Domain Properties of:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\mathrm{d}t$$

Time Domain Properties of:

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\mathrm{d}f$$

	pure real	pure imag.	even	odd	$X(f) = X^*(-f)$	$X(f) = -X^*(-f)$	pure real, even, $X(f) = X^*(-f)$	pure imag., odd, $X(f) = X^*(-f)$	pure real, odd, $X(f) = -X^*(-f)$	pure imag., even, $X(f) = -X^*(-f)$
pure real					✗					
pure imag.						✗				
even			✗							
odd				✗						
$x(t) = x^*(-t)$	✗									
$x(t) = -x^*(-t)$		✗								
pure real, even, $x(t) = x^*(-t)$							✗			
pure imag., odd, $x(t) = x^*(-t)$									✗	
pure real, odd, $x(t) = -x^*(-t)$								✗		
pure imag., even, $x(t) = -x^*(-t)$										✗

Figure 1: Symmetry Properties of Fourier Transforms.

(b) $x(t)$ is purely imag. $\Leftrightarrow X(f) = -X^*(-f), \forall f \in \mathcal{R}$. Recall that the odd-conjugate symmetry of the Fourier Transform is equivalent to the following ...

$$\Leftrightarrow \left\{ \begin{array}{ccc} |X(f)| & = & |X(-f)| \\ \text{and} & & \\ \angle X(f) & = & \angle X(-f) \end{array} \quad \forall f \in \mathcal{R} \right\} \Leftrightarrow \left\{ \begin{array}{ccc} \mathrm{Re}X(f) & = & -\mathrm{Re}X(-f) \\ \text{and} & & \\ \mathrm{Im}X(f) & = & \mathrm{Im}X(-f) \end{array} \quad \forall f \in \mathcal{R} \right\}$$

Table XIII.2—Properties of Fourier Transforms

	$x(t)=\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$	$X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$
Conjugate Symmetry	$\Im m[x(t)]=0$ (i.e., $x(t)$ is real)	$X(f)=X^*(-f)$ (i.e., $\Re e[X(f)]=\Re e[X(-f)]$, $\Im m[X(f)]=-\Im m[X(-f)]$)
Even Symmetry	$x(t)=x(-t)$	$X(f)=X(-f)$
Odd Symmetry	$x(t)=-x(-t)$	$X(f)=-X(-f)$
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(f)+bX_2(f)$
Duality	$X(t)$	$x(-f)$
Scale Change	$x(at)$	$\frac{1}{\lvert a\rvert}X(f/a)$
Time Delay	$x(t-t_0)$	$e^{-j2\pi ft_0}X(f)$
Times $e^{j2\pi f_0 t}$	$e^{j2\pi f_0 t}x(t)$	$X(f-f_0)$
Differentiation	$\frac{dx(t)}{dt}$	$j2\pi fX(f)$
Times t	$tx(t)$	$\frac{1}{-j2\pi}\frac{dX(f)}{df}$
Convolution	$\int_{-\infty}^{\infty} w(\tau)v(t-\tau)\,d\tau$	$W(f)V(f)$
Product	$w(t)v(t)$	$\int_{-\infty}^{\infty} W(\nu)V(f-\nu)\,d\nu$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{X(f)}{j2\pi f}+\frac{X(0)\delta(f)}{2}$

Other formulas:

$$X(0)=\int_{-\infty}^{\infty} x(t)\,dt; \qquad x(0)=\int_{-\infty}^{\infty} X(f)\,df$$

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt=\int_{-\infty}^{\infty} |X(f)|^2\,df \qquad \text{(Parseval)}$$

$$\int_{-\infty}^{\infty} x(t)y^*(t+\tau)e^{-j2\pi\nu t}\,dt=\int_{-\infty}^{\infty} X(f+\nu)Y^*(f)e^{-j2\pi f\tau}\,df$$

Figure 2: Some Properties of Fourier Transforms. (From [4])

Table XIII.1—Short Table of Fourier Transforms

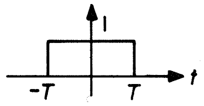
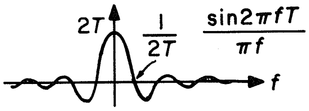
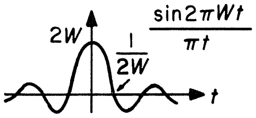
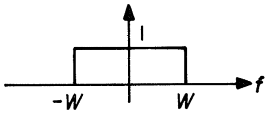
	$x(t)=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}\,df$		$X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}\,dt$
a)	$\delta(t)$	$\Longleftrightarrow$	1
b)	1	$\Longleftrightarrow$	$\delta(f)$
c)	$e^{-\pi(t/\tau)^2}$	$\Longleftrightarrow$	$\tau e^{-\pi(f\tau)^2}$
d)	$e^{-\alpha t}u(t)$	$\Longleftrightarrow$	$\dfrac{1}{\alpha+j2\pi f},\quad \alpha>0$
e)	$u(t)$	$\Longleftrightarrow$	$\dfrac{\delta(f)}{2}+\dfrac{1}{j2\pi f}$
f)	$\operatorname{sgn} t=\begin{cases}1, & t>0\\ -1, & t<0\end{cases}$	$\Longleftrightarrow$	$\dfrac{1}{j\pi f}$
g)	$\dfrac{1}{\pi t}$	$\Longleftrightarrow$	$-j\operatorname{sgn} f$
h)	$e^{-\alpha\|t\|}$	$\Longleftrightarrow$	$\dfrac{2\alpha}{\alpha^2+(2\pi f)^2}$
i)	$e^{j2\pi f_0 t}$	$\Longleftrightarrow$	$\delta(f-f_0)$
j)	$\sin 2\pi f_0 t$	$\Longleftrightarrow$	$\dfrac{\delta(f-f_0)-\delta(f+f_0)}{2j}$
k)	$\cos 2\pi f_0 t$	$\Longleftrightarrow$	$\dfrac{\delta(f-f_0)+\delta(f+f_0)}{2}$
l)	$e^{j2\pi f_0 t}u(t)$	$\Longleftrightarrow$	$\dfrac{\delta(f-f_0)}{2}+\dfrac{1}{j2\pi}\left[\dfrac{1}{f-f_0}\right]$
m)	$\sin 2\pi f_0 t\, u(t)$	$\Longleftrightarrow$	$\dfrac{\delta(f-f_0)-\delta(f+f_0)}{4j}+\dfrac{1}{2\pi}\left[\dfrac{f_0}{f_0^2-f^2}\right]$
n)	$\cos 2\pi f_0 t\, u(t)$	$\Longleftrightarrow$	$\dfrac{\delta(f-f_0)+\delta(f+f_0)}{4}+\dfrac{1}{j2\pi}\left[\dfrac{f}{f^2-f_0^2}\right]$
o)	$\dot{\delta}(t)$	$\Longleftrightarrow$	$j2\pi f$
p)	$te^{-\alpha t}u(t),\quad a>0$	$\Longleftrightarrow$	$\dfrac{1}{(\alpha+j2\pi f)^2}$
q)	t	$\Longleftrightarrow$	$\dfrac{j\dot{\delta}(f)}{2\pi}$
r)		$\Longleftrightarrow$	
s)		$\Longleftrightarrow$	
t)	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\Longleftrightarrow$	$\dfrac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(f-\dfrac{n}{T}\right)$

Figure 3: Some Fourier Transform Pairs (From [4])

B Notation and Properties of Second Order Processes

B.1 Definitions

55. A second-order random variable is one with a finite variance. A second-order random process is a random process made up of second-order random variables.

56. Let $X(t)$ and $Y(t)$ be a pair of jointly second-order, (possibly) complex-valued random processes. Define second-order moment functions:

    (a) *Mean:* $m_X(t) = \mathrm{E}\{X(t)\}$.

    (b) *Autocorrelation functions:*

    $$\begin{array}{r}R_X(t,s) = \mathrm{E}\{X^*(t)X(s)\}\\ \tilde{R}_X(t,s) = \mathrm{E}\{X(t)X(s)\}\end{array}.$$

    (c) *Cross correlation functions:*

    $$\begin{array}{r}R_{XY}(t,s) = \mathrm{E}\{X^*(t)Y(s)\}\\ \tilde{R}_{XY}(t,s) = \mathrm{E}\{X(t)Y(s)\}\end{array}.$$

    (d) *Autocovariance functions:*

    $$\begin{array}{r}C_X(t,s) = \mathrm{E}\{(X(t)-m_X(t))^*(X(s)-m_X(s))\}\\ \tilde{C}_X(t,s) = \mathrm{E}\{(X(t)-m_X(t))(X(s)-m_X(s))\}\end{array}.$$

    (e) *Cross covariance functions:*

    $$\begin{array}{r}C_{XY}(t,s) = \mathrm{E}\{(X(t)-m_X(t))^*(Y(s)-m_Y(s))\}\\ \tilde{C}_{XY}(t,s) = \mathrm{E}\{(X(t)-m_X(t))(Y(s)-m_Y(s))\}\end{array}.$$

57. Documentation of correlation and covariance properties:

    (a) Note that $C_{XY}(t,s) = R_{XY}(t,s) - m_X^*(t)m_Y(s)$, etc.

    (b) The (instantaneous) power of a second-order random process $X(t)$ is defined to be

    $$R_X(t,t) = C_X(t,t) + |m_X(t)|^2.$$

    (c) Autocorrelations and autocovariances are non-negative definite functions on $\mathcal{R}^2$.

    (d) A cross correlation/covariance must satisfy $R_{XY}(t,s) = R_{YX}^*(s,t)$ and:

    $$[\mathrm{Re}\{R_{XY}(t,s)\}]^2 \le R_{XX}(t,t)R_{YY}(s,s).$$

58. X is said to be *wide-sense stationary* if $m_X(t)$, $R_X(t,t+u)$, and $\tilde{R}_X(t,t+u)$ are constant w.r.t. t. In this case we define

$$\begin{aligned}\mu_X &\overset{\mathrm{def}}{=} m_X(t)\\ \rho_X(u) &\overset{\mathrm{def}}{=} R_X(t,t+u)\\ \tilde{\rho}_X(u) &\overset{\mathrm{def}}{=} \tilde{R}_X(t,t+u).\end{aligned}$$

59. X and Y are said to be jointly WSS if they are individually WSS and if $R_{XY}(t, t+u)$ and $\tilde{R}_{XY}(t, t+u)$ are constant w.r.t. t.

B.2 LTI Filtering

60. The output of a linear (possibly time-varying), bounded-input/bounded-output stable system driven by a second-order random process is well-defined and second-order.

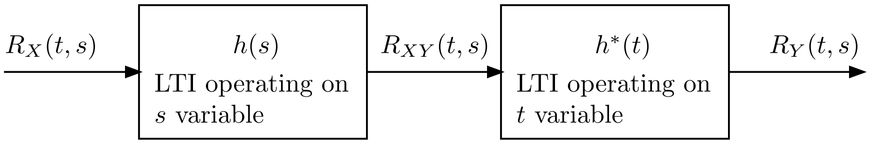
61. Although our favorite idealized brickwall filters are not BIBO they may be approximated arbitrarily closely by BIBO systems.

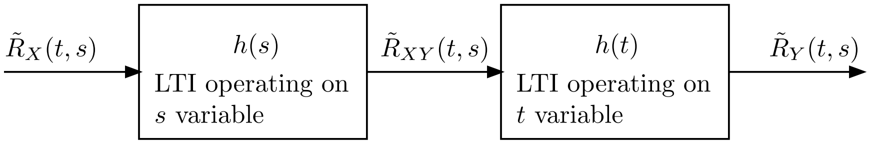
62. Two second-order random processes are said to be equivalent if they are equal in mean-square, i.e., if
$$\mathrm{E}\{|X(t) - Y(t)|^2\} = 0$$
for all t. The outputs of LTI BIBO systems driven by second-order processes are only defined up to this notion of equivalence.

63. We say that a second-order random process X has frequency content only in a set $f \in F$ if it is equal in mean-square to
$$W(t) = \int h_F(t-u)X(u)\mathrm{d}u; \quad H_F(f) = 1_F(f).$$

64. If X is a second-order process which is input to a BIBO stable LTI system with impulse response $h(t)$, then the system output Y is also a second-order process and
$$\begin{aligned}
m_Y(t) &= \int h(t-u)m_X(u)\mathrm{d}u \\
R_{XY}(t,s) &= \int h(s-u)R_X(t,u)\mathrm{d}u = [h * R_X(t,\cdot)](s) \qquad (4) \\
\tilde{R}_{XY}(t,s) &= \int h(s-u)\tilde{R}_X(t,u)\mathrm{d}u = [h * \tilde{R}_X(t,\cdot)](s) \\
R_Y(t,s) &= \int h^*(t-u)R_{XY}(u,s)\mathrm{d}u = [h^* * R_{XY}(\cdot,s)](t) \qquad (5) \\
&= \int\int h^*(t-u)h(s-v)R_X(u,v)\mathrm{d}v\mathrm{d}u \\
\tilde{R}_Y(t,s) &= \int h(t-u)\tilde{R}_{XY}(u,s)\mathrm{d}u = [h * \tilde{R}_{XY}(\cdot,s)](t) \\
&= \int\int h(t-u)h(s-v)\tilde{R}_X(u,v)\mathrm{d}v\mathrm{d}u
\end{aligned}$$
where h^* is just the pointwise conjugate of h.

65. In form of block diagrams we illustrate the previous equations:



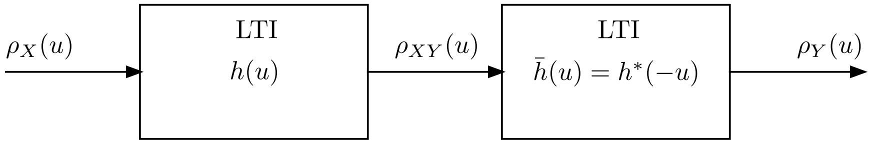


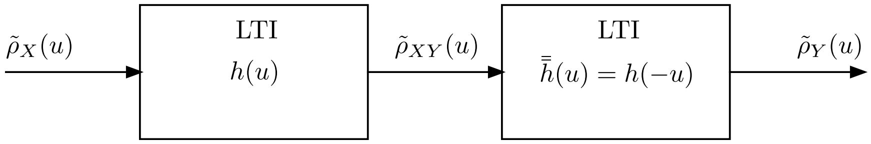
66. If X is WSS then Y is also WSS with

$$\begin{aligned}\mu_Y &= \mu_X \int h(t)\mathrm{d}t \\ \rho_Y(u) &= \left[\bar{h} * h * \rho_X\right](u) \\ \tilde{\rho}_Y(u) &= \left[\bar{\bar{h}} * h * \tilde{\rho}_X\right](u)\end{aligned}$$

where $\bar{\bar{h}}(t) \overset{\text{def}}{=} h(-t)$ and $\bar{h}(t) \overset{\text{def}}{=} h^*(-t)$. In terms of transfer functions and power spectral densities we have:

$$\begin{aligned}\mu_Y &= \mu_X H(0) \\ S_Y(f) &= |H(f)|^2 S_X(f) \\ \tilde{S}_Y(f) &= H(f)H(-f)\tilde{S}_X(f).\end{aligned}$$





C Block Diagram Calculus to Identify the Complex Baseband Models

60. To verify the form of the equivalent complex baseband models given previously we go through a series of block diagram manipulation steps out of which falls the simple model desired. These are shown in the following sequence of figures starting from Figure 4 to Figure 9.

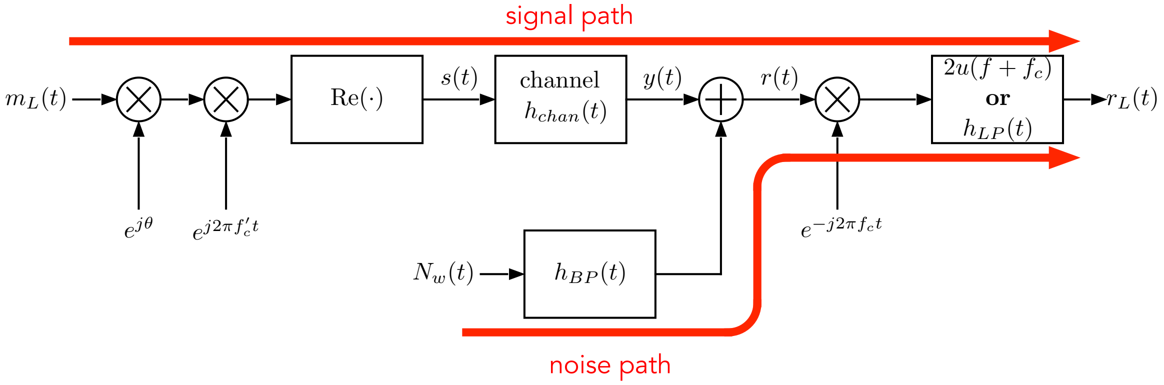


Figure 4: Signal and noise paths may be considered separately since the system in question is linear.

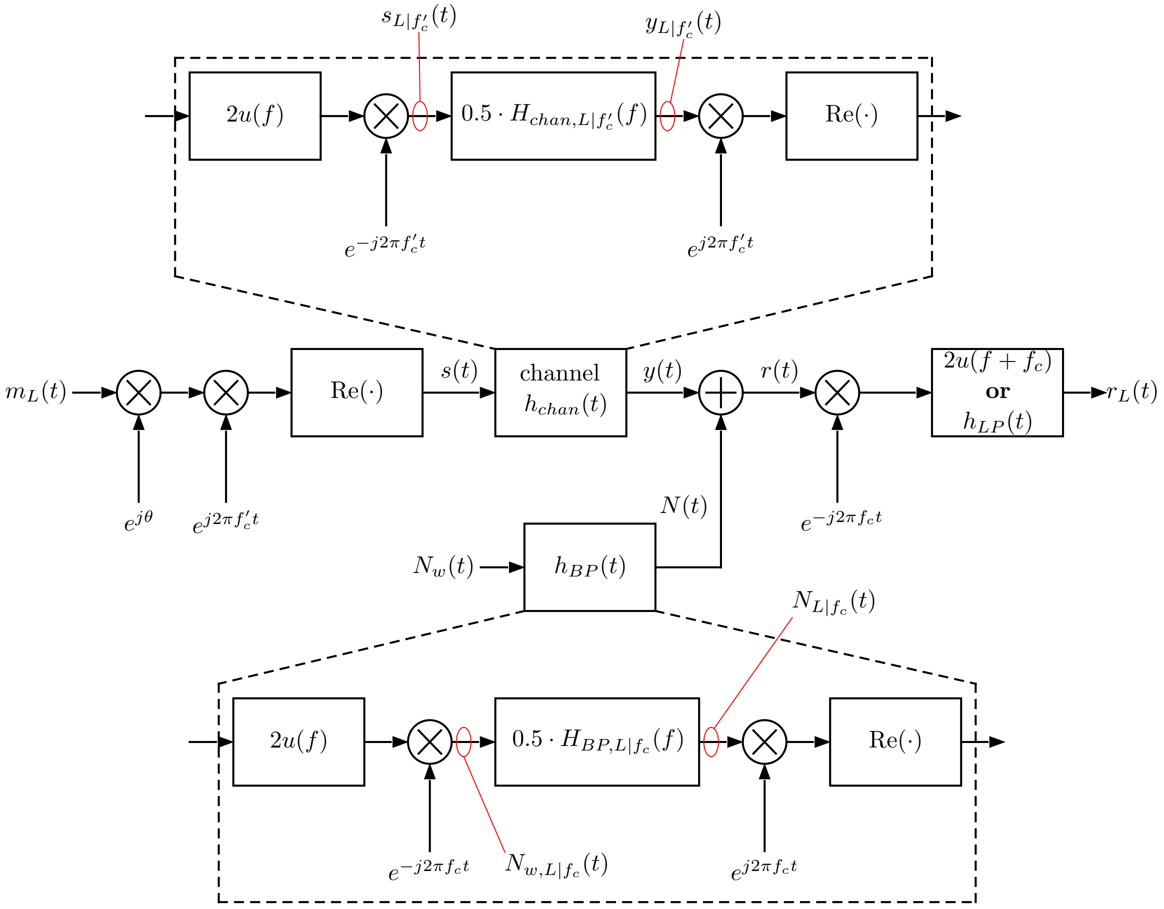


Figure 5: Observe the equivalent block diagrams for doing bandpass filtering in the complex baseband domain.

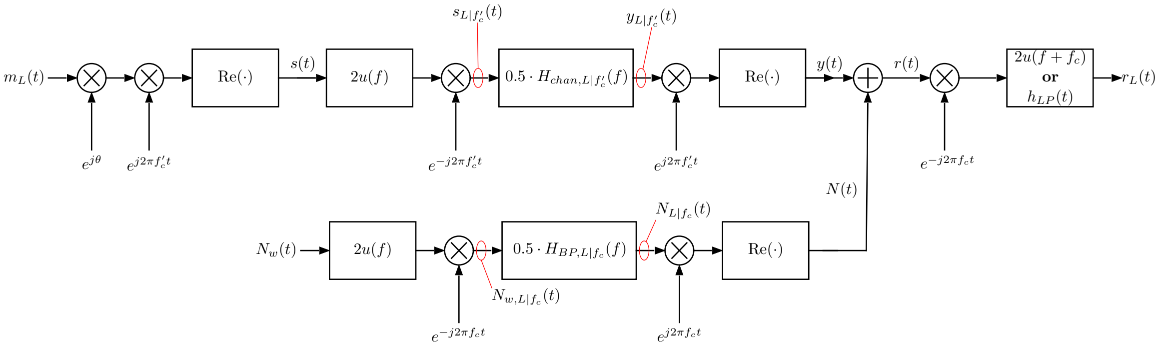


Figure 6: Insert the equivalent block diagrams for doing bandpass filtering in the complex baseband domain.

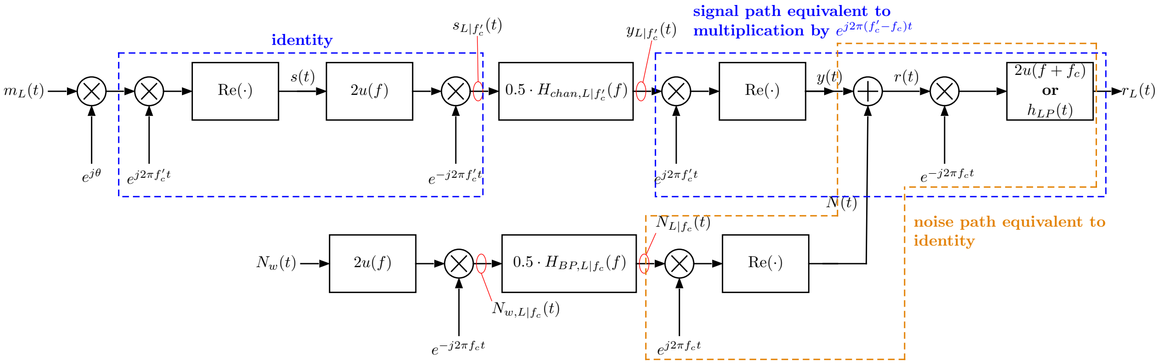


Figure 7: Identify simplifications in the previous block diagram.

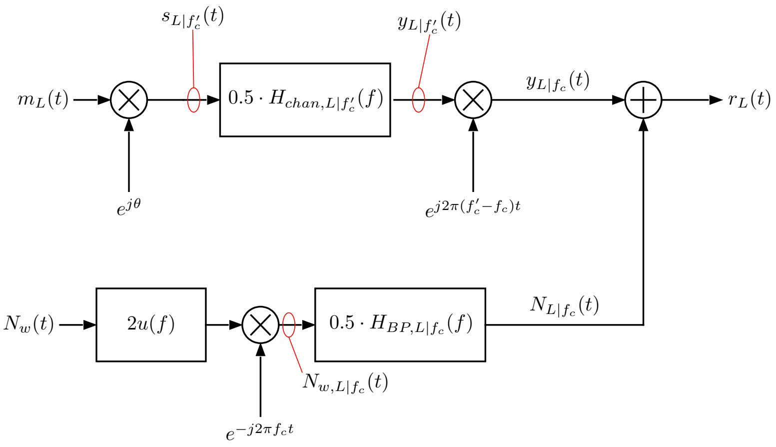


Figure 8: Replace blocks with simplifications.

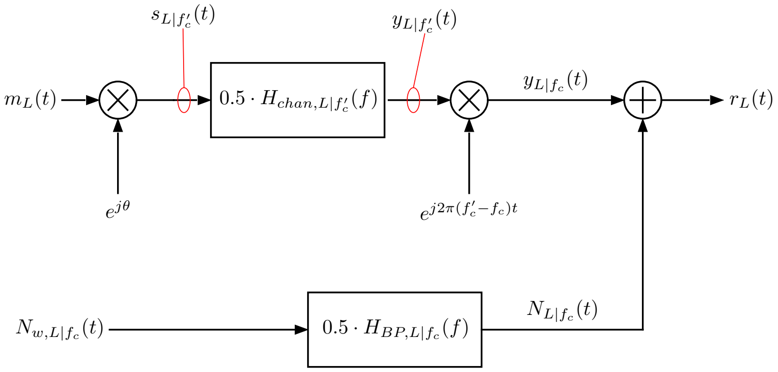


Figure 9: Model – complex baseband at input and complex baseband at output.

References

[1] H. Meyr, M. Moenclaey, and S. A. Fechtel. *Digital Communication Receivers.* Wiley, New York, 1998.

[2] Athanasios Papoulis. *The Fourier Integral and its Applications.* McGraw-Hill, New York, 1962.

[3] F. D. Neeser and J. L. Massey. Proper complex random processes with applications to information theory. *IEEE Info. Theory*, 39:1293–1302, July 1993.

[4] W. M. Siebert. *Circuits, Signals, and Systems.* The MIT Press, Cambridge, MA, 1986.