ECE 440 **Exam 3** **11/18/22**

Name: Solution

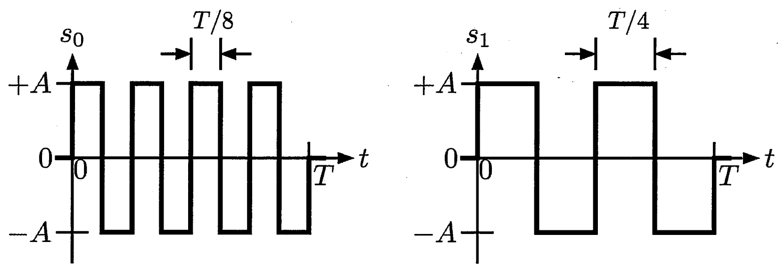
General Instructions:

- You have 50 minutes to complete the exam.
- Write your name on every page of the exam.
- Please do not write on the backs of pages.
- The exam is closed book.
- You are allowed both sides of three 8.5 by 11 inch sheets of paper for your personal notes in addition to the instructor supplied formula sheet.
- Calculators are allowed.
- Your work must be explained to receive full credit. All plots must be carefully drawn with axes labeled.
- Point values for each problem are as indicated. The exam totals 100 points.
- Please do not leave early as it is disruptive to those working around you.

Do not open the exam until you are told to begin.

Name: ______________________

Problem 1. *Binary Baseband Data Transmission.* [15 pts. total] A binary baseband data transmission system uses the signal set $s_0(t)$ and $s_1(t)$ shown below.



The channel is an additive white Gaussian noise channel with psd height $N_0/2$. The Bayes criterion is to be used with equally likely priors, i.e., $\pi_0 = \Pr[s_0 \text{ is trans.}] = \pi_1 = \Pr[s_1 \text{ is trans.}] = 0.5$. Find the average probability of error in terms of A, T, N_0, and the function $Q(\cdot)$. Assume that the receiver has been optimally designed. Explain your work and show any calculations needed.

In binary signalling case with equally likely signals s_0 and s_1 the average error probability is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}(1-\rho)}\right)$$

where

$$E_b = \frac{\mathcal{E}_0 + \mathcal{E}_1}{2} \quad \text{(average signal energy)}$$

$$\rho = \frac{\langle s_0, s_1 \rangle}{E_b} \quad \left(\begin{array}{c}\text{normalized correlation} \\ -1 \le \rho \le 1\end{array}\right)$$

Trivial to see $\langle s_0, s_1 \rangle = 0$ and $\mathcal{E}_0 = \mathcal{E}_1 = A^2 T$

$\therefore \rho = 0$

$$P_e = Q\left(\sqrt{\frac{A^2 T}{N_0}}\right).$$

Name: ______________________

Problem 2. *From Desired P_e to SNR to Required Tx Power* [20 pts. total]

Table G.1 A Short Table of Q-Function Values

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0	0.5	1.5	0.066807	3.0	0.0013499
0.1	0.46017	1.6	0.054799	3.1	0.00096760
0.2	0.42074	1.7	0.044565	3.2	0.00068714
0.3	0.38209	1.8	0.035930	3.3	0.00048342
0.4	0.34458	1.9	0.028717	3.4	0.00033693
0.5	0.30854	2.0	0.022750	3.5	0.00023263
0.6	0.27425	2.1	0.017864	3.6	0.00015911
0.7	0.24196	2.2	0.013903	3.7	0.00010780
0.8	0.21186	2.3	0.010724	3.8	7.2348×10^{-5}
0.9	0.18406	2.4	0.0081975	3.9	4.8096×10^{-5}
1.0	0.15866	2.5	0.0062097	4.0	3.1671×10^{-5}
1.1	0.13567	2.6	0.0046612	4.1	2.0658×10^{-5}
1.2	0.11507	2.7	0.0034670	4.2	1.3346×10^{-5}
1.3	0.096800	2.8	0.0025551	4.3	8.5399×10^{-6}
1.4	0.080757	2.9	0.0018658	4.4	5.4125×10^{-6}

From class we know that the probability of error of an optimally designed binary communication system with equally likely signals $s_0(t)$ and $s_1(t)$ is given by

$$P_e = Q\left(\sqrt{E_b(1-\rho)/N_0}\right)$$

where E_b is the average energy per bit (i.e., the average of the energies of the signals $s_0(t)$ and $s_1(t)$), ρ is a real number between -1 and 1 representing the normalized correlation between $s_0(t)$ and $s_1(t)$, and $N_0 = kT$ where $k = 1.38 \times 10^{-23}$ J/K and T is the temperature in Kelvin. In this problem we want to explore the relationship between power, energy, noise level, and transmission rate.

Recent news articles about the Voyager 1 spacecraft (currently about 15 billion km from earth) mention that the power in the signal as received on earth is about an attowatt (10^{-18} W). Assuming this received power, that the receiver temperature is $T = 300$ K, and that the probability of a bit error must be kept below 10^{-4}, what is the maximum bit rate that can be supported assuming ...

(a) [10 pts.] ... orthogonal signaling. $\Longrightarrow \rho = 0$

Power in received signal is ...

$P_{Rx} = E_b \cdot \frac{1}{T_b}$ $\quad T_b$ = duration of 1 bit $\Longrightarrow E_b = T_b \cdot P_{Rx}$

$= T_b \cdot 10^{-18}$ J

$N_0 = kT = (1.38 \times 10^{-23}\ \text{J/K})(300\text{K})$

$= 4.14 \times 10^{-21}$ J

$\Longrightarrow \dfrac{E_b}{N_0} = \dfrac{T_b \times 10^{-18}}{4.14 \times 10^{-21}} = \left(\dfrac{T_b}{4.14}\right) \times 10^3$

Problem 2. (cont'd.) Name: ____________________

Want $P_e = Q\left(\sqrt{\frac{E_b}{N_0}(1-\rho)}\right) \le 10^{-4}$ ∴ Reading the table need

$$\sqrt{\frac{E_b(1-\rho)}{N_0}} \geq 3.8 \implies \frac{E_b}{N_0} \geq \frac{(3.8)^2}{1-\rho}$$

$$= \begin{cases} (3.8)^2 & \text{if } \rho = 0 \\ \frac{(3.8)^2}{2} & \text{if } \rho = -1 \end{cases}$$

$(3.8)^2 \to 14.44$

$\frac{(3.8)^2}{2} \to 7.22$

(b) [10 pts.] ... antipodal signaling.

Therefore for part (a) need

$$\frac{1000 T_b}{4.14} \geq 14.44$$

$$T_b \geq \frac{(14.44)(4.14)}{1000} = 59.8 \text{ ms}$$

$$\iff \frac{1}{T_b} \le 16.7 \text{ bits/sec}$$

For part (b) need

$$\frac{1000 T_b}{4.14} \geq 7.22$$

$$T_b \geq 29.9 \text{ ms}$$

$$\iff \frac{1}{T_b} \le 33.4 \text{ bits/sec.}$$

Name: ____________________

Problem 3. *Matlab Simulation of AM Communication System.* [30 pts. total] This problem has four parts contained in the code below.

```
%% Matlab Code for Exam 3 Fall 2022.
% Comments below correspond to the questions in Parts (a), (b), (c), and
% (d) in the problem statement.

time = .3;
Ts = 1/10000;
t = Ts:Ts:time; lent=length(t);
fc = 1000;

fbe=[0 0.1 0.2 1];
damps=[1 1 0 0];
b=firpm(100,fbe,damps);
[H,omega] = freqz(b,1,500);

% Part (a) TO DO: Draw a picture of what the figure below should look like
% but with the frequencies corresponding to continuous time ...

figure(1)
plot(omega/(2*pi),20*log10(abs(H)),'LineWidth',2)
ax = gca;
xlabel('Give a descriptive label')
ylabel('Give a descriptive label')
title('Give a descriptive title')
grid on
ax.FontSize = 12;

% Part (b) TO DO: Draw a block diagram for the creation of the time-domain
% signal v(t) corresponding to vector v below ...

fm1 = 300;
fm2 = 100;

w = cos(2*pi*fm1*t) + 0.5*cos(2*pi*fm2*t);
c = cos(2*pi*fc*t);
v = c .* w;

figure(2)
subplot(2,1,1), plot(t,w)
ylabel('Label me'); title('Label me');
subplot(2,1,2), plot(t,v)
ylabel('Label me'); title('Label me');
```

```
[ssf,wfxs] = Plotspec(w,Ts);
[ssf,vfxs] = Plotspec(v,Ts);

% Part (b) TO DO: Draw a picture of what the figure below would look like
% and label the axes ...

figure(3)
subplot(2,1,1), plot(ssf,abs(wfxs))
ylabel('Label me.'); title('Label me.');
subplot(2,1,2), plot(ssf,abs(vfxs))
ylabel('Label me.'); title('Label me.');

% Part (c) TO DO: Draw a block diagram for the creation of the time-domain
% signal m(t) corresponding to vector m below.

sigma = 0.15;

x = v + sigma*randn(size(w));
y = x .* c;

m=2*filter(b,1,y);

% Part (d) TO DO: Compute the value of the approximate baseband
% signal-to-noise power ratio of the time-domain signal m(t) ...

figure(4)
subplot(2,1,1), plot(t,w)
ylabel('Label me'); title('Label me');
subplot(2,1,2), plot(t,m)
ylabel('Label me'); title('Label me');

function [ssf,fxs] = Plotspec(x,Ts)
% Plotspec is a local function to calculate the spectrum of an input signal

    N=length(x);                          % length of the signal x
    t=Ts*(1:N);                           % define a time vector
    ssf=(-N/2:N/2-1)/(Ts*N);              % frequency vector
    fx=Ts*fft(x(1:N));                    % do DFT/FFT
    fxs=fftshift(fx);                     % shift it for plotting

end
```

Problem 3. (cont'd.) Name: ____________

(a) [5 pts.] The Matlab code marked "Part (a)" in the listing creates a low pass filter. Sketch the approximate magnitude response of the filter with respect to its effect on continuous-time frequencies.

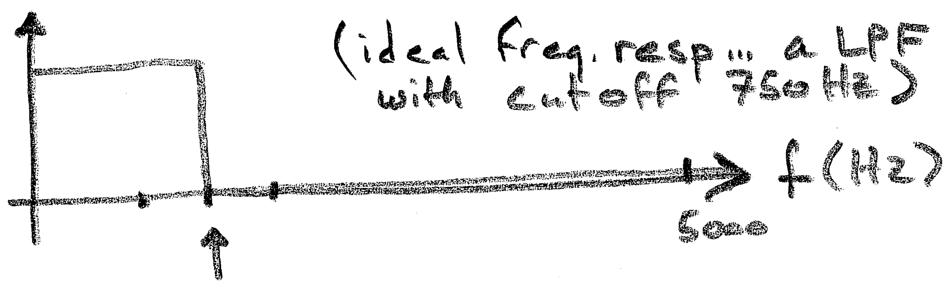
The specification of the frequency response is contained in

fbe = [0 0.1 0.2 1]

damps = [1 1 0 0]

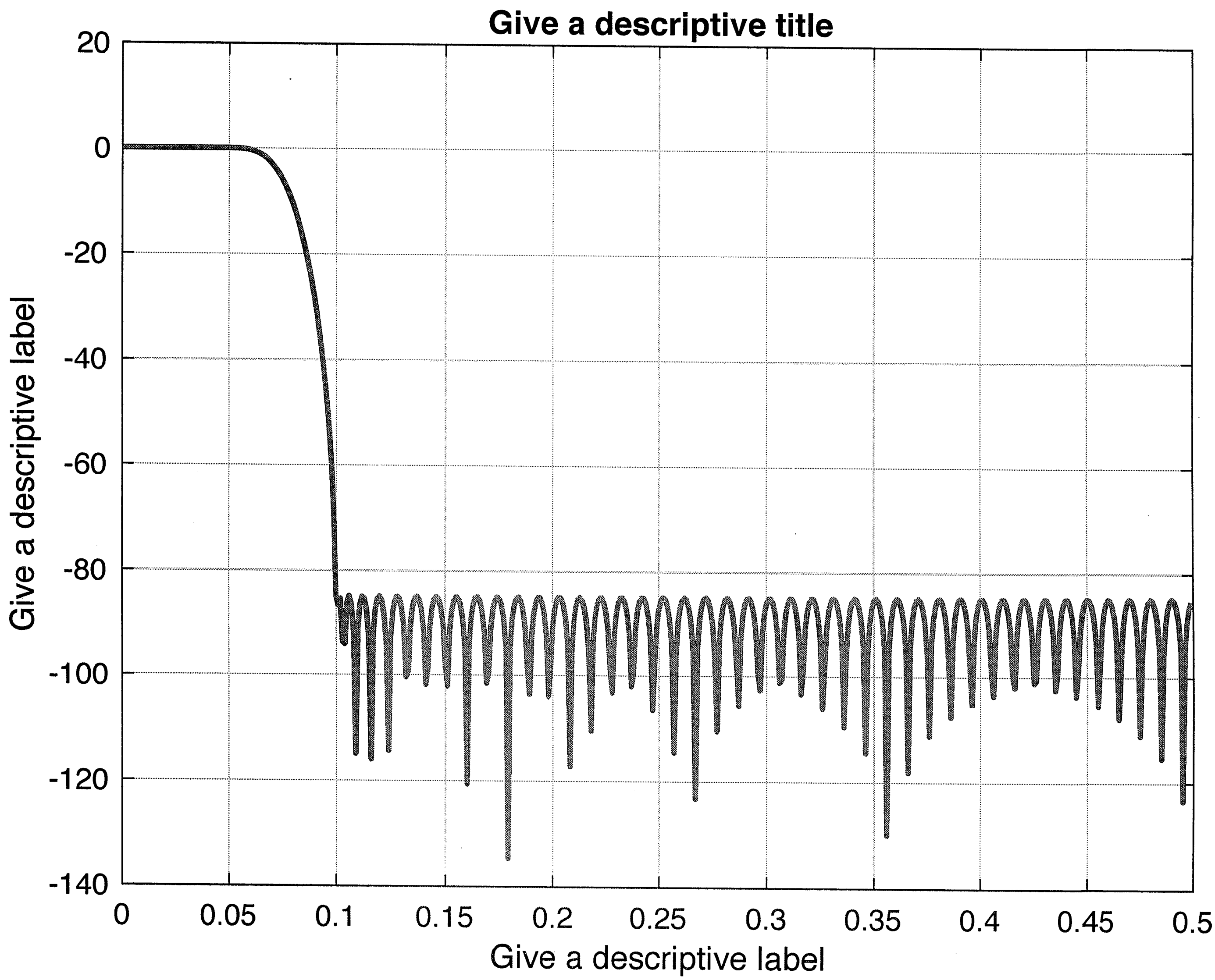
which corresponds to ...

$T_s = 1/10000$ sec $\Rightarrow f_s = 10$ kHz $\Rightarrow \frac{f_s}{2} = 5$ kHz (foldover)



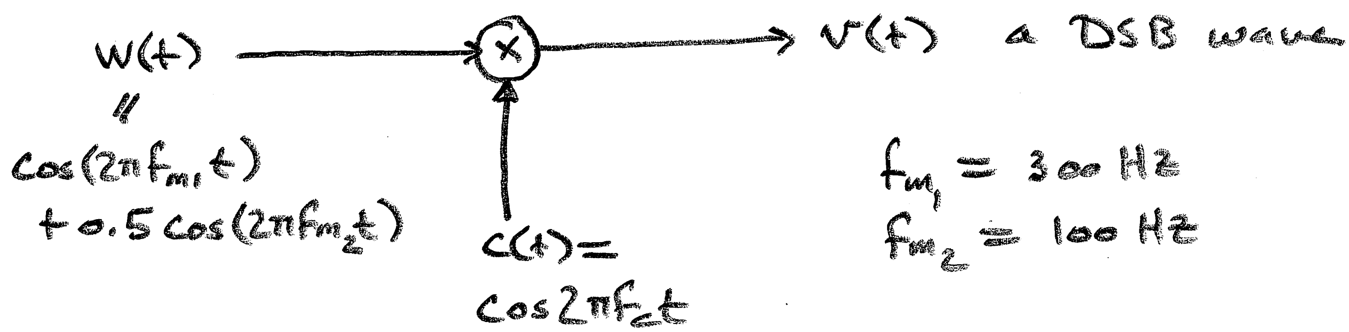
$0.15 \times 5000 \cong 750$ Hz

The actual figure
for part (a)

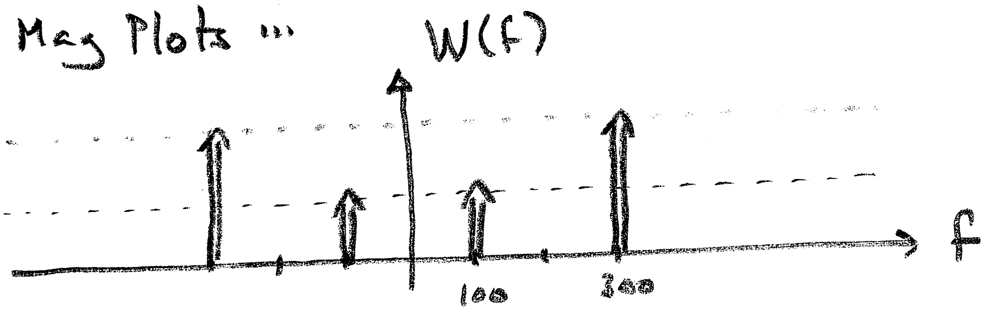


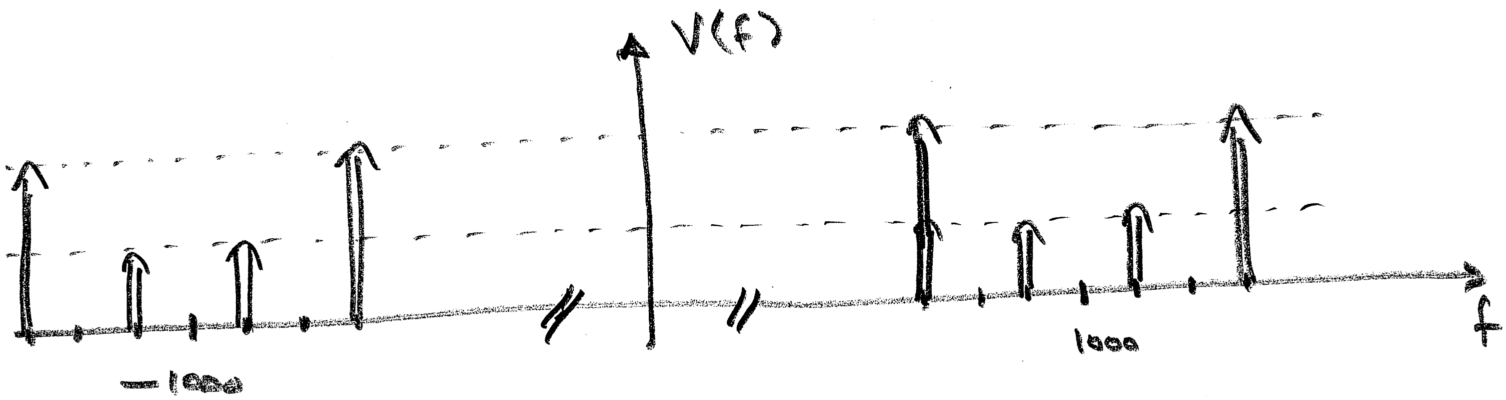
Problem 3. (cont'd.) Name: ______________________

(b) [10 pts.] For the Matlab code marked "Part (b)" in the listing draw a block diagram that illustrates the creation of the time-domain signal $v(t)$. Then also make an approximate spectral magnitude plot for $W(f)$ and $V(f)$.

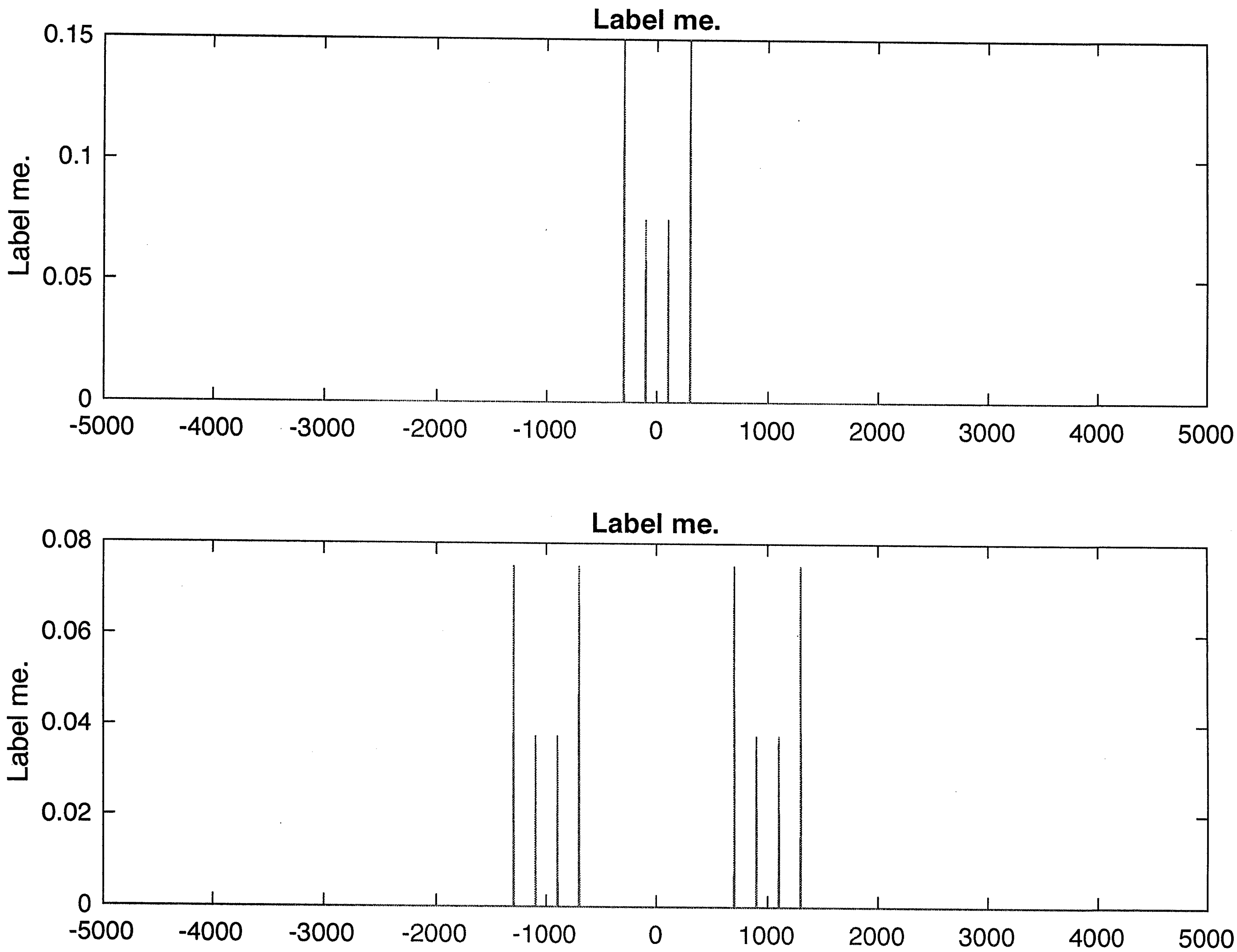


Spec Mag Plots ...



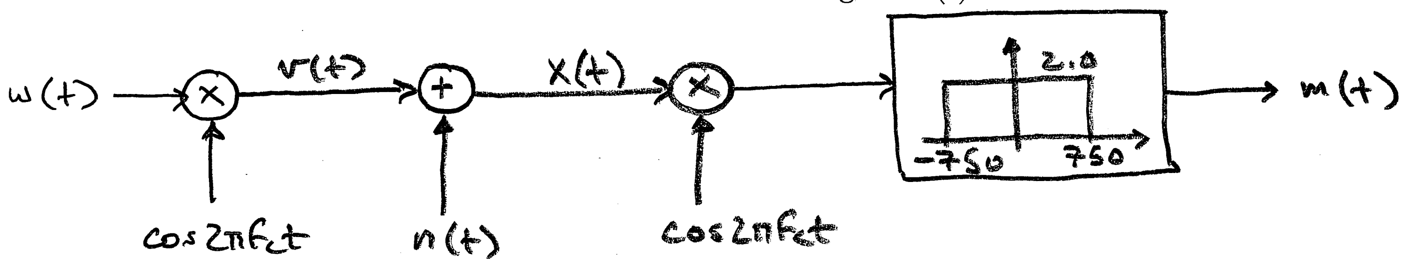


The actual figure for part (b).



Problem 3. (cont'd.) Name: ____________________

(c) [7 pts.] For the Matlab code marked "Part (c)" in the listing draw a block diagram that illustrates the creation of the time-domain signal $m(t)$.



(d) [8 pts.] For the values given in the Matlab code, what would be the approximate baseband signal-to-noise power ratio of the time-domain signal $m(t)$.

Due to the gain of 2 in the LPF $\quad Pow\{m(t)\} = Pow\{w(t)\}$

$$= \tfrac{1}{2}1^2 + \tfrac{1}{2}(0.5)^2$$

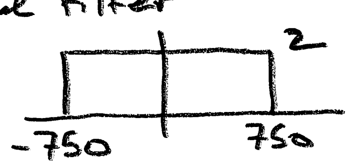
$$= 0.5 + 0.5 \cdot 0.25$$

$$= 0.5(1.25)$$

$\sigma = 0.15 \Rightarrow \sigma^2 = (0.15)^2 = \frac{N_o}{2} \cdot 2 \cdot 5000 = 5000\, N_o$

is the power in the noise, which corrupts the signal $v(t)$

Final filter



$\Rightarrow$ Equiv. noise power corrupting $m(t)$ is $\sigma_{BB}^2 = 750\, N_o = \frac{750}{5000}(0.15)^2$

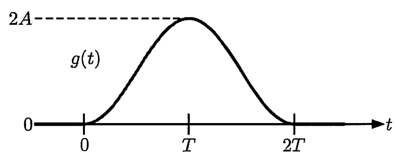
$$\cong 0.003$$

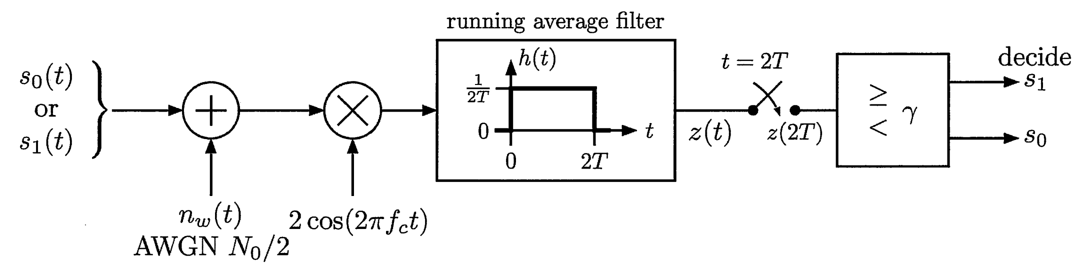
$\therefore$ SNR @ the demod output $= \frac{(0.5)(1.25)}{0.003} = 208.33 = 23.2\text{ dB}.$

Name: ______________________

Problem 4. *A Sub-Optimal ASK Receiver.* [35 points] The purpose of this problem is to step through the development of a reasonable (though non-optimal) receiver for ASK. The signals are $s_0(t) = 0$, $s_1(t) = +g(t)\cos(2\pi f_c t)$, where $g(t)$ is the time-domain raised cosine shaped pulse:

$$g(t) = \begin{cases} A(1+\cos(\pi(t-T)/T) & 0 \le t \le 2T \\ 0 & \text{else} \end{cases} .$$





Assume that $f_c \gg 1/T$ (which will suggest a certain approximation simplifying the results below). The receive filter is not matched and therefore non-optimal. Its output is a running average of its input with an averaging window of length $2T$. Note that the downconversion via multiplication by $2\cos(2\pi f_c t)$ occurs before the matched filter in this architecture.

(a) [8 pts.] Assuming that the transmitted signal is actually $s_1(t)$ and ignoring the noise find the message related part of $z(2T)$.

$s_1(t) = g(t)\cos 2\pi f_c t \Rightarrow$ signal-only part at the input to $h(t)$ is $2g(t)\cos^2 2\pi f_c t$

Therefore with s_1 as input and ignoring noise ...

$$z(t) = \int h(t-\tau)\, 2g(\tau)\cos^2 2\pi f_c \tau \, d\tau$$

$$= \frac{1}{2T}\int_{\max(0,t-2T)}^{t} g(\tau)\, d\tau + \underbrace{\frac{1}{2T}\int_{\max(0,t-2T)}^{t} g(\tau)\cos 2\pi 2f_c\tau \, d\tau}_{\approx 0 \text{ since } f_c \gg 1/T}$$

$$\therefore\ z(2T) = \frac{1}{2T}\int_{0}^{2T} A\left[1+\cos(\pi(t-T)/T)\right] d\tau$$

$= A$ --- draw a picture to see this.

Problem 4. (cont'd.) Name: ____________________

(b) [4 pts.] Repeat part (a) assuming the transmitted signal is actually $s_0(t)$.

$s_0(t) = 0 \implies z(2T) = 0$ with this signal and no noise

(c) [15 pts.] Ignoring the signals, find the mean and variance of the noise-related part of $z(2T)$.

The noise in the output is $n(t) = h(t) * 2n_w(t)\cos 2\pi f_c t$. Its mean is obviously equal to zero. To calculate its variance we need to compute

$$\mathrm{Var}\{n(2T)\} = \int_0^{2T}\int_0^{2T} h(\tau)h(s)\frac{N_0}{2}\delta(s-\tau)\cdot 4\cos 2\pi f_c(2T-\tau)\cos 2\pi f_c(2T-s)\,ds\,d\tau$$

$$= 2N_0\int_0^{2T} h^2(\tau)\cos^2(2\pi f_c(2T-\tau))\,d\tau = 2N_0\frac{1}{2}\left(\frac{1}{2T}\right)^2\cdot 2T$$

$$= N_0/2T$$

Problem 4. (cont'd.) Name: ____________

(d) [8 pts.] Assuming equally likely signals $s_0(t)$ and $s_1(t)$ find the best threshold and the resulting average probability of a bit error in terms of A, T, N_0, and the function $Q(\cdot)$.

Under s_1 $\quad z(2T) \sim N(A, N_0/2T)$

Under s_0 $\quad z(2T) \sim N(0, N_0/2T)$

The minimax threshold would then be at $A/2$ and the average error probability would equal either of the conditional error probabilities

$$P_e = P_{e|s_0} = P\left(z(2T) > \frac{A}{2} \,\middle|\, s_0 \text{ was transmitted}\right)$$

$$= P\left(\frac{z(2T)}{\sqrt{N_0/2T}} > \frac{A/2}{\sqrt{N_0/2T}} \,\middle|\, s_0 \text{ was transmitted}\right)$$

$$= Q\left(\frac{A/2}{\sqrt{N_0/2T}}\right) = Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right)$$