ECE 44000 (Fall 2022)
HW 5 (Probs. 18–24)
Due: 10/12/2022

Problem 18: Z&T (7th Edition) Problem 6.16.

4.16. Given the Gaussian random variable with the pdf

$$f_X(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$

let $Y = X^2$. Find the pdf of Y. *Hint*: Note that $Y = X^2$ is symmetrical about $X = 0$ and that it is impossible for Y to be less than zero.

Problem 19: Z&T (5th Edition) Problem 4.28[1].

4.28. A random variable X is defined by

$$f_X(x) = 4e^{-8|x|}$$

The random variable Y is related to X by $Y = 2 + 3X$.

(a) Determine $E[X]$, $E[X^2]$, and σ_X^2.

(b) Determine $f_Y(y)$.

(c) Determine $E[Y]$, $E[Y^2]$, and σ_Y^2.

(d) If you used $f_Y(y)$ in part (c), repeat that part using only $f_X(x)$.

Problem 20:
A DSB-SC modulated signal is transmitted over an additive noise channel where the psd of the noise is given by

$$S_n(f) = \begin{cases} 1 - \frac{|f|}{400} & |f| < 400 \\ 0 & |f| \geq 400 \end{cases}$$

where the units of f are kHz and the units of $S_n(f)$ are μJ. The message bandwidth is 4 kHz and the carrier frequency is 200 kHz. Assuming that the average power of the modulated wave is 10 W, determine the output signal-to-noise ratio of the receiver.

[1] A minor variation of Problem 6.28 in the 7th edition.

Problem 21: Z&T (7th Edition) Problems 7.3 and 7.5.

5.3. A random process is composed of sample functions that are square waves, each with constant amplitude A, period T_0, and random delay τ as sketched in Figure 5.15. The pdf of τ is

$$f(\tau) = \begin{cases} 1/T_0, & |\tau| \leq T_0/2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch several typical sample functions.

(b) Write the first-order pdf for this random process at some arbitrary time t_0. (*Hint*: Because of the random delay τ, the pdf is independent of t_0. Also, it might be easier to deduce the cdf and differentiate it to get the pdf.)

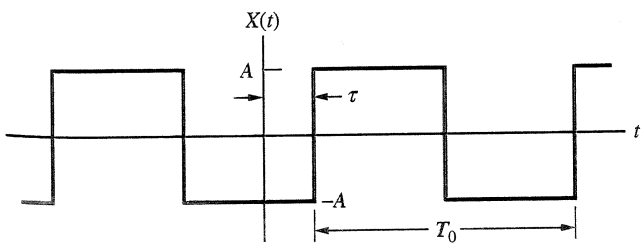


FIGURE 5.15

5.5. Consider the random process of Problem 5.3.

(a) Find the time-average mean and the autocorrelation function.

(b) Find the ensemble-average mean and the autocorrelation function.

(c) Is this process wide-sense stationary? Why?

Problem 22: Z&T (7th Edition) Problem 7.19.

6.19. An ideal finite-time integrator is characterized by the input-output relationship

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\alpha)\,d\alpha$$

a. Justify that its impulse response is $h(t) = \frac{1}{T}[u(t) - u(t-T)]$.

b. Obtain its frequency response function. Sketch it.

c. The input is white noise with two-sided power spectral density $N_0/2$. Find the power spectral density of the output of the filter.

d. Show that the autocorrelation function of the output is

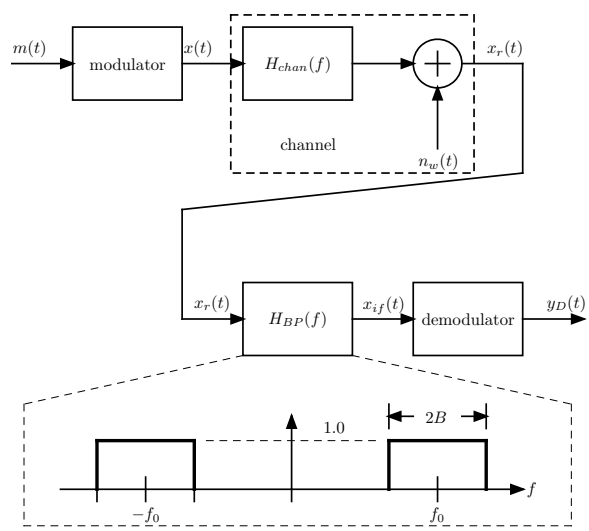
$$R_0(\tau) = \frac{N_0}{2T}\Lambda(\tau/T)$$

where $\Lambda(x)$ is the unit-area triangular function defined in Chapter 2.

e. What is the equivalent noise bandwidth of the integrator?

f. Show that the result for the output noise power obtained using the equivalent noise bandwidth found in part (e) coincides with the result found from the autocorrelation function of the output found in part (d).

Problem 23: A general model for passband communications is based on the block diagram shown below:



The message $m(t)$ will be modeled as a WSS random process with autocorrelation/psd $R_m(\tau) \leftrightarrow S_m(f)$ and bandlimited

$$S_m(f) = 0 \text{ for } f > W.$$

The noise $n_w(t)$ is also modeled as a WSS random process which is Gaussian, white, i.e.,

$$R_{n_w}(\tau) = \frac{N_0}{2}\delta(\tau)$$

$$\updownarrow$$

$$S_{n_w}(f) = \frac{N_0}{2} \text{ for } -\infty < f < \infty,$$

and statistically independent of $m(t)$ and any other random parameters of the communications model. For the purposes of this problem we will also assume that $H_{chan}(f) = 1$.

The purpose of this problem is to compute the signal-to-noise (SNR) power ratio at IF (i.e., associated with $x_{if}(t)$ in the block diagram). Since $m(t)$ is low-pass, the modulated signal $x(t)$ will be bandpass. In addition, we will assume that the bandpass filter $H_{BP}(f)$ in the block diagram passes the signal part without distortion, i.e., $x * h_{BP}(t) = x(t)$. Therefore,

$$\begin{aligned} x_{if}(t) &= x * h_{BP}(t) + n_w * h_{BP}(t) \\ &= x(t) + n(t) \end{aligned}$$

where $n(t)$ is a bandpass Gaussian noise with zero mean and psd of height $N_0/2$ and shape identical to that of $H_{BP}(f)$.

Then the SNR at IF is defined to be

$$\begin{aligned} \text{SNR}_{if} &\stackrel{\text{def}}{=} \frac{\text{power}[x(t)]}{\text{power}[n(t)]} \\ &= \frac{R_x(0)}{2N_0 B} \end{aligned}$$

assuming $x(t)$ is WSS and we can find its autocorrelation and/or psd.

(a) For AM DSB-SC assume

$$x(t) = A_c m(t) \cos(2\pi f_c t + \Theta)$$

where Θ is uniform on $[0, 2\pi)$ and statistically independent of message and noise.

- Find the auto-correlation and psd of $x(t)$. For a representative shape for $S_m(f)$ sketch a plot of $S_x(f)$.
- Specify the BPF so that $x * h_{BP}(t) = x(t)$ and the noise $n(t)$ has minimum power.
- Compute SNR_{if}.

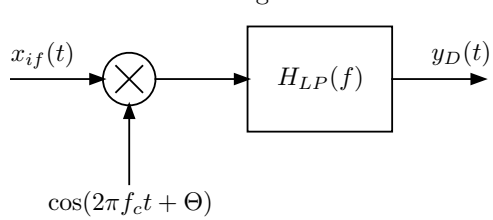
(b) Repeat (a) for AM LC

$$x(t) = A_c[1 + k_a m(t)] \cos(2\pi f_c t + \Theta).$$

(c) Repeat (a) for SSB

$$x(t) = 0.5 A_c m(t) \cos(2\pi f_c t + \Theta) \pm 0.5 A_c \hat{m}(t) \sin(2\pi f_c t + \Theta).$$

Problems 24: The three linear modulations AM DSB-SC, AM LC, and SSB have the identical coherent demodulator architecture shown in the figure below:



Use the standard assumptions on message, noise, and phase angle and compute the SNR at the demodulator output for the three linear modulations above.

4.16. Given the Gaussian random variable with the pdf

$$f_X(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}$$

let $Y = X^2$. Find the pdf of Y. *Hint*: Note that $Y = X^2$ is symmetrical about $X = 0$ and that it is impossible for Y to be less than zero.

$X \sim N(0, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

$Y = X^2$

Find cdf of Y and take the derivative. First

$$F_Y(y) = P_r(Y \le y) = P_r(X^2 \le y) = P_r(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) = F_X(\sqrt{y}) - [1 - F_X(\sqrt{y})]$$

(since pdf of X is symm. about $x = 0$)

$$= 2F_X(\sqrt{y}) - 1 \qquad (\text{if } y \geq 0, \text{ else } F_Y(y) = 0)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\{2F_X(\sqrt{y}) - 1\}$$

$$= 2F_X'(\sqrt{y})\,\tfrac{1}{2} y^{-1/2} = \frac{1}{\sqrt{y}} f_X(\sqrt{y}) \qquad y \geq 0$$

$$\therefore \quad f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\sigma} \frac{1}{\sqrt{y}} e^{-y/2\sigma^2} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

4.28. A random variable X is defined by

$$f_X(x) = 4e^{-8|x|}$$

The random variable Y is related to X by $Y = 2 + 3X$.

(a) Determine $E[X]$, $E[X^2]$, and σ_X^2.

(b) Determine $f_Y(y)$.

(c) Determine $E[Y]$, $E[Y^2]$, and σ_Y^2.

(d) If you used $f_Y(y)$ in part (c), repeat that part using only $f_X(x)$.

(a)

$$EX = \int_{-\infty}^{\infty} x\, 4e^{-8|x|}\, dx = \int_0^{\infty} 4x e^{-8x}\, dx + \int_{-\infty}^{0} 4x e^{-8|x|}\, dx$$

$\downarrow$ $z = -x$

$dz = -dx$

$$\int_{+\infty}^{0} -4z e^{-8z} (-dz)$$

$$= \int_0^{\infty} 4x e^{-8x}\, dx - \int_0^{\infty} 4z e^{-8z}\, dz = 0$$ (the integrals converge absolutely)

$$E[X^2] = \sigma_X^2 = \int_{-\infty}^{\infty} x^2 4e^{-8|x|}\, dx = 2\int_0^{\infty} x^2 4e^{-8x}\, dx$$

$$= 8\int_0^{\infty} x^2 e^{-8x}\, dx$$

$$= 8\left[-\frac{x^2}{8} - \frac{2x}{64} - \frac{2}{8^3}\right] e^{-8x}\Bigg|_{x=0}^{\infty}$$

$$= \frac{8 \cdot 2}{8^3} = \frac{2}{64} = \frac{1}{32}$$

(b) Determine $f_Y(y)$. $Y = 2 + 3X$

$$F_Y(y) = P_r(Y \le y) = P_r(2 + 3X \le y) = P_r(3X \le y - 2)$$

$$= P_r\left(X \le \frac{y-2}{3}\right) = F_X\left(\frac{y-2}{3}\right)$$

$$f_Y(y) = \frac{d}{dy} F_X\left(\frac{y-2}{3}\right) = f_X\left(\frac{y-2}{3}\right)\frac{1}{3}$$

$$= \frac{4}{3} e^{-8\left|\frac{y-2}{3}\right|}$$

(c) $EY = E(2+3X) = 2+3EX = 2+3\cdot 0 = 2$

$EY^2 = E(2+3X)^2 = E(4+12X+9X^2)$

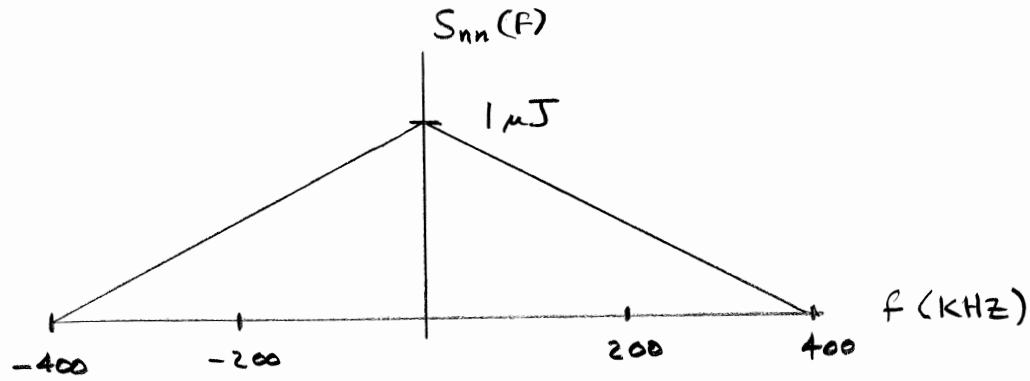
$= 4 + 12EX + 9EX^2$

$= 4 + 12\cdot 0 + 9\cdot 1/32 = 4 + \frac{9}{32}$

(d) No need

Problem 44 Problem 18

DSB-SC signal transmitted over additive noise channel where the noise is <u>not</u> white and instead has a psd

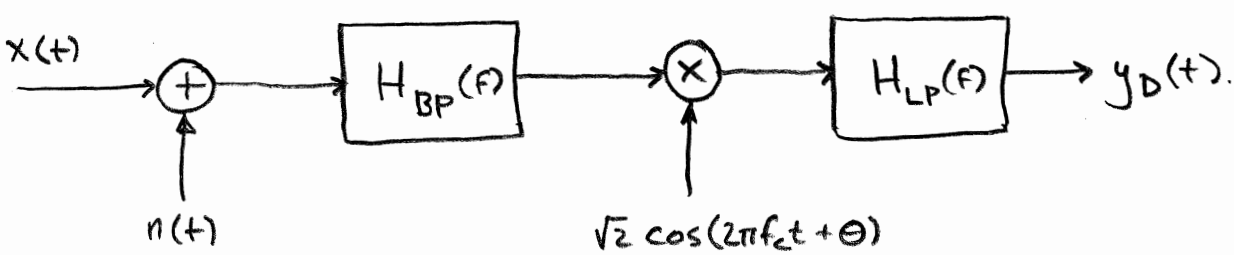


(notes: The problem statement had a typo when it indicated the noise psd was bandlimited to 400 Hz rather than 400kHz as shown above.

The unit of a power spectral density is W/Hz, but this is equivalent to W·s = J.)

Message BW = 4KHz.
f_c = 200KHz
Assume average power of modulated wave is 10W.
Find SNR at the output of the standard receiver.

The standard receiver from class notes (or text) is

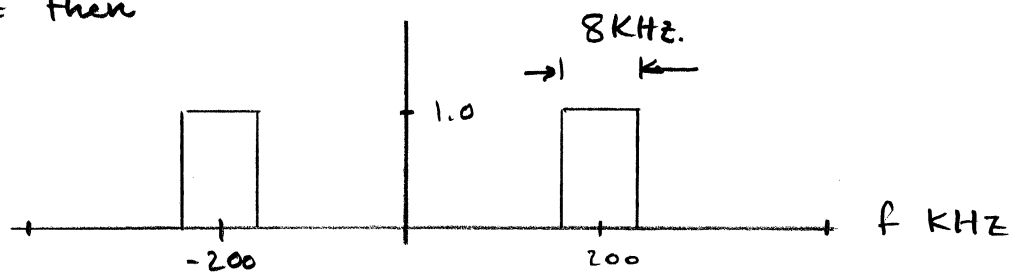


where $x(t) = \sqrt{2} A_c m(t) \cos(2\pi f_c t + \theta)$. In class notes we showed

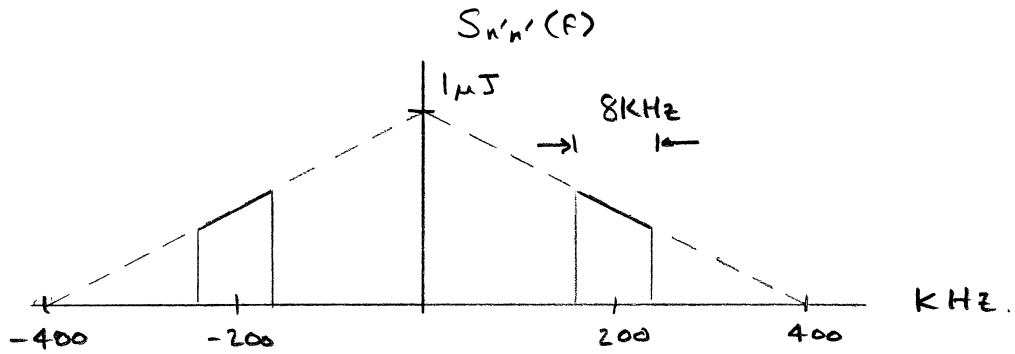
$$\text{Power}\{x(t)\} = R_{xx}(0) = A_c^2 P_m = 10W$$

given.

The BP filter should be chosen to just pass the modulated signal. If the message $m(t)$ has baseband bandwidth of 4Hz then



Therefore, at the output of the BP filter the noise $n'(t) = h_{BP} * n(t)$ will have psd



A little reflection will show

$$\text{Power}\{n'(t)\} = 2 \cdot \tfrac{1}{2}\mu J \cdot 8\,KHz$$
$$= 8\,mW$$

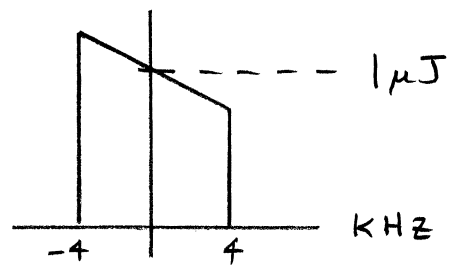
$$\therefore\ SNR_T^{opt} = \frac{10\,W}{8\,mW} = \frac{10}{8} \cdot 1000 = \frac{5}{4} \cdot 1000 = 1250$$
$$(\approx 31.0\,dB).$$

At the output of the detector we showed (class notes) that

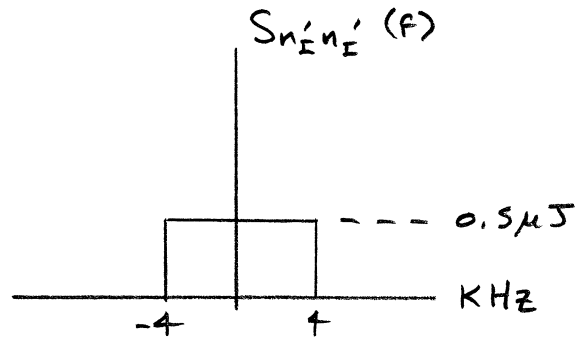
$$y_D(t) = \underbrace{A_c m(t)}_{\text{this term has power } 10W} + \underbrace{n'_I(t)}_{\text{must find power}}$$

To find $n_I'(t)$'s psd, first find that of $n_L'(t)$ which is

$$S_{n_L'n_L'}(f) = 2u(f+f_c)\,S_{n'n'}(f+f_c)$$



$$S_{n_I'n_I'}(f) = \frac{S_{n_L'n_L'}(f) + S_{n_L'n_L'}(-f)}{4}$$



$$\text{Power}\{n_I'(t)\} = 0.5\,\mu J \cdot 8\,\text{kHz} = 4\,\text{mW}$$

$$\therefore\ SNR_D^{opt} = \frac{10\,W}{4\,mW} = 2 \cdot 1250 = 2500$$

$$(\approx 34.0\,dB)$$

5.3. A random process is composed of sample functions that are square waves, each with constant amplitude A, period T_0, and random delay τ as sketched in Figure 5.15. The pdf of τ is

$$f(\tau) = \begin{cases} 1/T_0, & |\tau| \le T_0/2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch several typical sample functions.

(b) Write the first-order pdf for this random process at some arbitrary time t_0. (*Hint*: Because of the random delay τ, the pdf is independent of t_0. Also, it might be easier to deduce the cdf and differentiate it to get the pdf.)

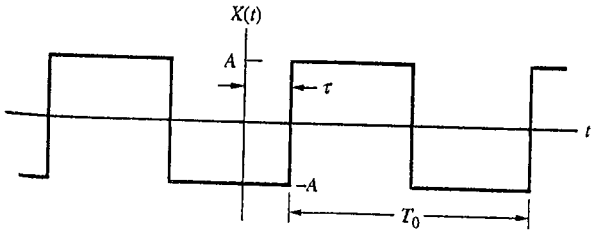


FIGURE 5.15

5.5. Consider the random process of Problem 5.3.

(a) Find the time-average mean and the autocorrelation function.

(b) Find the ensemble-average mean and the autocorrelation function.

(c) Is this process wide-sense stationary? Why?

5.3a A typical sample function is shown in Fig 5.15. Picking different values for τ will give all possible sample functions.

5.3b Fixing any time $t = t_0$ makes clear that the rv $X(t_0)$ is either equal to $+A$ or to $-A$ so the pdf will be discrete. As indicated in the hint $X(0)$ and $X(t_0)$ are identically distributed.

$$Pr(X(0) = -A) = Pr(\tau > 0) = 1/2$$

$$Pr(X(0) = +A) = Pr(\tau < 0) = 1/2$$

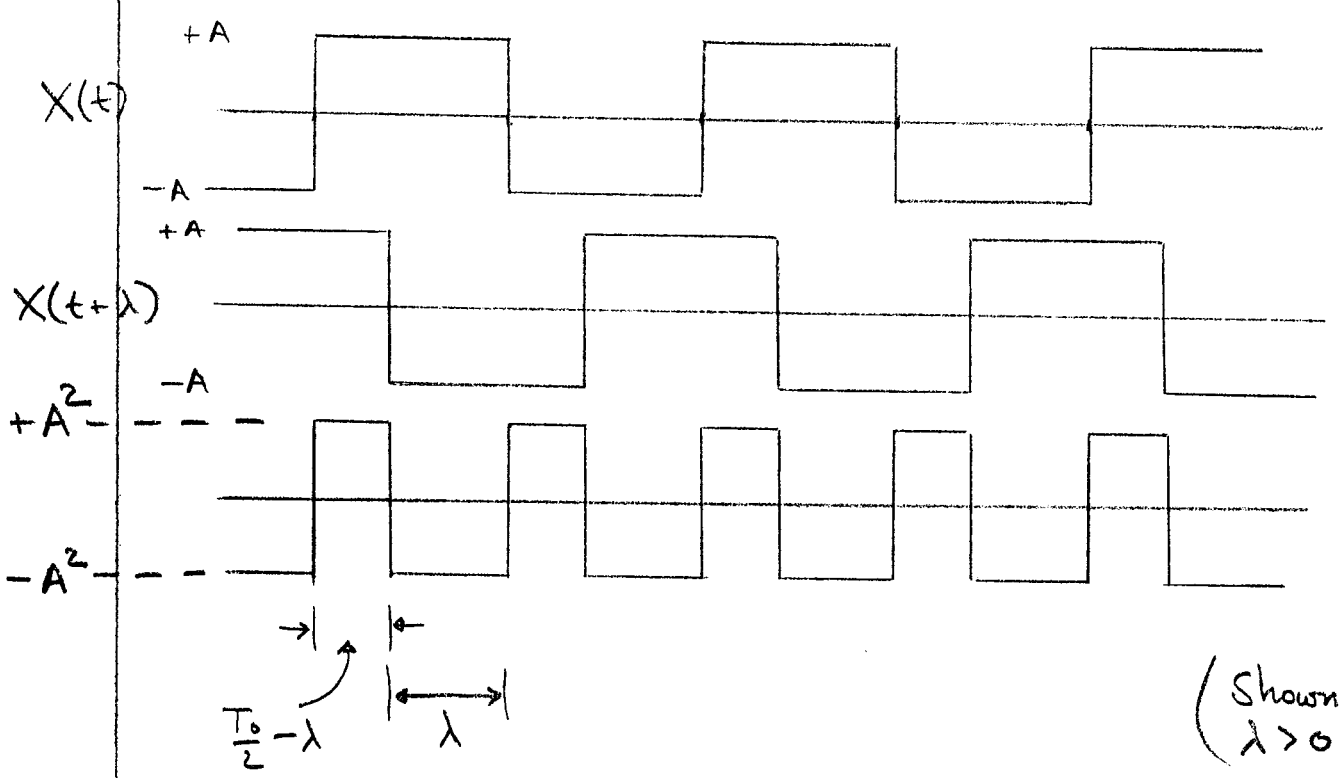
(5.5a) By inspection the time-average mean is zero since every sample function has mean zero.

For the time average auto correlation

$$R(\lambda) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,X(t+\lambda)\,dt$$

$$= \frac{1}{T_0}\int_{0}^{T_0} X(t)\,X(t+\lambda)\,dt \quad \text{since periodic}$$

We can do this integral over any period and therefore it will not depend on the particular sample functions value of τ. Thus, compute $R(\lambda)$ of the sample funct. where $\tau = 0$:



(Shown for $\lambda > 0$.)

Computing the average:

$$R(\lambda) = \frac{2}{T_0}\left[-A^2\lambda + A^2\left(\frac{T_0}{2} - \lambda\right)\right]$$

$$= A^2\left[1 - \frac{4\lambda}{T_0}\right] \qquad 0 \le \lambda \le T_0/2$$

By symmetry

$$R(\lambda) = \begin{cases} A^2\left[1 - \frac{4|\lambda|}{T_0}\right] & 0 \le |\lambda| \le T_0/2 \\ \text{periodically extend} & |\lambda| > T_0/2 \end{cases}$$

5.5b Ensemble mean is just the mean of the pmf found in 5.3b

$$\therefore EX(0) = EX(t) = 0$$

To compute

$$E\{X(t)X(t+\lambda)\}$$

we need the joint pmf of the rvs $X(t)$ and $X(t+\lambda)$. It is no loss of generality to set $t=0$.

This part of the problem is a little confusing and yet it is very important to understand the ensemble average calculation, which is why we take the trouble rather than simply leaning on the time average calculation.

Note $X(0)X(\lambda)$ is a discrete random variable taking values

$$+A^2 \quad \text{and} \quad -A^2$$

The correlation is

$$R(\lambda) = E\{X(0)X(\lambda)\}$$
$$= +A^2 \Pr(X(0)X(\lambda) = +A^2) - A^2 \Pr(X(0)X(\lambda) = -A^2)$$

hence it comes down to calculating the probabilities.

In the probability calculations the basic rv is τ, which is unif dist on $[-T_0/2, T_0/2]$. In addition, we compute $R(\lambda)$ for $0 \le \lambda \le T_0/2$ recognizing that it will be periodic in λ of period T_0 <u>and</u> symm. about $\lambda = 0$.

<u>Re: $\Pr(X(0)X(\lambda) = +A^2)$</u>

$$= \Pr(X(0)X(\lambda) = A^2 \mid X(0) = A)\Pr(X(0) = A)$$

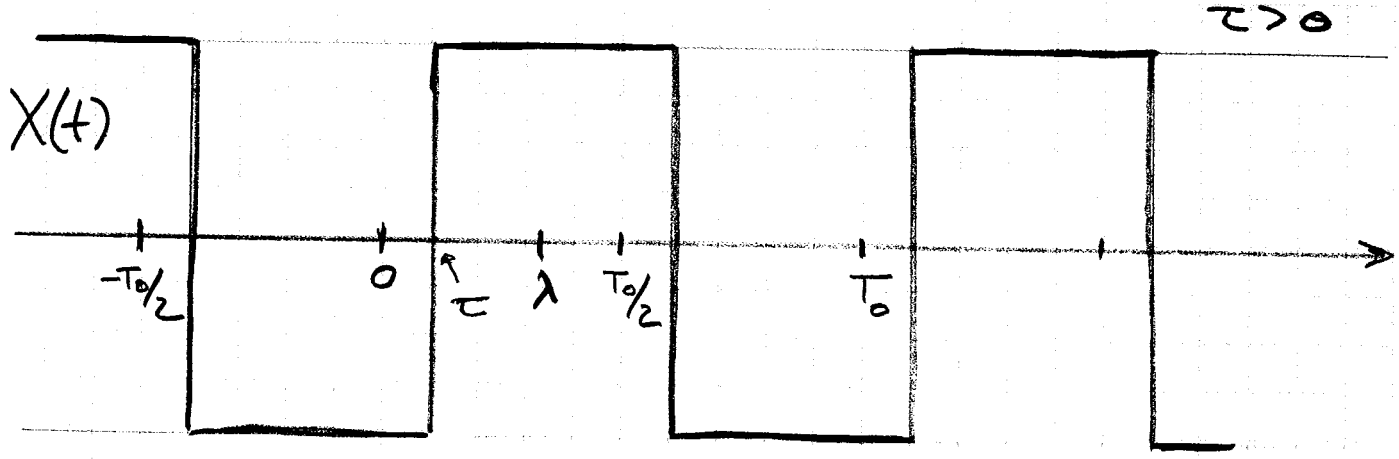
$$+ \Pr(X(0)X(\lambda) = A^2 \mid X(0) = -A)\Pr(X(0) = -A)$$

$$= \Pr(X(\lambda) = A \mid X(0) = A)\tfrac{1}{2} + \Pr(X(\lambda) = -A \mid X(0) = -A)\tfrac{1}{2}$$

Need to translate this conditional prob. into a statement about rv. τ.

Recall the picture:

Drawn for $\tau > 0$



Then can see

$$\{X(0) = +A\} = \{-T_0/2 < \tau < 0\}$$
$$\{X(0) = -A\} = \{0 < \tau < T_0/2\}$$

Now consider λ in the range $0 \le \lambda \le T_0/2$ and the r.v. $X(\lambda)$

$$\{X(\lambda) = +A\} = \{\lambda - T_0/2 < \tau < \lambda\}$$

$$\{X(\lambda) = -A\} = \left\{-\frac{T_0}{2} < \tau < \lambda - T_0/2\right\} \cup \{\lambda < \tau < T_0/2\}$$

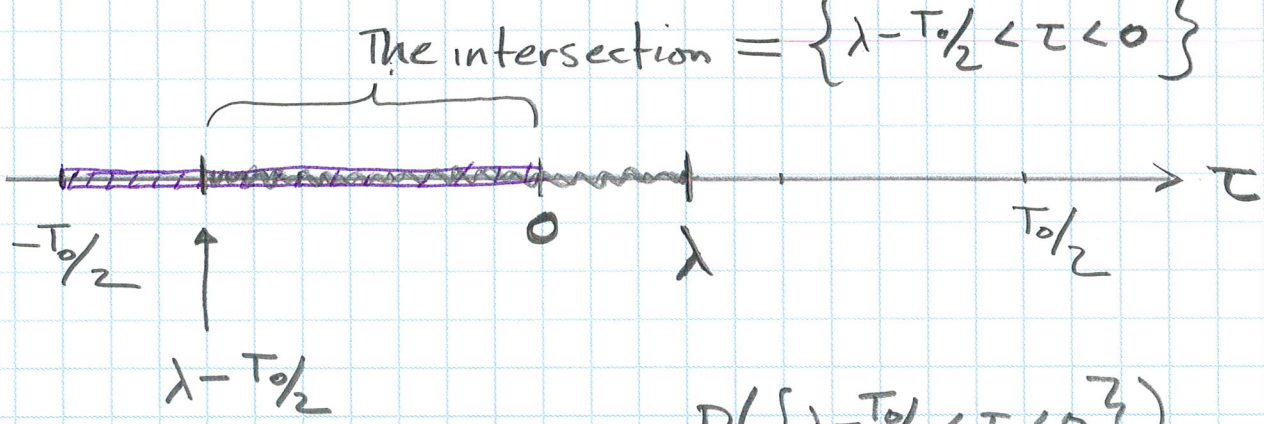
Then calculating the conditional probability

$$P(X(\lambda) = +A \mid X(0) = A)$$

$$= P\left(\left\{\lambda - \frac{T_0}{2} < \tau < \lambda\right\} \middle| \left\{-\frac{T_0}{2} < \tau < 0\right\}\right)$$

$$= \frac{P\left(\left\{\lambda - \frac{T_0}{2} < \tau < \lambda\right\} \cap \left\{-\frac{T_0}{2} < \tau < 0\right\}\right)}{P\left(\left\{-\frac{T_0}{2} < \tau < 0\right\}\right)}$$

To visualize the intersection draw another picture...



$$\therefore P(X(\lambda) = +A \mid X(0) = +A) = \frac{P(\{\lambda - T_0/2 < \tau < 0\})}{P(\{-T_0/2 < \tau < 0\})}$$

$$= \frac{(T_0/2 - \lambda)\frac{1}{T_0}}{(T_0/2)\frac{1}{T_0}}$$

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$$\Pr\left(X(\lambda)=A \mid X(0)=A\right) = \frac{\frac{1}{2}-\frac{\lambda}{T_0}}{\frac{1}{2}}$$

$$= 1-\frac{2\lambda}{T_0}$$

$$= \Pr\left(X(\lambda)=-A \mid X(0)=-A\right)$$

A symm. arg. will show

$\therefore$ Have the first part of $R(\lambda)$.

<u>Re: $\Pr\left(X(0)X(\lambda)=-A^2\right)$</u>

$$= \Pr\left(X(0)X(\lambda)=-A^2 \mid X(0)=A\right)\Pr\left(X(0)=A\right)$$
$$+ \Pr\left(X(0)X(\lambda)=-A^2 \mid X(0)=-A\right)\Pr\left(X(0)=-A\right)$$

$$= \Pr\left(X(\lambda)=-A \mid X(0)=A\right)\tfrac{1}{2} + \Pr\left(X(\lambda)=+A \mid X(0)=-A\right)\tfrac{1}{2}$$

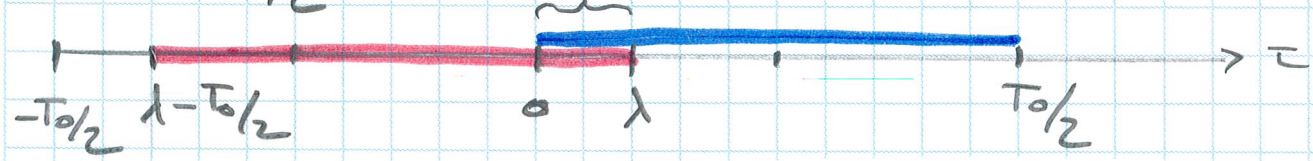
Same pictures/event definitions hold as in prev. case.

$$P\left(X(\lambda)=+A \mid X(0)=-A\right)$$

$$= \frac{P\left(\left\{\lambda-\frac{T_0}{2}<\tau<\lambda\right\}\cap\left\{0<\tau<T_0/2\right\}\right)}{P\left(\left\{0<\tau<T_0/2\right\}\right)}$$

$$= \frac{\lambda/T_0}{1/2} = 2\lambda/T_0$$

the intersection



By symm.

$$\Pr\left(X(\lambda) = -A \mid X(0) = +A\right) = \frac{2\lambda}{T_0}$$

Putting it all together:

$$R(\lambda) = A^2\left[\frac{1}{2}\left(1 - \frac{2\lambda}{T_0}\right) + \frac{1}{2}\left(1 - \frac{2\lambda}{T_0}\right)\right]$$

$$- A^2\left[\frac{1}{2}\,\frac{2\lambda}{T_0} + \frac{1}{2}\,\frac{2\lambda}{T_0}\right]$$

$$= A^2\left(1 - \frac{2\lambda}{T_0}\right) - A^2\,\frac{2\lambda}{T_0}$$

$$= A^2\left(1 - \frac{4\lambda}{T_0}\right)$$

5.5c Yes. It is WSS.

6.19. An ideal finite-time integrator is characterized by the input-output relationship

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\alpha)\,d\alpha$$

a. Justify that its impulse response is $h(t) = \frac{1}{T}[u(t) - u(t-T)]$.

b. Obtain its frequency response function. Sketch it.

c. The input is white noise with two-sided power spectral density $N_0/2$. Find the power spectral density of the output of the filter.

d. Show that the autocorrelation function of the output is

$$R_0(\tau) = \frac{N_0}{2T}\Lambda(\tau/T)$$

where $\Lambda(x)$ is the unit-area triangular function defined in Chapter 2.

e. What is the equivalent noise bandwidth of the integrator?

f. Show that the result for the output noise power obtained using the equivalent noise bandwidth found in part (e) coincides with the result found from the autocorrelation function of the output found in part (d).

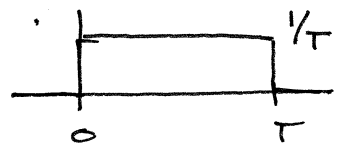
$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\alpha)\, d\alpha$$

(a) Let $X(t) = \delta(t)$. Then the output is the impulse response

$$h(t) = \frac{1}{T}\int_{t-T}^{t} \delta(\alpha)\, d\alpha = \begin{cases} \frac{1}{T} & t-T < 0 < t \\ 0 & \text{else} \end{cases}$$

Examine the condition:

$$\begin{aligned} t-T < 0 < t &\iff -T < -t < 0 \\ &\iff T > t > 0 \\ &\iff 0 < t < T \end{aligned}$$

$\therefore\ h(t) =$ 

$$= \frac{1}{T}\left[u(t) - u(t-T)\right]$$

(b)
$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi f t}\, dt = \frac{1}{T}\int_{0}^{T} e^{-j2\pi f t}\, dt$$

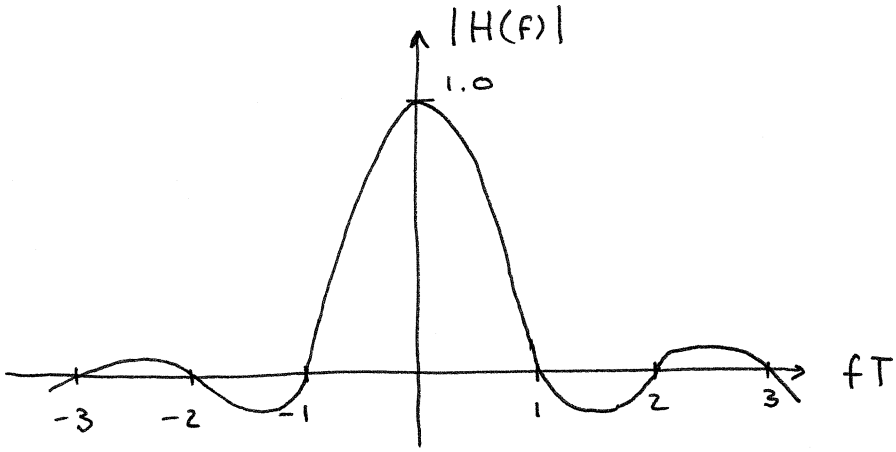
$$= \frac{1}{T}\,\frac{1}{-j2\pi f}\, e^{-j2\pi f t}\Big|_{0}^{T}$$

$$= \frac{1}{j2\pi f T}\left[1 - e^{-j2\pi f T}\right]$$

$$= \frac{e^{+j\pi f T} - e^{-j\pi f T}}{j2}\,\frac{1}{\pi f T}\, e^{-j\pi f T}$$

$$= \frac{\sin(\pi f T)}{\pi f T}\, e^{-j\pi f T} = \operatorname{sinc}(fT)\, e^{-j\pi f T}$$

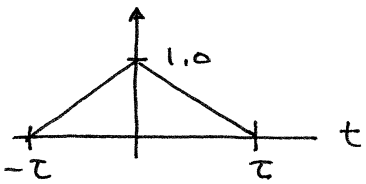
TOPS. 35500



(c) Output psd is

$$S_o(f) = |H(f)|^2 \frac{N_o}{2} = \frac{N_o}{2} \operatorname{sinc}^2(fT)$$

(d) The autocorrelation of the output is the inverse F.T. of the psd. From a table of FTs:



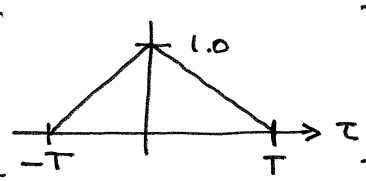
$$\longleftrightarrow \quad \tau \operatorname{sinc}^2(f\tau)$$

Therefore since

$$S_o(f) = \frac{N_o}{2} \frac{1}{T} \left(T \operatorname{sinc}^2(fT) \right)$$

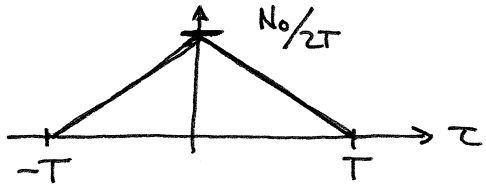
$$\updownarrow$$

$$R_o(\tau) = \frac{N_o}{2} \frac{1}{T} \cdot \left\{ \right.$$

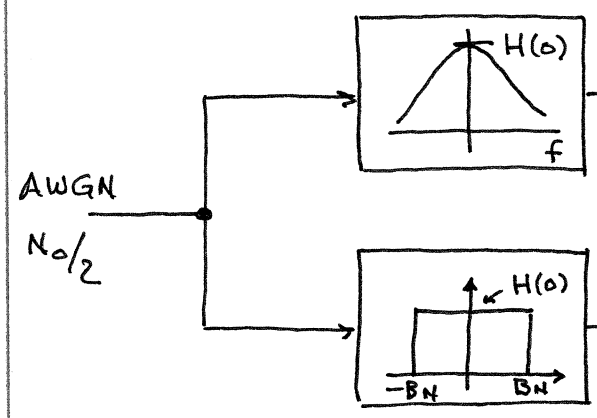


$$\left. \right\}$$

$$=$$



(e) For a lowpass filter $H(f)$ such as we have here the noise equivalent BW B_N is the bandwidth of an ideal lowpass filter of gain $H(0)$ which has the same noise power in its output as does the original LP filter when both are driven by AWGN.



$$\text{power} = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df$$

$$\text{power} = \frac{N_0}{2}\cdot 2B_N\,|H(0)|^2 = N_0 B_N\,|H(0)|^2$$

$$\Longrightarrow \quad B_N = \frac{1}{2}\,\frac{1}{|H(0)|^2}\int_{-\infty}^{\infty} |H(f)|^2\, df$$

$$= \frac{1}{|H(0)|^2}\int_{0}^{\infty} |H(f)|^2\, df \qquad \text{(if mag. resp. of } H \text{ is symmetric)}$$

Now in the case at hand $|H(0)| = 1.0$ but nobody wants to directly evaluate

$$\int_{-\infty}^{\infty} |H(f)|^2\, df$$

for sinc function H. Better to use Parseval:

$$\int_{-\infty}^{\infty} |H(f)|^2\, df = \int_{-\infty}^{\infty} |h(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} \left(\text{[rectangle of height } 1/T \text{ on } 0 \le t \le T\text{]} \right)^2 dt = \left(\frac{1}{T}\right)^2 \cdot T = \frac{1}{T}$$

$$B_N = \frac{1}{|H(0)|^2} \int_0^\infty |H(f)|^2 \, df$$

$$= \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{T} = \frac{1}{2T} \text{ Hz.}$$

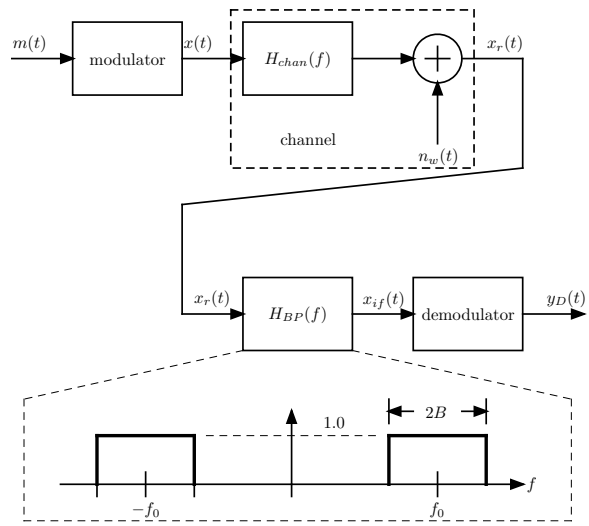
(f) From N.E.B. power $= N_0 \frac{1}{2T} = \frac{N_0}{2T}$

From $R_0(\tau)\Big|_{\tau=0} = \frac{N_0}{2T}$

Of course they agree (Lame question).

TOPS. 35500

Problem 26 A general model for passband communications is based on the block diagram shown below:



The message $m(t)$ will be modeled as a WSS random process with autocorrelation/psd $R_m(\tau) \leftrightarrow S_m(f)$ and bandlimited

$$S_m(f) = 0 \text{ for } f > W.$$

The noise $n_w(t)$ is also modeled as a WSS random process which is Gaussian, white, i.e.,

$$\begin{aligned} R_{n_w}(\tau) &= \frac{N_0}{2}\delta(\tau) \\ &\updownarrow \\ S_{n_w}(f) &= \frac{N_0}{2} \text{ for } -\infty < f < \infty, \end{aligned}$$

and statistically independent of $m(t)$ and any other random parameters of the communications model. For the purposes of this problem we will also assume that $H_{chan}(f) = 1$.

The purpose of this problem is to compute the signal-to-noise (SNR) power ratio at IF (i.e., associated with $x_{if}(t)$ in the block diagram). Since $m(t)$ is low-pass, the modulated signal $x(t)$ will be bandpass. In addition, we will assume that the bandpass filter $H_{BP}(f)$ in the block diagram passes the signal part without distortion, i.e., $x * h_{BP}(t) = x(t)$. Therefore,

$$\begin{aligned} x_{if}(t) &= x * h_{BP}(t) + n_w * h_{BP}(t) \\ &= x(t) + n(t) \end{aligned}$$

where $n(t)$ is a bandpass Gaussian noise with zero mean and psd of height $N_0/2$ and shape identical to that of $H_{BP}(f)$.

Then the SNR at IF is defined to be

$$\begin{aligned}\mathrm{SNR}_{if} &\stackrel{\text{def}}{=} \frac{\text{power}[x(t)]}{\text{power}[n(t)]} \\ &= \frac{R_x(0)}{2N_0B}\end{aligned}$$

assuming $x(t)$ is WSS and we can find its autocorrelation and/or psd.

(a) For AM DSB-SC assume

$$x(t) = A_c m(t)\cos(2\pi f_c t + \Theta)$$

where Θ is uniform on $[0, 2\pi)$ and statistically independent of message and noise.

- Find the auto-correlation and psd of $x(t)$. For a representative shape for $S_m(f)$ sketch a plot of $S_x(f)$.
- Specify the BPF so that $x * h_{BP}(t) = x(t)$ and the noise $n(t)$ has minimum power.
- Compute SNR_{if}.

(b) Repeat (a) for AM LC

$$x(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t + \Theta).$$

(c) Repeat (a) for SSB

$$x(t) = 0.5A_c m(t)\cos(2\pi f_c t + \Theta) \pm\ 0.5A_c \hat{m}(t)\sin(2\pi f_c t + \Theta).$$

Solution:

Part (a): AM DSB-SC

$$x(t) = A_c m(t)\cos(2\pi f_c t + \Theta).$$

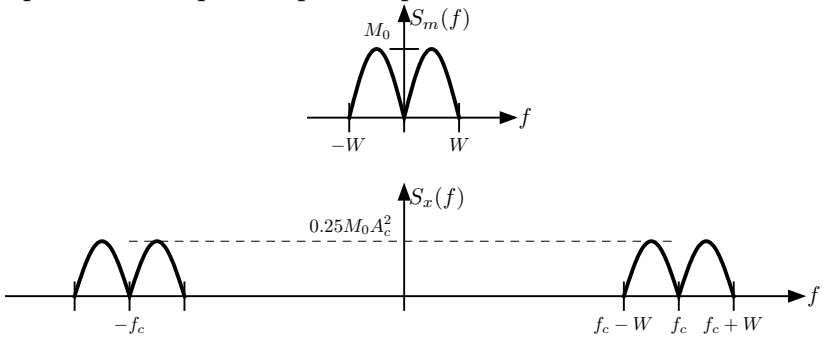
- Autocorrelation. As shown previously or by reference to the Modulation Theorem we have

$$R_x(\tau) = 0.5A_c^2 R_m(\tau)\cos(2\pi f_c \tau).$$

- Power spectrum. Then taking the Fourier Transform of the autocorrelation

$$S_x(f) = 0.25A_c^2\left[S_m(f-f_c) + S_m(f+f_c)\right].$$

A representative plot of power spectrum is shown below.



- From the psd plot it is clear that the BPF of minimum bandwidth, which just passes the signal is (we assume that $f_c > W > 0$)

$$H_{BP}(f) = \begin{cases} 1 & f_c - W \le f \le f_c + W \\ 1 & -f_c - W \le f \le -f_c + W \\ 0 & \text{otherwise} \end{cases}.$$

- Then we may compute the SNR at IF using the following steps
  * power$[x(t)] = R_x(0) = 0.5A_c^2 P_m$.
  * With the transmission BW of the IF BPF $= 2W \Rightarrow$ the minimum noise power $= 2N_0W$.
  * Taking the ratio:

$$\text{SNR}_{if} = \frac{A_c^2 P_m}{4N_0W}.$$

Part (b): AM-LC

$$x(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t + \Theta).$$

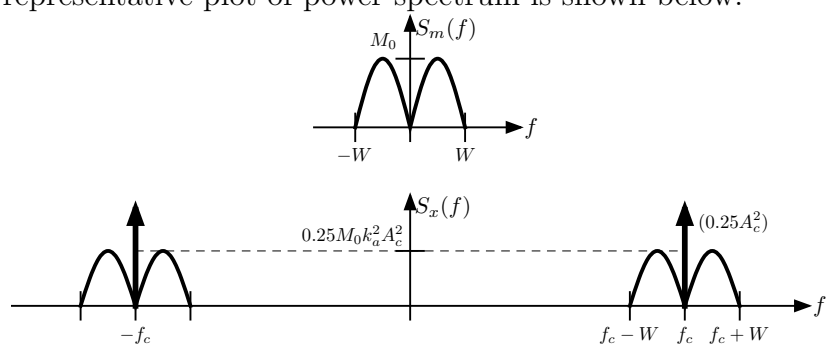
- Autocorrelation: In the same fashion as for AM-DSB we find

$$R_x(\tau) = 0.5A_c^2[1 + k_a^2 R_m(\tau)]\cos(2\pi f_c \tau).$$

- Power spectrum: Taking the Fourier Transform of the autocorrelation

$$\begin{aligned} S_x(f) &= 0.25A_c^2\left[\delta(f - f_c) + \delta(f + f_c)\right] \\ &\quad + 0.25k_a^2 A_c^2\left[S_m(f - f_c) + S_m(f + f_c)\right]. \end{aligned}$$

A representative plot of power spectrum is shown below.



- Since the psd above occupies the same band as was occupied by AM-DSB, the minimum BW BPF is identical to that used previously.

- Then we may compute the SNR at IF using the following steps

  * power$[x(t)] = R_x(0) = 0.5A_c^2(1 + k_a^2 P_m)$.
  * Transmission BW $= 2W \Rightarrow$ minimum noise power $= 2N_0W$.
  * Ratio:

$$\text{SNR}_{if} = \frac{A_c^2(1 + k_a^2 P_m)}{4N_0W}.$$

Part (c): SSB

$$x(t) = 0.5A_c m(t)\cos(2\pi f_c t + \Theta) \pm\ 0.5A_c \hat{m}(t)\sin(2\pi f_c t + \Theta)$$

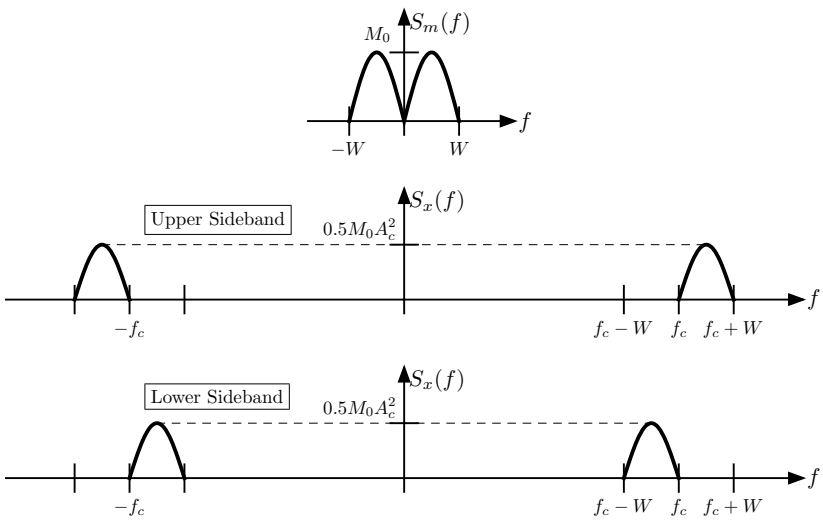
- Autocorrelation: ($\hat{R}_m$ is the HT of R_m)

$$R_x(\tau) = 0.25A_c^2 R_m(\tau)\cos(2\pi f_c \tau) \pm\ 0.25A_c^2 \hat{R}_m(\tau)\sin(2\pi f_c \tau).$$

- Power spectrum:

$$\begin{aligned} S_x(f) &= 0.125A_c^2\left[S_m(f - f_c) + S_m(f + f_c)\right] \\ &\quad \mp 0.125A_c^2\left[\text{sgn}(f - f_c)S_m(f - f_c)\right. \\ &\quad \left. - \text{sgn}(f + f_c)S_m(f + f_c)\right]. \end{aligned}$$

(In $\pm$ and $\mp$ above, top choice is for lower sideband SSB, bottom choice is for upper.) A representative plot of power spectrum is shown below.



- From the psd plot it is clear that the BPF of minimum bandwidth, which just passes the upper sideband signal is (we assume that $f_c > W > 0$)

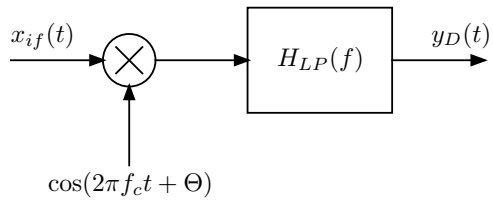
$$H_{BP}(f) = \begin{cases} 1 & f_c \leq f \leq f_c + W \\ 1 & -f_c - W \leq f \leq -f_c \\ 0 & \text{otherwise} \end{cases} .$$

- <u>Computing SNR at IF:</u>
  * power$[x(t)] = R_x(0) = 0.25A_c^2 P_m$.
  * Transmission BW $= W \Rightarrow$ minimum noise power $= N_0 W$.
  * Ratio:

$$\text{SNR}_{if} = \frac{A_c^2 P_m}{4N_0 W}.$$

Problem 27

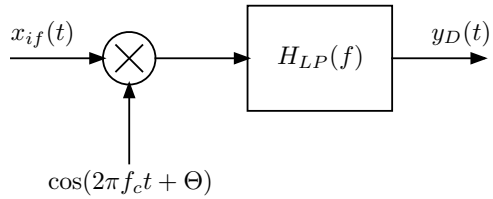
The three linear modulations AM DSB-SC, AM LC, and SSB have the identical coherent demodulator architecture shown in the figure below:



Use the standard assumptions on message, noise, and phase angle and compute the SNR at the demodulator output for the three linear modulations above.

Solution: For the three linear modulations the coherent demodulators are identical:

$$x_{if}(t) = x(t) + n(t).$$



<u>AM-DSB</u>

$$x(t) = A_c m(t) \cos(2\pi f_c t + \Theta).$$

- <u>Signal component of y_D:</u>

$$\begin{aligned} A_c m(t) \cos^2(2\pi f_c t + \Theta) &= 0.5 A_c m(t) \\ &\quad + 0.5 A_c m(t) \cos(2\pi(2f_c)t + 2\Theta). \end{aligned}$$

  * Passes LPF $\Rightarrow 0.5 A_c m(t)$.
  * Power in signal component: $0.25 A_c^2 P_m$.

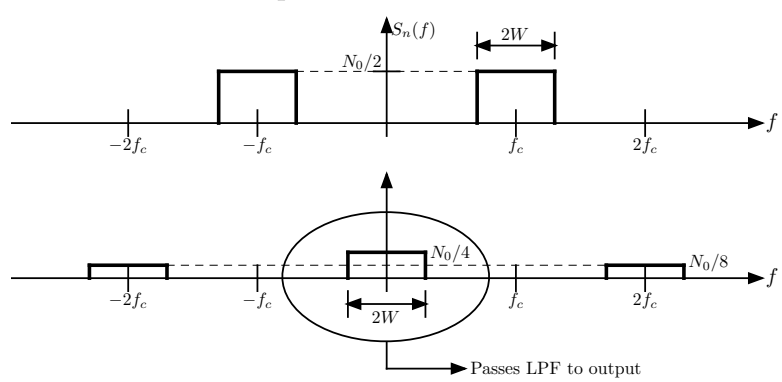
- <u>Noise component of y_D:</u>

$$n(t) \cos(2\pi f_c t + \Theta)$$

  * Above is WSS with

$$\begin{aligned} R(\tau) &= 0.5 R_n(\tau) \cos(2\pi f_c \tau) \\ &\updownarrow \\ S(f) &= 0.25 S_n(f - f_c) + 0.25 S_n(f + f_c) \end{aligned}$$

  * Power in noise component: $0.5 N_0 W$.



- <u>Demodulated SNR:</u>

$$\mathrm{SNR}_D = \frac{0.25 A_c^2 P_m}{0.5 N_0 W}.$$

AM-LC

$$x(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t + \Theta).$$

- Signal component of y_D:
  - * Passes LPF $\Rightarrow 0.5A_c[1 + k_a m(t)]$.
  - * Passes dc block $\Rightarrow 0.5A_c k_a m(t)$.
  - * Power in desired signal component: $0.25A_c^2 k_a^2 P_m$.
- Noise component of y_D:
  - * Identical to DSB case.
  - * Power in noise component: $0.5N_0W$.
- Demodulated SNR:

$$\text{SNR}_D = \frac{0.25A_c^2 k_a^2 P_m}{0.5N_0W}.$$

AM-SSB

$$x(t) = 0.5A_c m(t)\cos(2\pi f_c t + \Theta) \pm\ 0.5A_c \hat{m}(t)\sin(2\pi f_c t + \Theta)$$

- Signal component of y_D:

$$\begin{aligned} x(t)\cos(2\pi f_c t + \Theta) &= 0.5A_c m(t)\cos^2(2\pi f_c t + \Theta) \\ &\quad \pm 0.5A_c \hat{m}(t)\cos(2\pi f_c t + \Theta)\sin(2\pi f_c t + \Theta) \end{aligned}$$

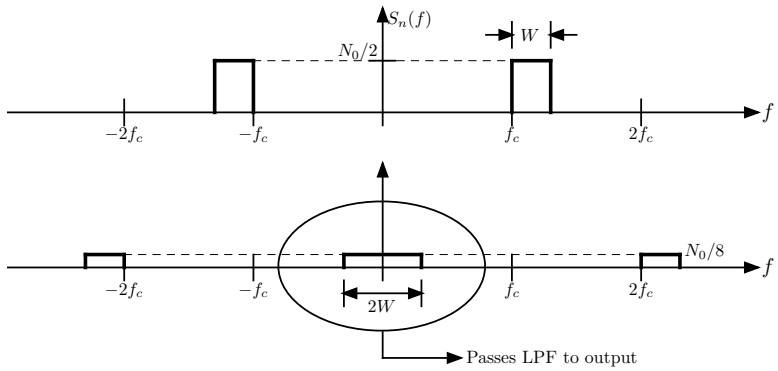
  - * Passes LPF $\Rightarrow 0.25A_c m(t)$.
  - * Power in signal component: $(0.25A_c)^2 P_m$.
- Noise component of y_D:

$$n(t)\cos(2\pi f_c t + \Theta)$$

  - * Above is WSS with

$$\begin{aligned} R(\tau) &= 0.5R_n(\tau)\cos(2\pi f_c \tau) \\ &\updownarrow \\ S(f) &= 0.25S_n(f - f_c) + 0.25S_n(f + f_c) \end{aligned}$$

  - * Power in noise component: $0.25N_0W$.



- <u>Demodulated SNR:</u>

$$\mathrm{SNR}_D = \frac{(0.25A_c)^2 P_m}{0.25 N_0 W}.$$