Session 28

HW Hints

Name: ______________________ PU ID: ____________

**ECE 302: Probabilistic Methods in Electrical and Computer Engineering**
**Spring 2023**
**Instructor: Prof. J. V. Krogmeier**

PURDUE UNIVERSITY

# Homework 6

Spring 2023
(Due Wednesday March 22 at 11:59pm on Gradescope)

All homework should include a brief statement in the box below, which gives a description of the sources you used for information (old homework files, the internet, books, papers, chatGPT, etc.). You should indicate if other people (besides Prof. Krogmeier or TA Fanyang Cheng) have worked with you and list their names. You do not need to reference the course text, handouts, or materials posted on the course web page. If none of the previous sources apply, then state "I did not receive help on this homework". If the statement is completed you will receive a 5 point bonus.

Each problem appears on a page by itself. There is a space to show your work and a space to indicate your final answer. If you need more room to show your work, you may insert page(s) after the problem definition page and before the subsequent problem. You must show work to receive full credit.

**Topics: Common PMFs (Conditional PMF, PDF, CDF (Chapters 3.4 and 4.2.2); Common PDFs (Ch 4.4); Functions of a Random Variable (Ch 4.5)**

## Statement:

HW 7 posted
E1 regrades due today.

# HW6 Hints

Covered in Session 22

**Exercise 1.** (From textbook, problems 4.30, 4.32)
A random variable $X$ has cdf:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - \frac{1}{4}e^{-2x} & \text{for } x \geq 0 \end{cases} = P(X \leq x)$$

(a) Find $F_X(x|A)$, where $A = \{X > 0\}$.

(b) Find $F_X(x|C)$, where $C = \{X = 0\}$

(c) Find $f_X(x|B)$ and $F_X(x|B)$, where $B = \{X > 0.25\}$


ⓐ $F(x|A) = \dfrac{P(\{X \leq x\} \cap \{X > 0\})}{P(X > 0)}$

Now $\{X \geq 0\} = \{X = 0\} \cup \{X > 0\}$. Here $P(X=0) = 3/4$ and therefore $P(X>0) = 1/4$.

Thus $\{X \leq x\} \cap \{X > 0\} = \begin{cases} \phi & \text{if } x \leq 0 \\ \{0 < X \leq x\} & \text{if } x > 0. \end{cases}$

$$\therefore F(x|A) = \begin{cases} 0 & \text{if } x \leq 0 \\ \dfrac{F(x) - F(0)}{1/4} = \dfrac{1 - \frac{1}{4}e^{-2x} - \frac{3}{4}}{1/4} = 1 - e^{-2x} & \text{if } x > 0 \end{cases}$$

If wanted $f_X(x|A)$

$a =$

we'd simply $f_X(x|A) = \dfrac{d}{dx}\left\{1 - e^{-2x}\right\}$

$b =$

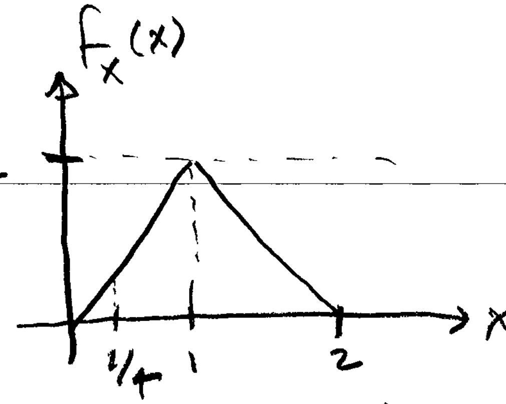
$= 2e^{-2x}$ for $x > 0$

$c =$

$(= 0$ for $x \leq 0)$.

**Exercise 2.**
Consider the random variable $X$ with PDF given by

$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$



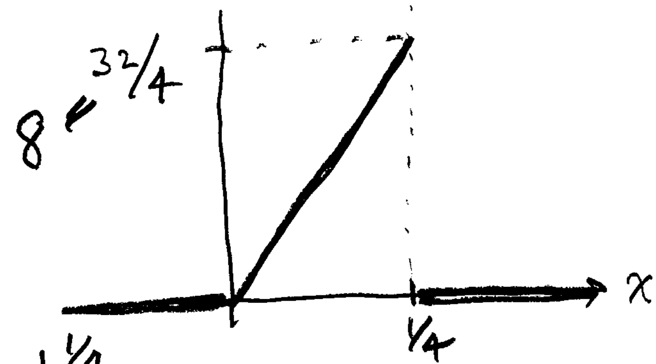

(a) Sketch $f_X(x)$. Label axes and relevant values.

(b) Find and sketch the conditional density $f_X(x|A)$ for the event $A = \{X < 1/4\}$.

(c) What is the conditional mean $E(X|B)$ for the event $B = \{2/3 < X < 4/3\}$? (Hint: you do not need to find $P(B)$ to solve part (c)!)

$$f_X(x|A) = \frac{f_X(x)}{P(X<1/4)}$$

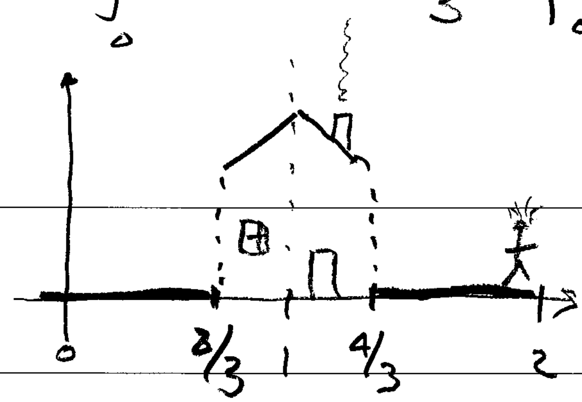
$$P(X<\tfrac{1}{4}) = \frac{1}{2}\,\frac{1}{4}\cdot\frac{1}{4} = 1/32$$




$$\frac{1}{2}h\frac{1}{4} = 1 \Rightarrow h = 8$$

$$E[X|A] = \int_0^{1/4} x \underbrace{f_X(x|A)}_{8x}\,dx$$

$$= \int_0^{1/4} 8x^2\,dx = \frac{8}{3}x^3\Big|_0^{1/4} = \frac{8}{3}\,\frac{1}{4^3}$$




$a =$

$b =$

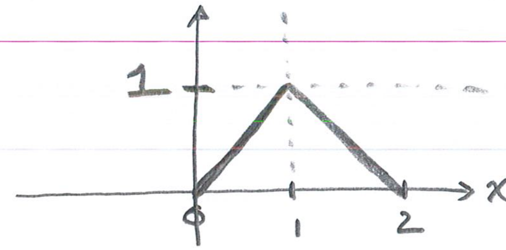
$c =$

$P(X<\frac{1}{4}) = \frac{1}{2}\,\frac{1}{4}\,\frac{1}{4} = \frac{1}{32}$ $\Longrightarrow$ $f_X(x|A) = \frac{d}{dx} F_X(x|A)$ ... easy calc.

**Exercise 2.**

Consider the random variable $X$ with PDF given by

$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$




(a) Sketch $f_X(x)$. Label axes and relevant values.

(b) Find and sketch the conditional density $f_X(x|A)$ for the event $A = \{X < 1/4\}$.

(c) What is the conditional mean $E(X|B)$ for the event $B = \{2/3 < X < 4/3\}$?
(Hint: you do not need to find $P(B)$ to solve part (c)!) $\Longrightarrow$ Not really that helpful.

To find cond. mean one would normally find the cond. pdf & average ...

$$E[X|B] = \int x\, f_X(x|B)\, dx$$

integral over the proper range ... here $\frac{2}{3} < x < \frac{4}{3}$

$f_X(x)$ is symm. wrt $x=1$

$B$ is symm. wrt $x=1$

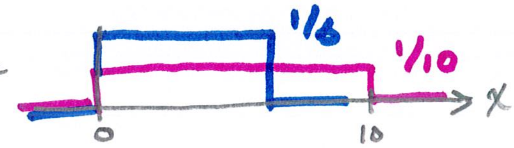
$\Longrightarrow f_X(x|B)$ is symm wrt $x=1$ $\Longrightarrow E[X|B] = 1.$

$a =$

$b =$

$c =$

Same as prev.

$f_X(x) =$ 


**Exercise 3.**
Let $X$ be a uniform random variable on the interval $(0, 10)$.

(a) Find $P(X \le 6)$ and $P(X > 8)$.

$P(X \le 6) = 6/10$

$P(X > 8) = 2/10$

(b) Compute the conditional PDF's of $f_X(x|X \le 6)$ and $f_x(x|X > 8)$.

(c) Find the conditional means of $E(X|X \le 6)$ and $E(X|X > 8)$.

(d) Find the conditional variances of $Var(X|X \le 6)$ and $Var(X|X > 8)$.

$$f_X(x|X \le 6) = \begin{cases} \dfrac{f_X(x)}{6/10} & 0 < x < 6 \\ 0 & \text{else} \end{cases}$$

see

$$E[X|X \le 6] = \int_{-\infty}^{\infty} x\, f_X(x|X \le 6)\, dx = 3$$

...

| |
|---|
| $a =$ |
| $b =$ |
| $c =$ |
| $d =$ |

$$\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X) \qquad X \sim N(5,16)$$

**Exercise 4.** (From textbook, problem 4.63 (a-c))
Let $X$ be a Gaussian random variable with mean 5 and variance 16.

(a) Find $P(X > 4)$, $P(X \geq 7)$, $P(2 < X < 7)$, $P(6 \leq X \leq 8)$.

(b) If $P(X < a) = 0.8869$, what is the value of $a$?

(c) If $P(X > b) = 0.11131$, what is the value of $b$?

$$X-5 \sim N(0,16)$$

$$\frac{X-5}{4} \sim N(0,1) \qquad \Phi(\cdot)$$

$$P(2 < X < 7) = P(\{2 < X < 7\}) = F_X(7) - F_X(2)$$

$$P(2-5 < X-5 < 7-5)$$

$$P\Big(\underbrace{\tfrac{2-5}{4}}_{-3/4} < \frac{X-5}{4} \leq \underbrace{\tfrac{7-5}{4}}_{2/4}\Big) = \Phi\left(\tfrac{1}{2}\right) - \Phi\left(-\tfrac{3}{4}\right)$$

$$= \Phi\left(\tfrac{1}{2}\right) - \left[1 - \Phi\left(\tfrac{3}{4}\right)\right]$$

$$= \Phi\left(\tfrac{1}{2}\right) + \Phi\left(\tfrac{3}{4}\right) - 1$$

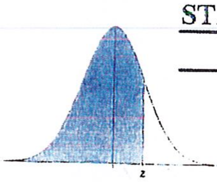
(b)

| | |
|---|---|
| $a =$ | $P\left(\frac{X-5}{4} < \frac{a-5}{4}\right) = 0.8869$ |
| $b =$ | $\Phi\left(\frac{a-5}{4}\right)$ |
| $c =$ | $\underbrace{\Phi^{-1}\left(\Phi\left(\frac{a-5}{4}\right)\right)}_{\frac{a-5}{4}} = \Phi^{-1}(.8869)$ |

1,21

1.21

Learn how to use a table like this!



STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

| Z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .50000 | .50399 | .50798 | .51197 | .51595 | .51994 | .52392 | .52790 | .53188 | .53586 |
| 0.1 | .53983 | .54380 | .54776 | .55172 | .55567 | .55962 | .56356 | .56749 | .57142 | .57535 |
| 0.2 | .57926 | .58317 | .58706 | .59095 | .59483 | .59871 | .60257 | .60642 | .61026 | .61409 |
| 0.3 | .61791 | .62172 | .62552 | .62930 | .63307 | .63683 | .64058 | .64431 | .64803 | .65173 |
| 0.4 | .65542 | .65910 | .66276 | .66640 | .67003 | .67364 | .67724 | .68082 | .68439 | .68793 |
| 0.5 | .69146 | .69497 | .69847 | .70194 | .70540 | .70884 | .71226 | .71566 | .71904 | .72240 |
| 0.6 | .72575 | .72907 | .73237 | .73565 | .73891 | .74215 | .74537 | .74857 | .75175 | .75490 |
| 0.7 | .75804 | .76115 | .76424 | .76730 | .77035 | .77337 | .77637 | .77935 | .78230 | .78524 |
| 0.8 | .78814 | .79103 | .79389 | .79673 | .79955 | .80234 | .80511 | .80785 | .81057 | .81327 |
| 0.9 | .81594 | .81859 | .82121 | .82381 | .82639 | .82894 | .83147 | .83398 | .83646 | .83891 |
| 1.0 | .84134 | .84375 | .84614 | .84849 | .85083 | .85314 | .85543 | .85769 | .85993 | .86214 |
| 1.1 | .86433 | .86650 | .86864 | .87076 | .87286 | .87493 | .87698 | .87900 | .88100 | .88298 |
| 1.2 | .88493 | .88686 | .88877 | .89065 | .89251 | .89435 | .89617 | .89796 | .89973 | .90147 |
| 1.3 | .90320 | .90490 | .90658 | .90824 | .90988 | .91149 | .91309 | .91466 | .91621 | .91774 |
| 1.4 | .91924 | .92073 | .92220 | .92364 | .92507 | .92647 | .92785 | .92922 | .93056 | .93189 |
| 1.5 | .93319 | .93448 | .93574 | .93699 | .93822 | .93943 | .94062 | .94179 | .94295 | .94408 |
| 1.6 | .94520 | .94630 | .94738 | .94845 | .94950 | .95053 | .95154 | .95254 | .95352 | .95449 |
| 1.7 | .95543 | .95637 | .95728 | .95818 | .95907 | .95994 | .96080 | .96164 | .96246 | .96327 |
| 1.8 | .96407 | .96485 | .96562 | .96638 | .96712 | .96784 | .96856 | .96926 | .96995 | .97062 |
| 1.9 | .97128 | .97193 | .97257 | .97320 | .97381 | .97441 | .97500 | .97558 | .97615 | .97670 |
| 2.0 | .97725 | .97778 | .97831 | .97882 | .97932 | .97982 | .98030 | .98077 | .98124 | .98169 |
| 2.1 | .98214 | .98257 | .98300 | .98341 | .98382 | .98422 | .98461 | .98500 | .98537 | .98574 |
| 2.2 | .98610 | .98645 | .98679 | .98713 | .98745 | .98778 | .98809 | .98840 | .98870 | .98899 |
| 2.3 | .98928 | .98956 | .98983 | .99010 | .99036 | .99061 | .99086 | .99111 | .99134 | .99158 |
| 2.4 | .99180 | .99202 | .99224 | .99245 | .99266 | .99286 | .99305 | .99324 | .99343 | .99361 |
| 2.5 | .99379 | .99396 | .99413 | .99430 | .99446 | .99461 | .99477 | .99492 | .99506 | .99520 |
| 2.6 | .99534 | .99547 | .99560 | .99573 | .99585 | .99598 | .99609 | .99621 | .99632 | .99643 |
| 2.7 | .99653 | .99664 | .99674 | .99683 | .99693 | .99702 | .99711 | .99720 | .99728 | .99736 |
| 2.8 | .99744 | .99752 | .99760 | .99767 | .99774 | .99781 | .99788 | .99795 | .99801 | .99807 |
| 2.9 | .99813 | .99819 | .99825 | .99831 | .99836 | .99841 | .99846 | .99851 | .99856 | .99861 |
| 3.0 | .99865 | .99869 | .99874 | .99878 | .99882 | .99886 | .99889 | .99893 | .99896 | .99900 |
| 3.1 | .99903 | .99906 | .99910 | .99913 | .99916 | .99918 | .99921 | .99924 | .99926 | .99929 |
| 3.2 | .99931 | .99934 | .99936 | .99938 | .99940 | .99942 | .99944 | .99946 | .99948 | .99950 |
| 3.3 | .99952 | .99953 | .99955 | .99957 | .99958 | .99960 | .99961 | .99962 | .99964 | .99965 |
| 3.4 | .99966 | .99968 | .99969 | .99970 | .99971 | .99972 | .99973 | .99974 | .99975 | .99976 |
| 3.5 | .99977 | .99978 | .99978 | .99979 | .99980 | .99981 | .99981 | .99982 | .99983 | .99983 |
| 3.6 | .99984 | .99985 | .99985 | .99986 | .99986 | .99987 | .99987 | .99988 | .99988 | .99989 |
| 3.7 | .99989 | .99990 | .99990 | .99990 | .99991 | .99991 | .99992 | .99992 | .99992 | .99992 |
| 3.8 | .99993 | .99993 | .99993 | .99994 | .99994 | .99994 | .99994 | .99995 | .99995 | .99995 |
| 3.9 | .99995 | .99995 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99997 | .99997 |

Law of Unconscious Statistician

**Exercise 6.** (FROM FINAL, SPRING 2016)
~~A device is deployed in a remote region. The time, $T$, to failure, is exponentially distributed with mean 3~~ years. The device will not be monitored during the first 2 years, so the time before failure can be discovered is $X = max(T, 2)$. What is $E(X)$?

(Hint: Consider separately what happens when $T < 2$ and $T \geq 2$. (That is to say, consider both conditions $C = \{T < 2\}$ and $C^c$. This allows you to apply principles we learned in class, and only do one integration. Without this approach, you may find the following integral (without limits) helpful.)

$$\int xe^{ax}\,dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$$

Way 1 Find $f_X(x)$ where $X = max(T,2)$

Then $E(X) = \int x f_X(x)\,dx$

LoU.S.

<u>Way 2</u>

$$E(g(T)) = \int_{-\infty}^{\infty} g(t) f_T(t)\,dt$$

$$f_T(t) = \begin{cases} \frac{1}{3}e^{-t/3} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= \int_0^2 \underbrace{g(t)}_{2} f_T(t)\,dt + \int_2^{\infty} \underbrace{g(t)}_{t} f_T(t)\,dt$$

$E(X) =$

**Exercise 7.**
Let random variable $X$ have pdf $f_X(x) = 1/(2x^2)$ for $|x| \geq 1$ and $f_X(x) = 0$ for $|x| < 1$. Let $Y = \sqrt{|X|}$. Find $E(Y)$.

$$f_X(x) = \begin{cases} \frac{1}{2x^2} & x \geq 1 \\ 0 & -1 < x < 1 \\ \frac{1}{2x^2} & x \leq -1 \end{cases}$$

$$E[Y] = E\left[\sqrt{|X|}\right] = \int_{-\infty}^{\infty} \sqrt{|x|}\, f_X(x)\, dx$$

$$= \int_{-\infty}^{-1} \sqrt{|x|}\, \frac{1}{2x^2}\, dx + 0 + \int_{1}^{\infty} \sqrt{|x|}\, \frac{1}{2x^2}\, dx$$

$$= 2 \int_{1}^{\infty} \sqrt{x}\, \frac{1}{2x^2}\, dx = 2 \int_{1}^{\infty} \frac{1}{2} x^{1/2} x^{-2}\, dx$$

$$= \int_{1}^{\infty} x^{-3/2}\, dx = (-2)x^{-1/2}\Big|_{1}^{\infty} = 2$$

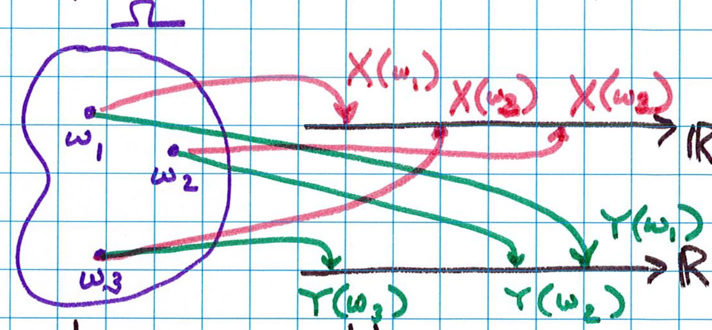
$E(Y) =$

# Pairs of Random Variables

$(\Omega, \mathcal{F}, P)$ a probability space. Consider two functions

$$X : \Omega \longrightarrow \mathbb{R}$$

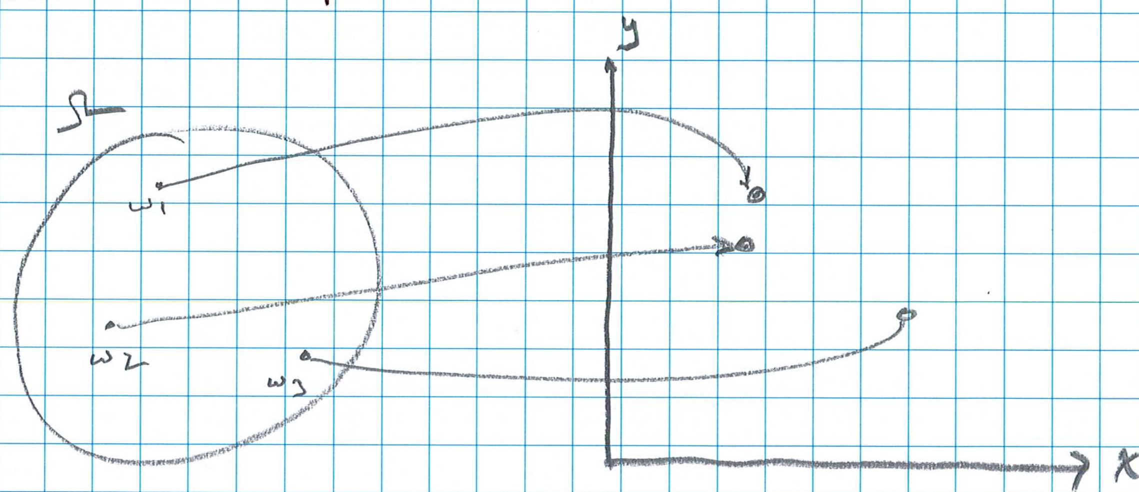
$$Y : \Omega \longrightarrow \mathbb{R}$$




which are individually random variables.

Such are called <u>jointly distributed rvs.</u>

<u>Equivalent Interpretation</u>




$$\begin{pmatrix} X \\ Y \end{pmatrix} : \Omega \longrightarrow \mathbb{R}^2$$

$$Z = X + iY \qquad Z : \Omega \rightarrow \mathbb{C}$$

**Example:** $\Omega = [0, 2\pi)$

$A \subset \Omega$, $A \in \mathcal{F}$ (Borel sets)

$$P(A) \triangleq \frac{\text{length}(A)}{2\pi}$$

Define $X(\theta) = \cos\theta$

$Y(\theta) = \sin\theta$

$$Z(\theta) = X(\theta) + jY(\theta)$$
$$= e^{j\theta}$$