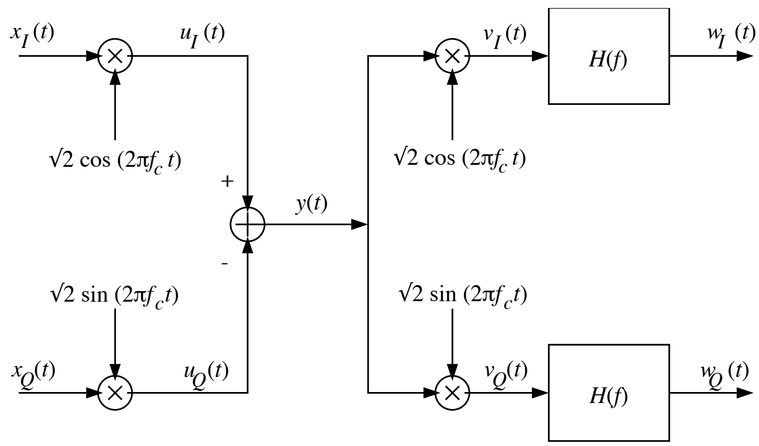
ECE 44000 (Fall 2022)
HW 1 (Probs. 1–4)
Due: 8/29/2022

Problem 1: Consider the following block diagram:



where

- The signals $x_I(t)$ and $x_Q(t)$ have continuous-time Fourier transforms $X_I(f)$ and $X_Q(f)$ which are both bandlimited: $X_I(f) = X_Q(f) = 0$ for $|f| > f_m$.
- $f_c \gg f_m$.
- The filter $H(f)$ is a low-pass filter:

$$H(f) = \begin{cases} 1 & |f| \le f_m \\ 0 & |f| > f_m \end{cases} .$$

Plot the Fourier transforms of $x_I(t)$, $\sqrt{2}\cos(2\pi f_c t)$, $u_I(t)$, $x_Q(t)$, $\sqrt{2}\sin(2\pi f_c t)$, $u_Q(t)$, $y(t)$, $v_I(t)$, $w_I(t)$, $v_Q(t)$, and $w_Q(t)$ when

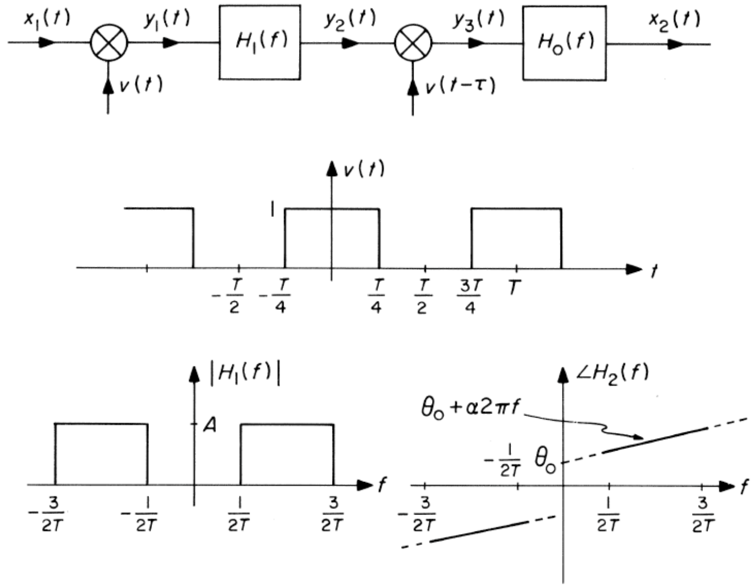
$$\begin{aligned} X_I(f) &= \begin{cases} \chi_I & |f| < f_m \\ 0 & \text{otherwise} \end{cases} \\ X_Q(f) &= \begin{cases} \chi_Q(1 - |f|/f_m) & |f| < f_m \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Problem 2: [From Siebert] The design of high-gain dc amplifiers (i.e., amplifiers whose pass band includes $f = 0$) frequently presents difficulties because small slow changes (due to aging, temperature fluctuations, supply voltage changes, etc.) in the quiescent operating points of the active elements can produce responses indistinguishable from those due to small desired signals. One way to circumvent these difficulties is to use a *chopper* to modulate the signal onto a carrier so that an ac coupled amplifier can be used instead. Another chopper is then used as a synchronous detector to restore the signal to its original frequency range. The scheme is illustrated in the figure below. Assume that the spectrum of the input signal $x_1(t)$ is restricted to $|f| < f_1$. The periodic time functions $v(t)$ and $v(t - \tau)$ describe the chopping action. $H_0(f)$ is an ideal lowpass filter with gain 1 over the pass band $|f| < f_1$; and $H_1(f)$ is a bandpass high-gain amplifier with characteristics shown.

(a) Find the first few terms of the Fourier series for $v(t)$ and $v(t - \tau)$.

(b) Sketch the spectra of $y_1(t)$ and $y_2(t)$ and label them accurately in terms of an assumed spectral shape for $X_1(f)$.

(c) Find an expression for $x_2(t)$ in terms of $x_1(t)$ and the system parameters.



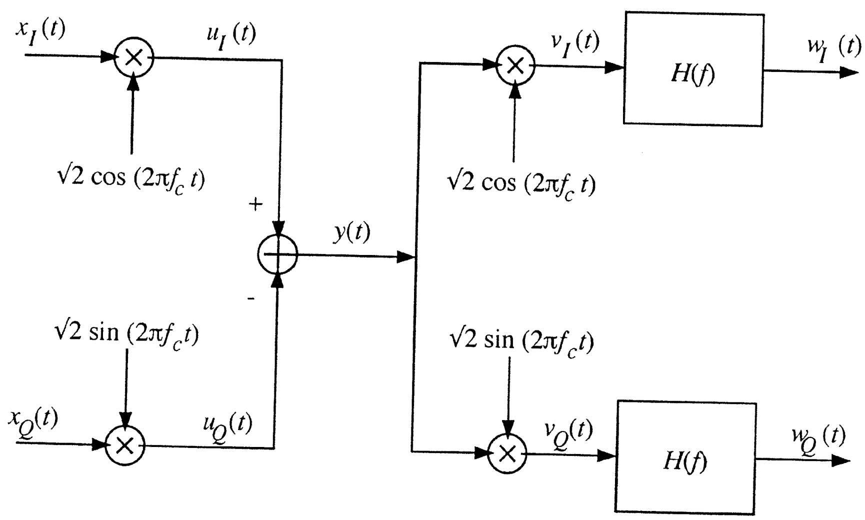
Problem 3: Z&T Computer Exercise 2.3 (page 111).

2.3. Write a computer program to evaluate the coefficients of the complex exponential Fourier series of a signal by using the FFT. Check it by evaluating the Fourier series coefficients of a square-wave and comparing your results with Computer Exercise 2.2.

Problem 4: Z&T Computer Exercise 2.4 (page 111).

2.4. How would you use the same approach as in Computer Exercise 2.3 to evaluate the Fourier transform of a pulse-type signal. How do the two outputs differ? Compute an approximation to the Fourier transform of a square pulse signal 1 unit wide and compare with the theoretical result.

Consider the following block diagram:



where

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Plot the Fourier transforms of $x_I(t)$, $\sqrt{2}\cos(2\pi f_c t)$, $u_I(t)$, $x_Q(t)$, $\sqrt{2}\sin(2\pi f_c t)$, $u_Q(t)$, $y(t)$, $v_I(t)$, $w_I(t)$, $v_Q(t)$, and $w_Q(t)$ when

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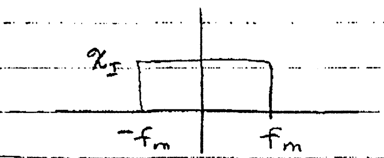
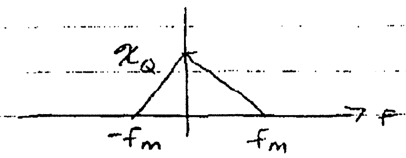
$|X_I(f)|$



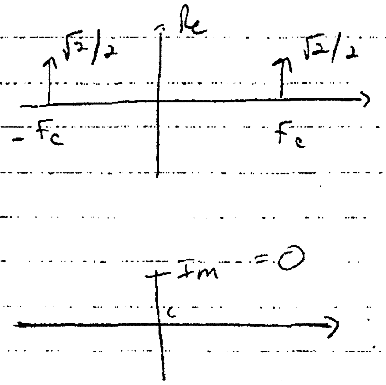
$|X_Q(f)|$



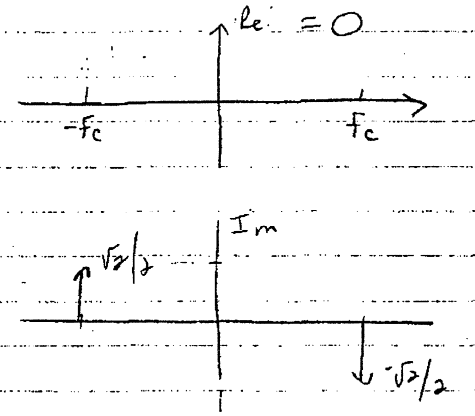
phases = 0

Can plot Re, Im, instead of mag, phase.

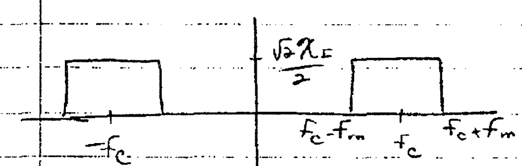
$\mathcal{F}\{\sqrt{2}\cos 2\pi f_c t\}$



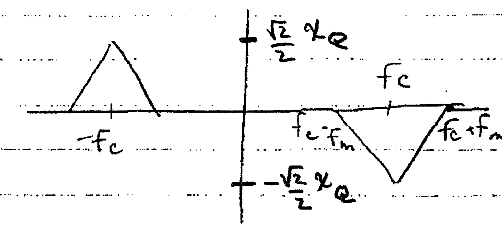
$\mathcal{F}\{\sqrt{2}\sin 2\pi f_c t\}$



$Re\{U_I(f)\}$



$Im\{U_Q(f)\}$



$Im\{U_I(f)\} = 0$

$Re\{U_Q(f)\} = 0$

Note:

$|j| = 1$

$\angle j = 90^\circ = \pi/2$

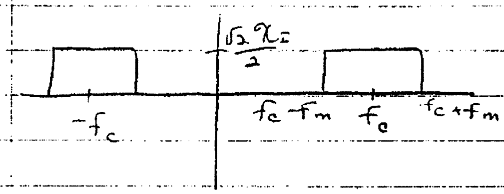
$\angle -j = -\pi/2$

(and multiplication in time-domain corresponds to convolution in freq. domain)

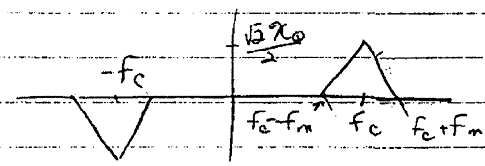
$y(t) = u_I(t) - u_Q(t)$

$Y(f)$: Is much clearer to plot $Re\{Y(f)\}$, $Im(Y(f))$ instead of $|Y(f)|$, $\measuredangle Y(f)$

$Re\{Y(f)\}$

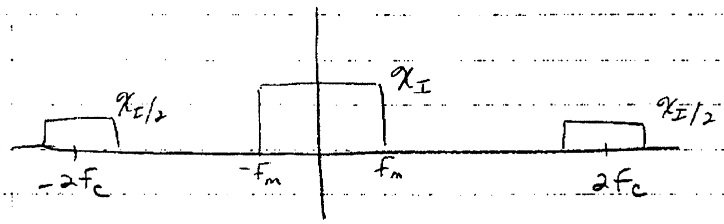


$Im\{Y(f)\}$

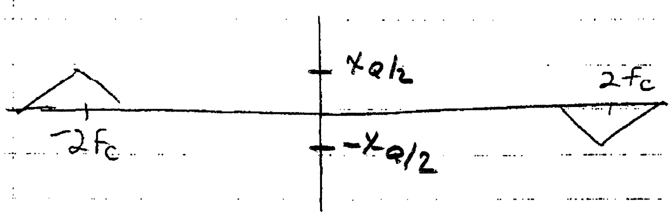


$U_I(f) = \frac{\sqrt{2}}{2}\left[Y(f - f_c) + Y(f + f_c)\right]$

$Re\{U_I(f)\}$



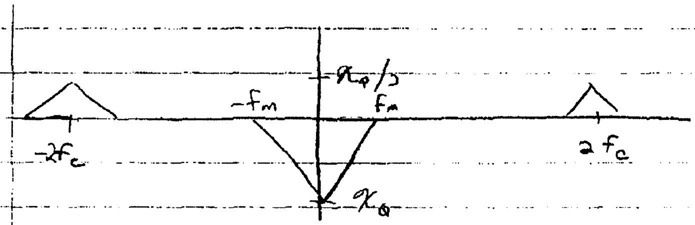
$Im\{U_I(f)\}$



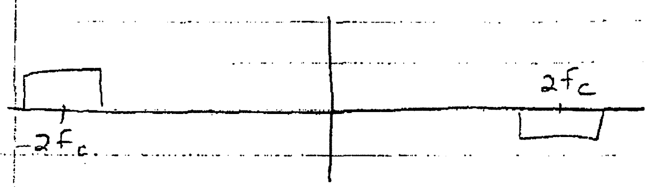
$Re\{V_Q(f)\}$

$$V_Q(f) = \frac{\sqrt{2}}{2j}\left[Y(f-f_c) - Y(f+f_c)\right]$$

$$= \frac{j\sqrt{2}}{2}\left[Y(f+f_c) - Y(f-f_c)\right]$$

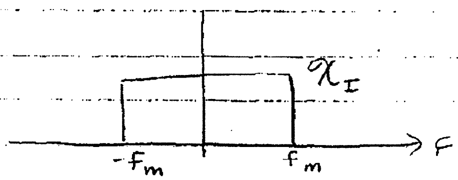


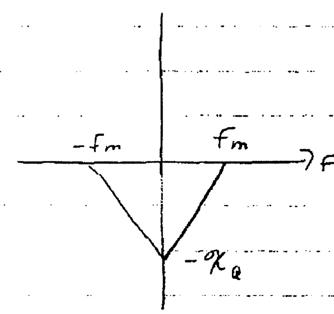
$Im\{V_Q(f)\}$



after LPF:

$Re\{W_I(f)\}$ $Re\{W_Q(f)\}$





$$Im\{W_I(f)\} = Im\{W_Q(f)\} = 0$$

The design of high-gain dc amplifiers (i.e., amplifiers whose pass band includes $f = 0$) frequently presents difficulties because small slow changes (due to aging, temperature fluctuations, supply voltage changes, etc.) in the quiescent operating points of the active elements can produce responses indistinguishable from those due to small desired signals. One way to circumvent these difficulties is to use a *chopper* to modulate the signal onto a carrier so that an ac coupled amplifier can be used instead. Another chopper is then used as a synchronous detector to restore the signal to its original frequency range. The scheme is illustrated in the figure below. Assume that the spectrum of the input signal $x_1(t)$ is restricted to $|f| < f_1$. The periodic time functions $v(t)$ and $v(t-\tau)$ describe the chopping action. $H_0(f)$ is an ideal lowpass filter with gain 1 over the pass band $|f| < f_1$; and $H_1(f)$ is a bandpass high-gain amplifier with characteristics shown.

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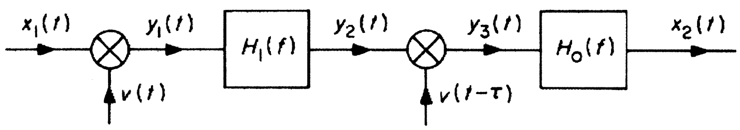
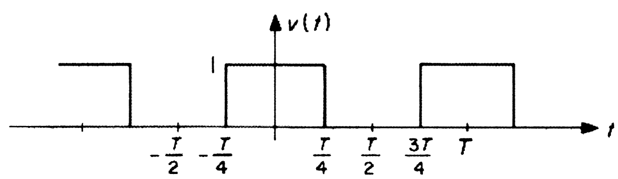
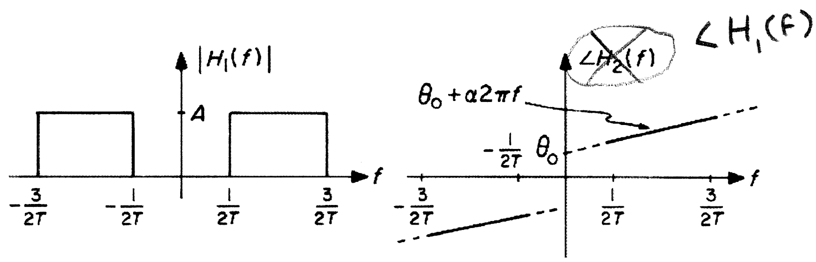






ⓐ Fourier Series for $v(t)$ and $v(t-\tau)$

$$V_k = \frac{1}{T}\int_{-T/4}^{T/4} e^{-j2\pi kt/T}\,dt \qquad \text{fund. freq.} = 1/T$$

$$= \frac{1}{T}\frac{1}{-j2\pi k/T} e^{-j2\pi kt/T}\Big|_{t=-T/4}^{T/4} = \frac{-1}{j2\pi k}\left[e^{-j\pi k/2} - e^{+j\pi k/2}\right]$$

$$= \frac{1}{\pi k}\sin \pi k/2 = \frac{1}{2}\,\mathrm{sinc}\,(k/2)$$

Can also write the F.S. for $v(t-\tau)$ in terms of that for $v(t)$.

$$v(t) = \sum_k V_k e^{j2\pi kt/T} \Rightarrow v(t-\tau) = \sum_k V_k e^{j2\pi k(t-\tau)/T}$$

$$= \sum_k V_k e^{-j2\pi k\tau/T} e^{j2\pi kt/T}$$

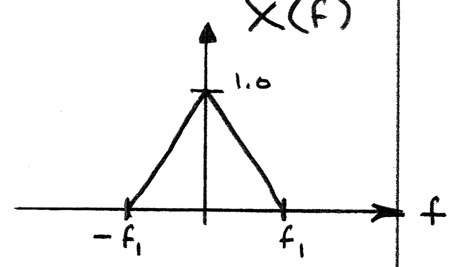
$$\therefore \quad v(t) \longleftrightarrow V_k = \tfrac{1}{2}\operatorname{sinc}(k/2)$$

$$v(t-\tau) \longleftrightarrow V_k e^{-j2\pi k\tau/T} = \tfrac{1}{2}\operatorname{sinc}(k/2)\, e^{-j2\pi k\tau/T}$$

(b) To get the continuous spectra for $y_1(t)$ and $y_2(t)$ we need to convert the F.S. for $v(t)$ to a F.T. Thus

$$V(f) = \sum_k V_k \,\delta(f - k/T)$$

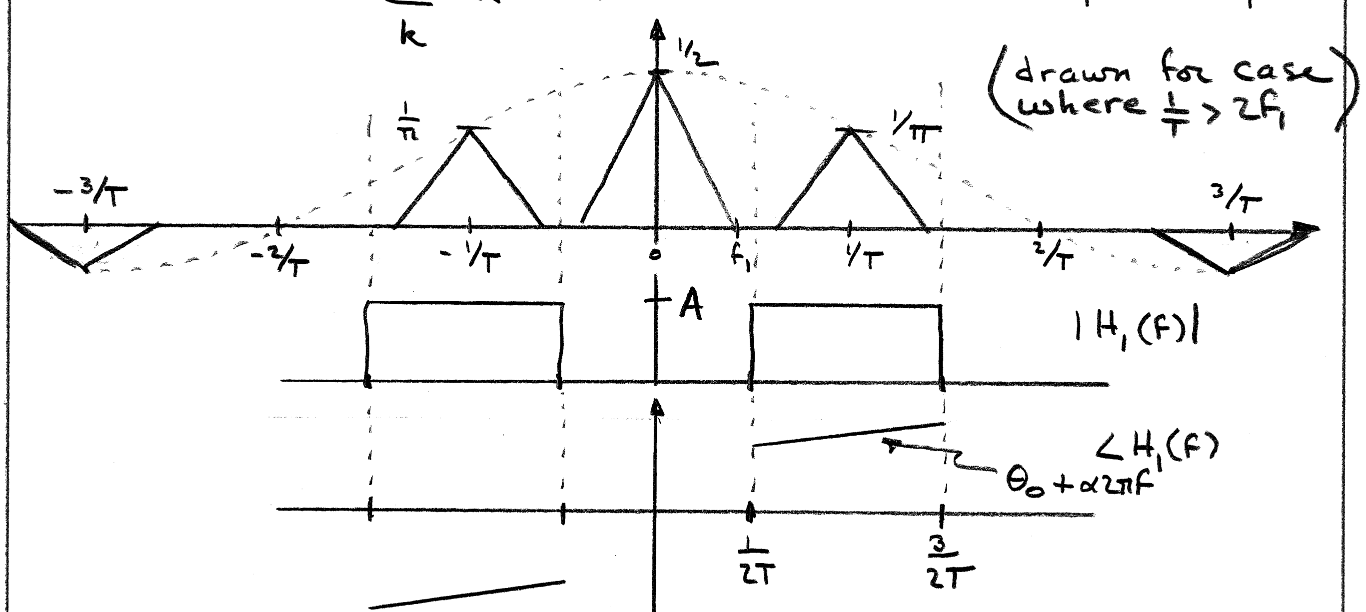
Then $y_1(t) = x(t)\,v(t)$. Since mult. in time corresp. to convolution in freq. we have



$$Y_1(f) = X * V(f)$$

$$= \sum_k V_k X(f - k/T)$$

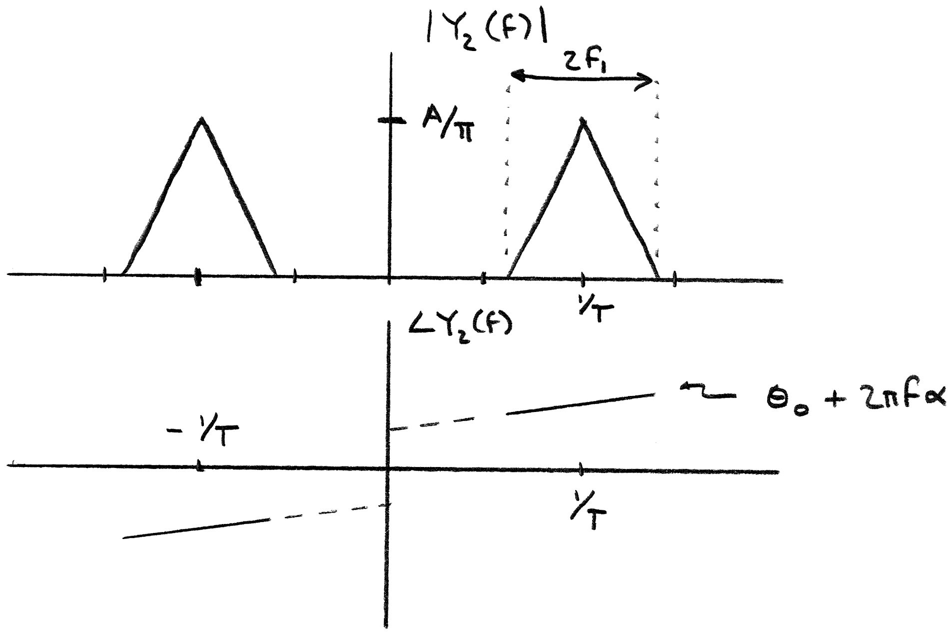
(drawn for case where $\frac{1}{T} > 2f_1$).



Therefore

$$Y_2(f) = AV_1\, X(f - 1/T)\, e^{+j(\theta_0 + 2\pi f \alpha)}$$
$$+ AV_{-1}\, X(f + \tfrac{1}{T})\, e^{+j(-\theta_0 + 2\pi f\alpha)}$$

$$= \frac{A}{\pi} X(f - \tfrac{1}{T})\, e^{+j\theta_0} e^{j2\pi f \alpha}$$
$$+ \frac{A}{\pi} X(f + \tfrac{1}{T})\, e^{-j\theta_0} e^{j2\pi f\alpha}$$



c) $y_3(t) = y_2(t)\, v(t - \tau)$

$$Y_3(f) = Y_2(f) * \sum_k V_k e^{-j2\pi k\tau/T} \delta(f - k/T)$$

$$= \sum_k V_k e^{-j2\pi k \tau/T}\, Y_2(f - k/T).$$

Only the $k = \pm 1$ terms in the above sum will fall into the passband of $H_0(f)$.

$$X_2(f) = H_0(f)\,Y_3(f)$$

$$= H_0(f)\left\{ V_1 e^{-j2\pi\tau/T} Y_2(f - 1/T) + V_{-1} e^{+j2\pi\tau/T} Y_2(f + 1/T) \right\}$$

Each Y_2 contains two terms coming from X, but only one will fall in the passband of H_0. Therefore

$$X_2(f) = \frac{A}{\pi^2} e^{-j2\pi\tau/T} X(f) e^{-j\theta_0} e^{j2\pi(f-1/T)\alpha} + \frac{A}{\pi^2} e^{+j2\pi\tau/T} X(f) e^{+j\theta_0} e^{j2\pi(f+1/T)\alpha}$$

$$= \frac{A}{\pi^2} X(f) e^{j2\pi f\alpha}\left[e^{+j(\theta_0 + 2\pi(\tau+\alpha)/T)} + e^{-j(\theta_0 + 2\pi(\tau+\alpha)/T)} \right]$$

$$= \frac{2A}{\pi^2} \cos\left[2\pi(\tau+\alpha)/T + \theta_0\right] X(f) e^{j2\pi f\alpha}$$

Would normally pick the delay τ s.t. the cosine term is equal to one. Note that a practical filter H_1 would have θ_0 and α negative so the desired delay τ would be positive as expected.

Then $x_2(t)$ is an amplified and delayed version of $x_1(t)$ as desired.

Z+T Sixth Edition, Computer Exercise 2.3, p. 110

2.3. Write a computer program to evaluate the coefficients of the complex exponential Fourier series of a signal by using the FFT. Check it by evaluating the Fourier series coefficients of a square-wave and comparing your results with Computer Exercise 2.2.

A periodic time function $x(t) = x(t+T_0)\ \forall t$ has a Fourier series with Fourier coefficients given by

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi k t/T_0}\, dt \qquad k \in \mathbb{Z}$$

Suppose we discretize the integral by sampling the integrand and summing the samples.

Let $t = nT$ where the sampling interval is a divisor of T_0 ie $T_0 = KT$.

Define

$$\tilde{X}_k = \frac{1}{KT}\sum_{n=0}^{K-1} x(nT)\, e^{-j2\pi k n T/T_0} \cdot T$$

$$(*) \quad = \frac{1}{K}\sum_{n=0}^{K-1} x(nT)\, e^{-j2\pi kn/K} = \frac{1}{K}\sum_{n=0}^{K-1} x(nT_0/K)\, e^{-j2\pi kn/K}$$

Since this discretization of the original integral is a Riemann sum approximation to the integral we know that for any particular fixed index k:

$$\lim_{K\to\infty} \tilde{X}_k = X_k$$

But it will turn out that the size of K needed to make $\tilde{X}_k \approx X_k$ will in general depend on k. This is obviously true because the approximating set of F.S. coeffs $\tilde{X}_k$ is periodic in k with period K while the exact seq. of F.S. coeffs X_k is not periodic.

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For a fixed K

$$\tilde{X}_k = \tilde{X}_{k+K} \qquad \forall\, k \in \mathbb{Z}$$

Therefore

$$\tilde{X}_{-1} = \tilde{X}_{K-1}$$

$$\tilde{X}_{-2} = \tilde{X}_{K-2}$$

$$\vdots$$

$$\tilde{X}_{-M} = \tilde{X}_{K-M}$$

Now supposing we want to approximately compute $2M+1$ of the F.S. coeffs

$$X_k \qquad -M \leq k \leq M$$

Then we would choose $K \geq 2M+1$ (perhaps need K even larger), calculate $\tilde{X}_k$ for $0 \leq k \leq K-1$ from (*) and choose

$$X_k \approx \tilde{X}_k \qquad 0 \leq k \leq M$$

$$X_{-k} \approx \tilde{X}_{K-k} \qquad 1 \leq k \leq M$$

Now (*) can be computed using the DFT/FFT but have to be a little bit careful.

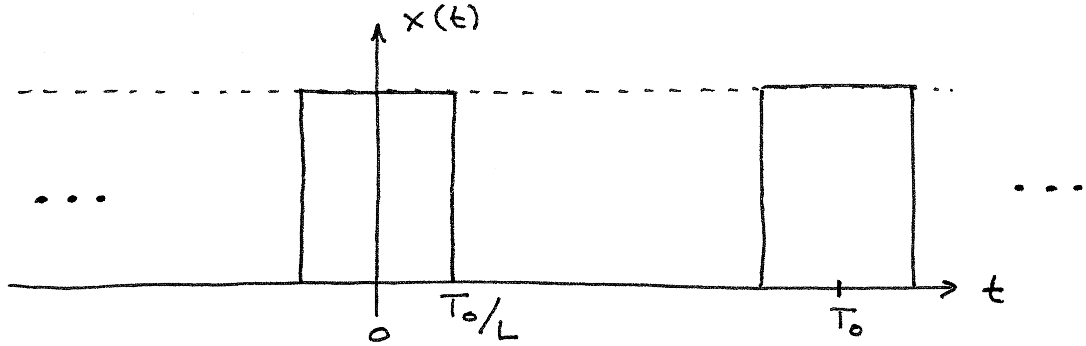
Recall the definition of the K-point DFT

- Forward $\quad G_k = \sum_{n=0}^{K-1} g_n e^{-j2\pi kn/K} \qquad 0 \leq k \leq K-1$
- Inverse $\quad g_n = \frac{1}{K} \sum_{k=0}^{K-1} G_k e^{+j2\pi kn/K} \qquad 0 \leq n \leq K-1$

Therefore (*) is a scaled Forward K-point DFT:

$$\tilde{X}_k = \frac{1}{K} \cdot \left\{ \begin{array}{l} \text{Forward K-point DFT} \\ \text{of } x(nT_0/K) \end{array} \right\}$$

Now we are supposed to work this out for a square wave. We did a square wave example in class, which we can generalize.

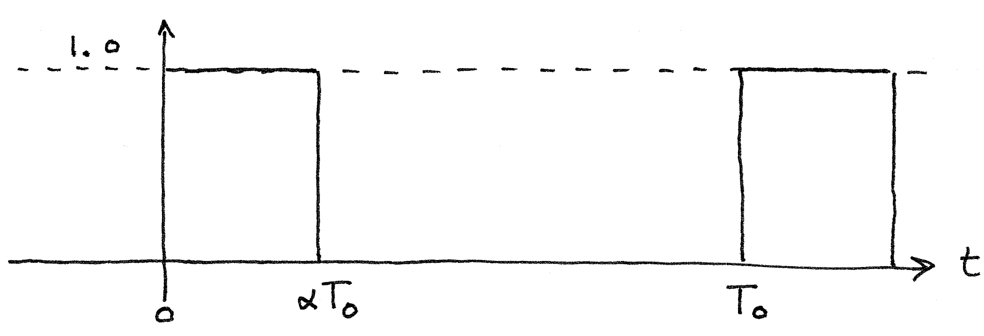


$L \geq 2$

For the case $L = 4$ the result from class was that

$$X_k = \begin{cases} 1/2 & k = 0 \\ 0 & k \text{ even, } k \neq 0 \\ \dfrac{(-1)^{(k-1)/2}}{\pi k} & k \text{ odd} \end{cases}$$

Because its slightly simpler to code I'll actually use the following periodic waveform (note that time shifts only change the phase of the F.S. coeffs and not their magnitudes and I'm mainly interested in the mag.).



α is the duty cycle of the pulse train: $0 < \alpha < 1$.

```
%EE 440 Spring 2010, Problem 9

K = 1024;                           %K pt. DFT
alpha = 0.5;                        %Square wave duty cycle
M = 50;                             %Compute 2*M+1 FS
coeffs
N = floor(alpha*K);                 %Number of nonzero time
samps
x = zeros(1,K);
x(1:N) = ones(1,N);

X = fft(x)/K;                       %Create FS coeffs
X_coeffs = [X(K-M+1:K) X(1:M+1)];   %Find the actual FS
coeff ests
index = -M:1:M;                     %For plotting

stem(index,abs(X_coeffs))
```

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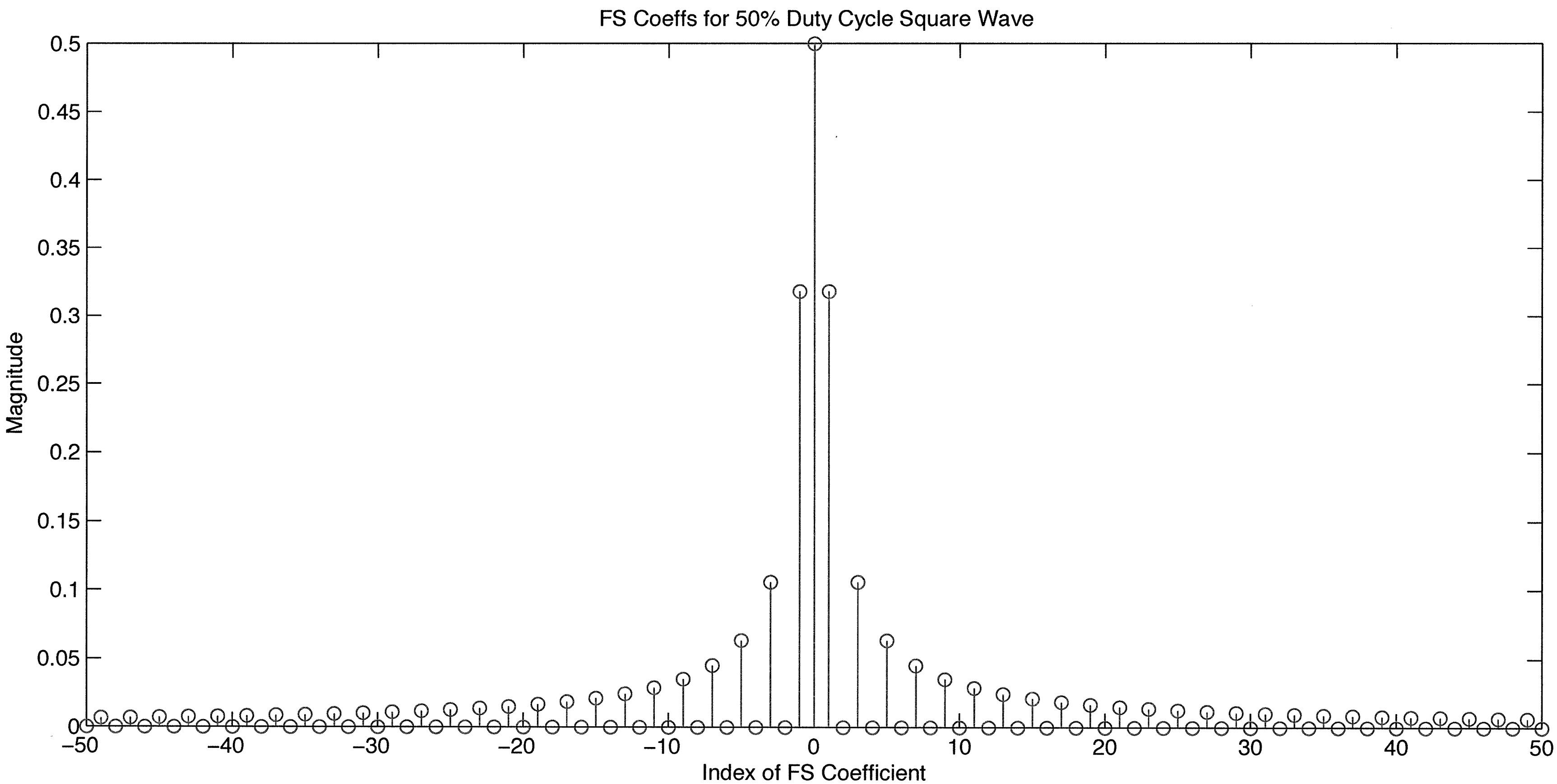


FS Coeffs for 50% Duty Cycle Square Wave
Magnitude
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
−50
−40
−30
−20
−10
0
10
20
30
40
50
Index of FS Coefficient

Z+T Sixth Edition, Computer Exercise 2.4, p. 110.

2.4. How would you use the same approach as in Computer Exercise 2.3 to evaluate the Fourier transform of a pulse-type signal. How do the two outputs differ? Compute an approximation to the Fourier transform of a square pulse signal 1 unit wide and compare with the theoretical result.

For simplicity I will write this for time-domain pulses which are causal. Since such numerical methods will only work for pulses of finite energy we would always be able to truncate the Fourier integral over ~~an~~ infinite interval $-\infty < t < \infty$ to a finite interval and still keep the error small. Then making a signal causal amounts to time shifting.

For a general causal pulse $x(t)$:

$$X(f) = \int_0^\infty x(t)\, e^{-j2\pi f t}\, dt$$

We can therefore pick a T_0 sufficiently large that

$$X(f) \approx \int_0^{T_0} x(t)\, e^{-j2\pi f t}\, dt$$

As in previous problem we let $t = nT$ and discretize the integral. Suppose $T_0 = KT$

$$X(f) \approx \sum_{n=0}^{K-1} x(nT)\, e^{-j2\pi f n T}\, T = T \sum_{n=0}^{K-1} x(nT)\, e^{-j2\pi f n T}$$

$$\approx \frac{T_0}{K} \sum_{n=0}^{K-1} x(nT_0/K)\, e^{-j2\pi f n T_0/K}$$

We will have to evaluate on some suff. fine freq. grid. Say

$$f = \frac{k}{T_0} \qquad k \in \mathbb{Z}$$

$$\Rightarrow X\left(\frac{k}{T_0}\right) = X(k\Delta f) = \frac{T_0}{K} \sum_{n=0}^{K-1} x(nT_0/K)\, e^{-j2\pi \frac{k}{T_0} n T_0/K}$$

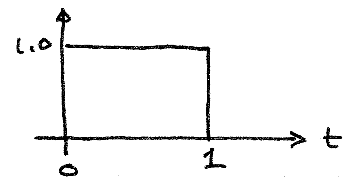
$\Delta f = 1/T_0$

$$= \frac{1}{\Delta f} \frac{1}{K} \sum_{n=0}^{K-1} x\left(\frac{n}{\Delta f K}\right) e^{-j2\pi kn/K}$$

Once again the desired result is computed from a K point DFT:

$$X(k\Delta f) = \frac{1}{K\Delta f} \cdot \left\{ \begin{array}{l} \text{Forward K-point} \\ \text{DFT of} \\ x(n/\Delta f K). \end{array} \right\}$$

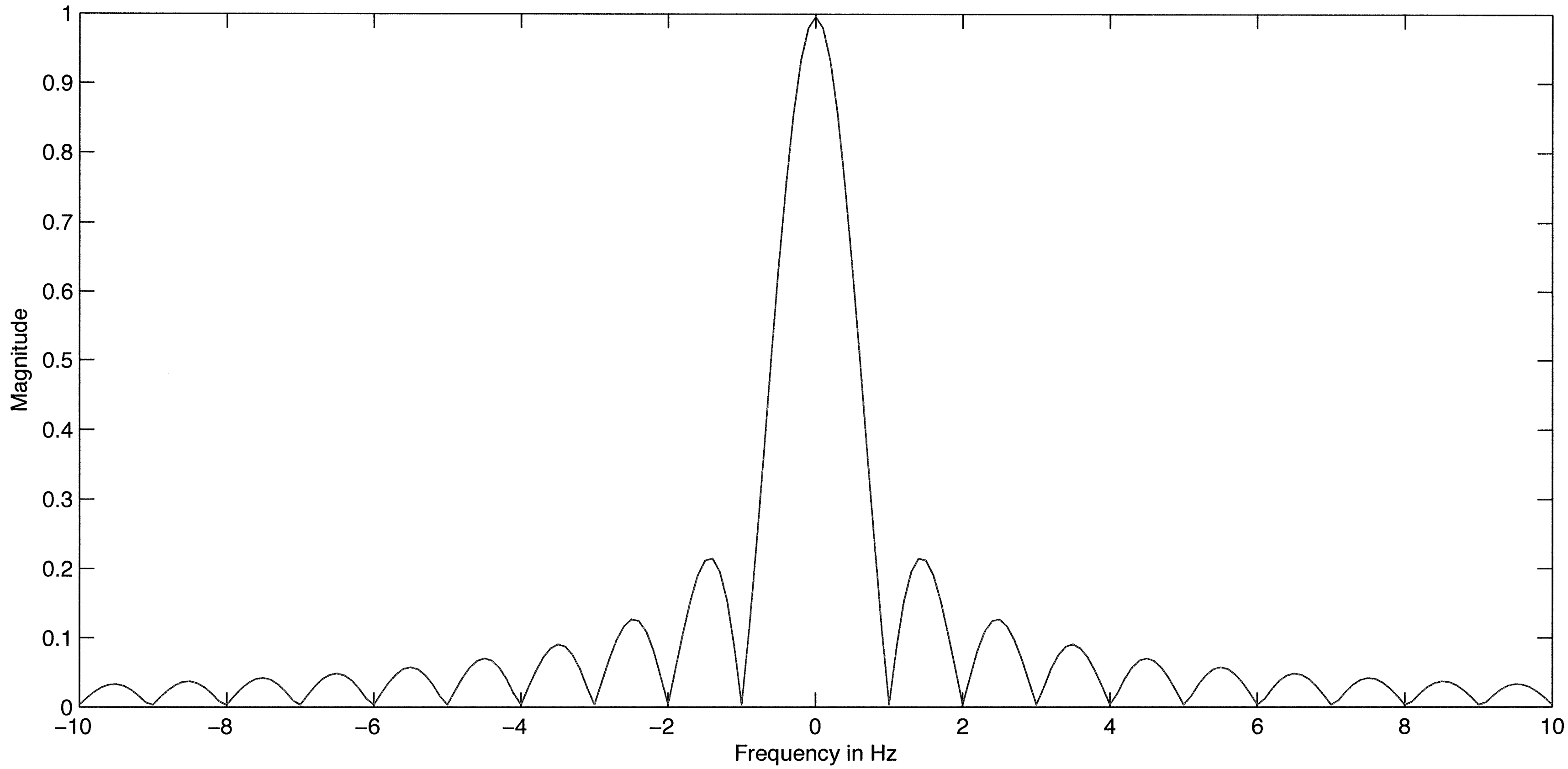
Example for Calculation



From a table of Fourier Transforms has a Fourier Transform magnitude

$$\left| \frac{\sin(\pi f)}{\pi f} \right|$$

```
%EE 440 Spring 2010, Problem 10

K = 1024;                                    %K pt. DFT
Delta_f = 0.1;                               %Frequency spacing in
Hz
N = floor(Delta_f*K);

x = zeros(1,K);
x(1:N) = ones(1,N);

X = fft(x)/(Delta_f*K);
Xs = fftshift(X);

index = 1:K;
f = (index - 1 - floor(K/2))*Delta_f;

plot(f,abs(Xs))
```


Magnitude
Frequency in Hz
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
−10
−8
−6
−4
−2
0
2
4
6
8
10