ECE 44000 HW #8 (Probs. 38 – 47) Due 12/2/22

Problem 38: Read the Section 5.5 of Z&T (7th edition). Do the problem below which comes from the 6th edition. The problem is similar to Probs. 5.15 and 5.18 of the 7th edition.

Z+T 6th Edition

4.14. Given the following channel pulse-response samples:

$P_c(-4T) = -0.001$ $p_c(-3T) = 0.001$ $p_c(-2T) = -0.01$ $p_c(-T) = 0.1$ $p_c(0) = 1.0$

$p_c(T) = 0.2$ $p_c(2T) = -0.02$ $p_c(3T) = 0.005$ $P_c(4T) = -0.003$

a. Find the tap coefficients for a three-tap zero-forcing equalizer.

b. Find the output samples for $mT = -2T, -T, 0, T$, and $2T$.

Problem 39: Read Section 5.5 of Z&T (7th edition) and work the problem below (similar to Prob. 5.17 of the 7th edition).

Z+T 6th Edition

4.16. A simple model for a multipath communications channel is shown in Figure 4.20(a).

a. Find $H_c(f) = Y(f)/X(f)$ for this channel and plot $|H_c(f)|$ for $\beta = 1$ and 0.5.

b. In order to equalize, or undo, the channel-induced distortion, an equalization filter is used. Ideally, its transfer function should be

$$H_{\text{eq}}(f) = \frac{1}{H_c(f)}$$

if the effects of noise are ignored and only distortion caused by the channel is considered. A tapped delay-line or transversal filter, as shown in Figure 4.20(b), is commonly used to approximate $H_{\text{eq}}(f)$. Write down a series expression for $H'_{\text{eq}}(f) = Z(f)/Y(f)$.

c. Using $(1+x)^{-1} = 1-x+x^2-x^3+\dots, |x|<1$, find a series expression for $1/H_c(f)$. Equating this with $H_{\text{eq}}(f)$ found in part (b), find the values for $\beta_1, \beta_2, \dots, \beta_N$, assuming $\tau_m = \Delta$.

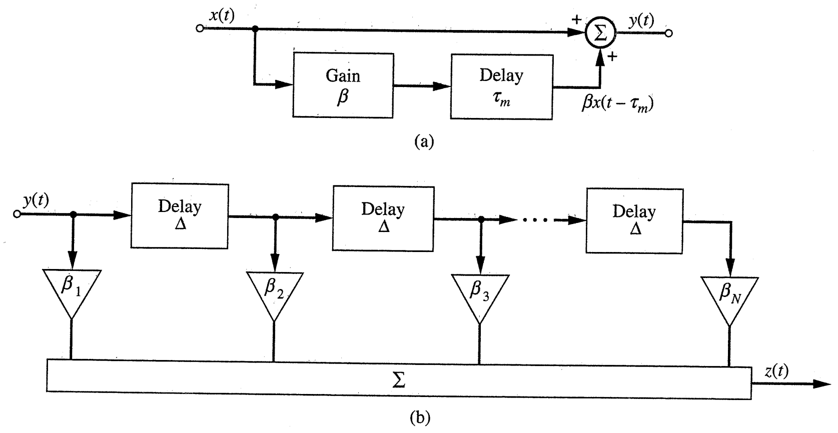


Figure 4.20

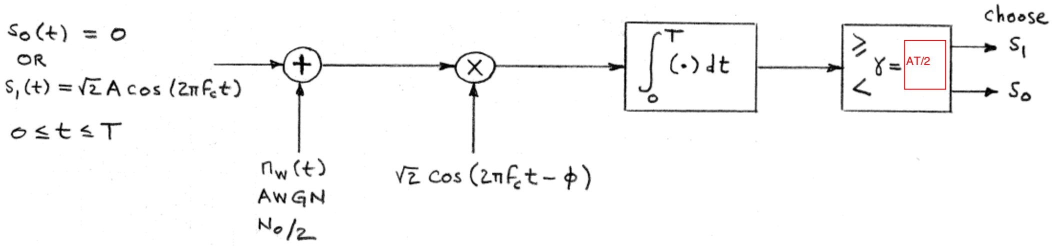
Problem 40: [Fall 2007 Final Exam] The figure below shows an ASK receiver intended as a coherent receiver but with a small phase offset ϕ. The two signals $s_0(t) = 0$ and $s_1(t) = \sqrt{2}A\cos(2\pi f_c t)$ are equally likely. The integration time T may be assumed to be an integer multiple of the period of the carrier for simplicity. The threshold has been set to $\gamma = AT/2$.

(a) For an arbitrary small value of ϕ compute the conditional probability $P_{e,0|\phi}$ of an error given that $s_0(t)$ was transmitted and the phase error was ϕ. Write your answer in terms of the Gaussian Q-function.

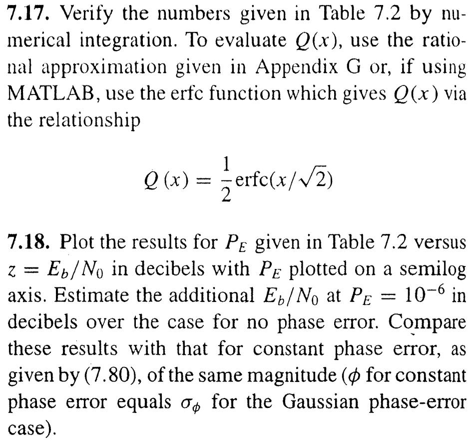
(b) For an arbitrary small value of ϕ compute the conditional probability $P_{e,1|\phi}$ of an error given that $s_1(t)$ was transmitted and the phase error was ϕ. Write your answer in terms of the Gaussian Q-function.

(c) Use the answers in (a) and (b) to find the expression for the unconditional probability of an error given phase error ϕ, $P_{e|\phi}$.

(d) If you had a prior model for the phase error ϕ in terms of its pdf $f(\phi)$ how would you go about finding the overall probability of error averaged over ϕ?



Problem 41: Z&T 6th edition Problems 7.17 and 7.18. See pages 419 and 420 of the 7th edition. The table in the problem statement is Table 9.2 in the 7th edition and Equation (7.80) in the 6th edition is Equation (9.81) in the 7th edition.



7.17. Verify the numbers given in Table 7.2 by numerical integration. To evaluate $Q(x)$, use the rational approximation given in Appendix G or, if using MATLAB, use the erfc function which gives $Q(x)$ via the relationship

$$Q(x) = \frac{1}{2}\operatorname{erfc}(x/\sqrt{2})$$

7.18. Plot the results for P_E given in Table 7.2 versus $z = E_b/N_0$ in decibels with P_E plotted on a semilog axis. Estimate the additional E_b/N_0 at $P_E = 10^{-6}$ in decibels over the case for no phase error. Compare these results with that for constant phase error, as given by (7.80), of the same magnitude (ϕ for constant phase error equals σ_ϕ for the Gaussian phase-error case).

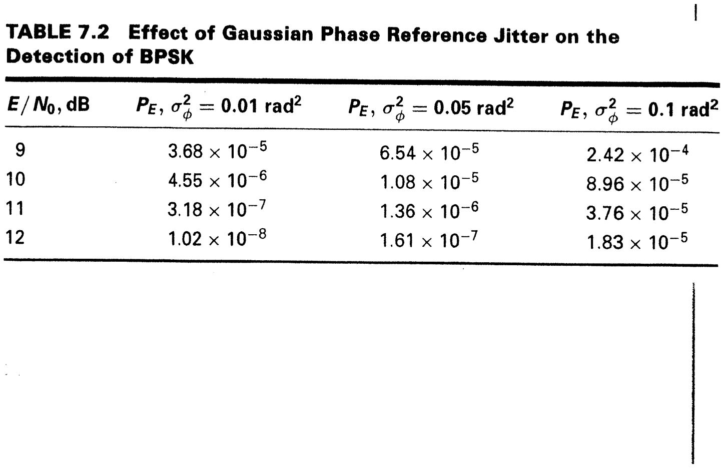


TABLE 7.2 Effect of Gaussian Phase Reference Jitter on the Detection of BPSK

E/N_0, dB	P_E, $\sigma_\phi^2 = 0.01$ rad^2	P_E, $\sigma_\phi^2 = 0.05$ rad^2	P_E, $\sigma_\phi^2 = 0.1$ rad^2
9	3.68×10^{-5}	6.54×10^{-5}	2.42×10^{-4}
10	4.55×10^{-6}	1.08×10^{-5}	8.96×10^{-5}
11	3.18×10^{-7}	1.36×10^{-6}	3.76×10^{-5}
12	1.02×10^{-8}	1.61×10^{-7}	1.83×10^{-5}

Problem 42: Z&T 7th edition Problem 9.4.

9.4 A receiver for baseband digital data has a threshold set at ϵ instead of zero. Rederive (9.8), (9.9), and (9.11) taking this into account. If $P(+A) = P(-A) = \frac{1}{2}$, find E_b/N_0 in decibels as a function of ϵ for $0 \le \epsilon/\sigma \le 1$ to give $P_E = 10^{-6}$, where σ^2 is the variance of N.

$$P(E|A) = \int_{\sqrt{2A^2T/N_0}}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du \triangleq Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) \tag{9.8}$$

$$P(E|-A) = \int_{AT}^{\infty} \frac{e^{-\eta^2/N_0T}}{\sqrt{\pi N_0 T}}\, d\eta \triangleq Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) \tag{9.9}$$

$$P_E = P(E|+A)P(+A) + P(E|-A)P(-A) \tag{9.10}$$

Problem 43: Z&T 7th edition Problem 9.25.

9.25

- **(a)** Consider the sequence 011 101 010 111. Differentially encode it and assume that the differentially encoded sequence is used to biphase modulate a sinusoidal carrier of arbitrary phase. Prove that the demodulator of Figure 9.17 properly gives back the original sequence.
- **(b)** Now invert the sequence (i.e., 1s become 0s and vice versa). What does the demodulator of Figure 9.17 give now?

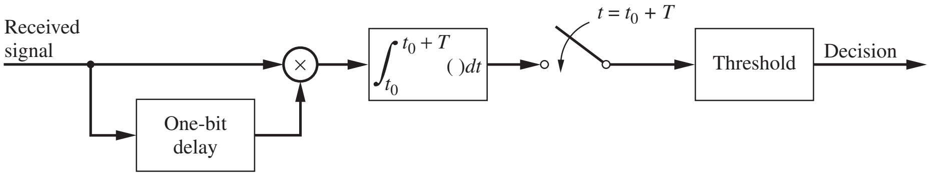


Figure 9.17
Demodulation of DPSK.

Problem 44: Z&T 7th edition Problem 9.26.

9.26

- **(a)** In the analysis of the optimum detector for DPSK, show that the random variables n_1, n_2, n_3, and n_4 have zero means and variances $N_0T/4$.
- **(b)** Show that w_1, w_2, w_3, and w_4 have zero means and variances $N_0T/8$.

Problem 45:

The problem below is a follow-on to the "3 tap zero-forcing equalizer" problem above. In that problem we were given the channel pulse response samples:

$$\begin{array}{lll} p_c(-4T) = -0.001 & p_c(-3T) = 0.001 & p_c(-2T) = -0.01 \\ p_c(-T) = 0.1 & p_c(0) = 1.0 & p_c(T) = 0.2 \\ p_c(2T) = -0.02 & p_c(3T) = 0.005 & p_c(4T) = -0.003 \end{array} .$$

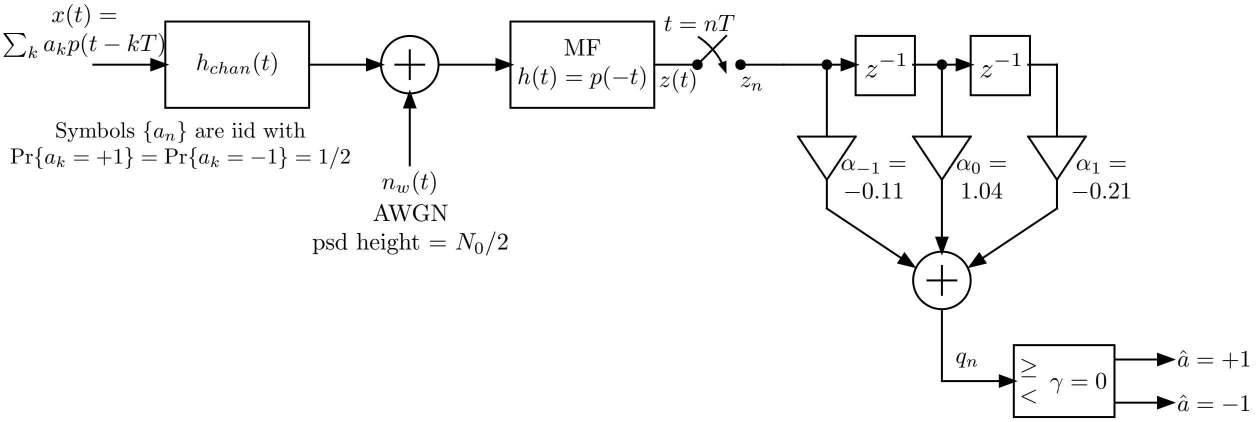
We found the coffecients of the equalizer (shown in the block diagram below rounded to two decimal places) by solving the matrix equation

$$\begin{bmatrix} p_c(0) & p_c(-T) & p_c(-2T) \\ p_c(T) & p_c(0) & p_c(-T) \\ p_c(2T) & p_c(T) & p_c(0) \end{bmatrix} \begin{bmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} .$$

We assume that the pulse and matched filter were originally designed to satisfy the Nyquist criterion, that is, $h * p(t) = R_{pp}(t)$ is $= 1$ when evaluated at $t = 0$ and $= 0$ when evaluated at $t = mT$, $m \neq 0$. However, the channel filter has destroyed the Nyquist property as seen from the samples of

$$p_c(t) = p * h_{chan} * h(t) = R_{pp} * h_{chan}(t),$$

which is why the equalizer is needed.



For simplicity assume that only three nonzero symbols are sent, namely a_{-1}, a_0, and a_1. In this problem we concentrate on the decision regarding the middle symbol, i.e., $\hat{a}_0$.

(a) For the equalizer shown it is q_1 which should be compared to a threshold to estimate a_0. Explain why.

(b) The equalizer output q_1 consists of a signal part and a noise part. Conditioning on (i.e., assuming) $a_{-1} = a_1 = +1$:

- Find the signal part as a function of the unknown symbol a_0.
- Find the noise part, i.e., characterize the noise random variable by stating its probability distribution and finding its mean and variance.

(c) Find the conditional error probabilities:

$$\Pr\{q_1 < 0 | a_{-1} = a_0 = a_1 = +1\}$$
$$\Pr\{q_1 \geq 0 | a_{-1} = a_1 = +1,\, a_0 = -1\}$$

in terms of either the cdf or the Q function of a zero mean, variance one Gaussian random variable.

(d) Now for comparison suppose we wire around the equalizer and apply z_n directly to the decision device. Assume the same setup as before (i.e., only three symbols are transmitted and condition on $a_{-1} = a_1 = +1$). Now we should compare z_0 to the threshold to estimate a_0.

  - z_0 consists of a signal part and a noise part. Find them.
  - Find the conditional error probabilities:

$$\begin{array}{l} \Pr\{z_0 < 0 | a_{-1} = a_0 = a_1 = +1\} \\ \Pr\{z_0 \geq 0 | a_{-1} = a_1 = +1,\, a_0 = -1\} \end{array}$$

  in terms of either the cdf or the Q function of a zero mean, variance one Gaussian random variable.

Problem 46:

This problem is a "follow on" to the problem above (again). The problem setup and assumptions are exactly the same as in the original problem except as noted below.

(a) In Part (c) of the previous problem we calculated two conditional error probabilities and found that they were independent of a_{-1}, a_0, and a_1 with

$$\begin{aligned} \Pr\{q_1 < 0 | a_{-1} = a_0 = a_1 = +1\} &= \Pr\{q_1 \geq 0 | a_{-1} = a_1 = +1,\, a_0 = -1\} \\ &= Q\left(\sqrt{\frac{2}{1.14 N_0}}\right) = Q\left(0.94\sqrt{\frac{2}{N_0}}\right). \end{aligned}$$

Based on this observation, what is the probability of error at the equalizer output averaged over all possible symbols a_{-1}, a_0, and a_1? Explain your reasoning.

(b) Similarly, in Part (d) of the previous problem we computed some conditional error probabilities for the non-equalized signal and found that they did depend on the values of a_{-1}, a_0, and a_1:

$$\begin{aligned} \Pr\{z_0 < 0 | a_{-1} = a_0 = a_1 = +1\} &= Q\left(1.3\sqrt{\frac{2}{N_0}}\right) \\ \Pr\{z_0 \geq 0 | a_{-1} = a_1 = +1,\, a_0 = -1\} &= Q\left(0.7\sqrt{\frac{2}{N_0}}\right). \end{aligned}$$

In order to calculate the average probability of error using the non-equalized signal we need to compute 6 more conditional probabilities. Show exactly which probabilities you need to compute and how you would use them to compute the average probability of error. Then pick one of them and compute it to show that you know what you are doing.

Problem 47: Z&T 7th edition Problem 9.30.

9.30 Find the probability of error for noncoherent ASK, with signal set

$$s_i(t) = \begin{cases} 0, & 0 \le t \le T,\ i = 1 \\ A\cos(2\pi f_c t + \theta), & 0 \le t \le T,\ i = 2 \end{cases}$$

where θ is a uniformly distributed random variable in $[0, 2\pi)$. White Gaussian noise of two-sided power spectral density $N_0/2$ is added to this signal in the channel. The receiver is a bandpass filter of bandwidth $2/T$ Hz centered on f_c, followed by an envelope detector, which is input to a sampler and threshold comparator. Assume that the signal, when present, is passed by the filter without distortion, and let the noise variance at the filter output be $\sigma_N^2 = N_0 B_T = 2N_0/T$.

Show that the envelope detector output with signal 1 present (i.e., zero signal) is Rayleigh-distributed, and that the envelope detector output with signal 2 present is Ricean-distributed. Assuming that the threshold is set at $A/2$, find an expression for the probability of error. You will not be able to integrate this expression. However, by making use of the approximation

$$I_0(v) \approx \frac{e^v}{\sqrt{2\pi v}},\ v \gg 1$$

you will be able to approximate the pdf of the sampler output for large signal-to-noise ratio as Gaussian and express the probability of error in terms of a Q-function. (*Hint*: Neglect the $v^{-1/2}$ in the above approximation.)

Show that the probability of error for the signal-to-noise ratio large is approximately

$$\begin{aligned} P_E &= \frac{1}{2}P(E \mid S+N) + \frac{1}{2}P(E \mid 0) \\ &\approx \frac{e^{-z}}{\sqrt{4\pi z}} + \frac{1}{2}e^{-z/2},\ z = \frac{A^2}{4\sigma_N^2} \gg 1 \end{aligned}$$

Note that $z = \frac{A^2}{4\sigma_N^2}$ is the average signal power- (the signal is 0 half the time) to-noise variance ratio. Plot the error probability versus the signal-to-noise ratio and compare with that for DPSK and noncoherent FSK.