ECE 30200 – Sec. 001 **4/3/23**

Exam 2

Name: Solution PUID: ______________

General Instructions:

- You have **60 minutes** to complete the exam.
- Write your name on every page of the exam.
- Please **do not** write on the backs of pages.
- The exam is **closed book** and **closed notes**.
- Calculators are **not allowed**.
- Problems contain several parts each, with space near the problem statement for work and to display your final answer. In some cases axes are provided for plots. Please **put a box around your final answer** to help the graders understand your work and to guide us in determining partial credit. To get partial credit, **work must be shown** and your thinking must be explained and well-ordered.
- Please do not leave early as it is disruptive to those working around you.

Do not open the exam until you are told to begin.

Name: ____________________

Problem 1. *Betting on Roulette.* [32 pts. total]

Roulette is a casino-based gambling game. The American style of roulette wheel has spaces numbered 0, 00, and 1, 2, ..., 36 (for 38 spaces). Spaces are also colored red, black, and green. There are 18 red spaces, 18 black spaces, and 2 green spaces. See the image below.

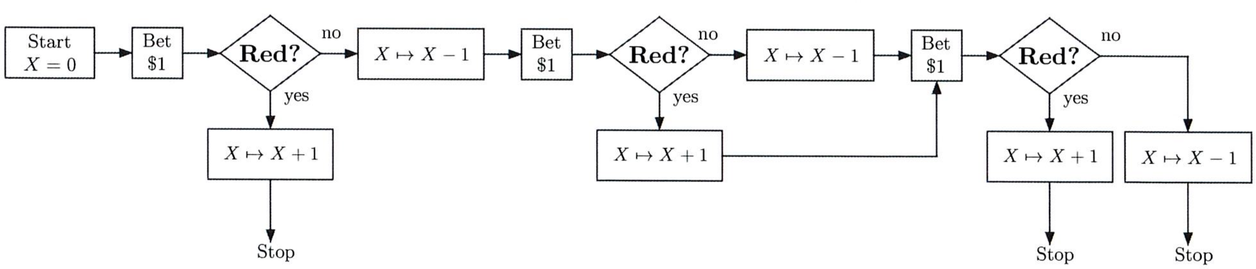
The machine is designed such that when the wheel is spun and a ball is introduced the probability of the ball coming to rest in any of the 38 spaces is equal to 1/38. Subsequent operations of the machine are independent.

Consider the following strategy (also diagrammed in the figure below).

1. You start by betting \$1 on red.

2. If the ball lands on ...

    - one of the 18 red spaces, you win the bet and you get your \$1 bet back plus \$1 of profit giving you \$1 more than when you started. At this point you quit.
    - one of the 20 non-red spaces, you lose the bet giving you \$1 less than when you started. But at this point you continue by placing \$1 bets on each of the next two spins of the wheel (irregardless of how they turn out) and then you quit.

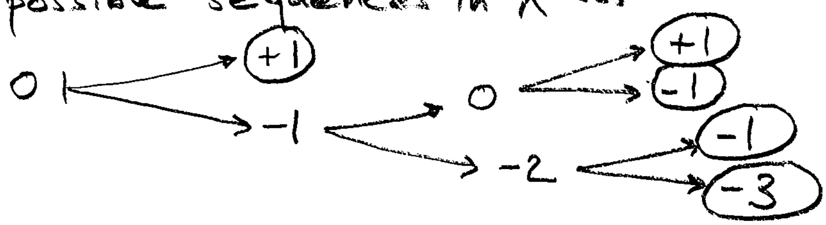


Let X denote your winnings when you finally quit.

Problem 1. (cont'd.) Name: ____________________

(a) [3 pts] X can take a finite number of positive and negative integer values. What values may X take with non-zero probability?

If we follow the block diagram from start to all of the possible ways to stop we would have the following possible sequences in X ...



at a "fork" upper branch corresponds to winning & lower branch to losing

The circle values are the possible ending states for winnings X is

$$X = +1, -1, -3$$

(b) [1 pt] Let p be the probability of winning on a single trial of the Roulette wheel when betting on red. What is the value of p as an exact fraction?

38 spaces, equally likely
18 are red

$$P\left(\left\{\begin{matrix}\text{winning on a} \\ \text{single trial}\end{matrix}\right\}\right) = \frac{18}{38} = \frac{9}{19} = p$$

Problem 1. (cont'd.) Name: ____________________

(c) [8 pts] Find $P(X > 0)$ in terms of p.

From the sequences of success and failure ending in the possible final values of X, which we listed in part (a)

$$\{X=+1\} = \{S\} \cup \{FSS\} = \{X>0\}$$

$$\therefore P(X>0) = P(\{S\}) + P(\{FSS\})$$
$$= p + (1-p)p^2$$

(d) [8 pts] Find $\mathrm{E}[X]$ in terms of p.

For this we need to look at how the other possible values of X occur

$$\{X=-1\} = \{FSF\} \cup \{FFS\}$$
$$\{X=-3\} = \{FFF\}$$
$$\therefore P(\{X=-1\}) = 2p(1-p)^2 \qquad P(\{X=-3\}) = (1-p)^3$$

Then
$$E[X] = (+1)\left[p+(1-p)p^2\right] + (-1)\left[2p(1-p)^2\right] + (-3)\left[(1-p)^3\right]$$

Of course we can simplify this expression, but it might just make this part harder to grade. Numerical evaluation is another matter, of course.

$p = 0.47$
$p^2 = 0.2209 \approx 0.22$
$p^3 = 0.103823 \approx 0.10$

$$\begin{array}{r} 2209 \\ 47 \\ \hline 15463 \\ 8836 \\ \hline 103823 \end{array} \qquad \begin{array}{r} 47 \\ 47 \\ \hline 329 \\ 188 \\ \hline 2209 \end{array}$$

Problem 1. (cont'd.) Name: ______________________

(e) [10 pts] It is enlightening to compute the values of $P(X > 0)$ and E[X] for the actual value of p found in (b). Since, you don't have a calculator, assume that $p = 0.47$ and $1 - p = 0.53$ and limit your calculations to 2 decimal places. Write down the values of the probability of positive winnings and the expected winnings.

$p + (1-p)p^2 = p + p^2 - p^3$
$2p(1-p)^2 = 2p(1-2p+p^2) = 2p - 4p^2 + 2p^3$
$(1-p)^3 = (1-p)(1-2p+p^2) = 1 - 2p + p^2 - p + 2p^2 - p^3$
$= 1 - 3p + 3p^2 - p^3$

$$\begin{array}{r} .47 \\ .22 \\ \hline .69 \\ -.10 \\ \hline .59 \end{array}$$
$\longrightarrow P(X=+1) = 0.59$
$= P(X>0)$

$2p = .94$
$2p^3 = .20$
$4p^2 = .88$

$$\begin{array}{r} .94 \\ .20 \\ \hline 1.14 \\ -.88 \\ \hline .26 \end{array}$$

$\therefore P(X=-1) = .26$

$3p = 1.41$
$3p^2 = .66$
$p^3 = .10$

$$\begin{array}{r} 1.66 \\ -1.41 \\ \hline .25 \\ -.10 \\ \hline .15 \end{array}$$

$\therefore P(X=-3) = .15$

Now for $E[X]$
$= (+1)(.59) + (-1)(.26) + (-3)(.15)$
$= -0.12$

$$\begin{array}{r} .59 \\ -.26 \\ \hline .33 \\ -.45 \\ \hline -.12 \end{array}$$

Summarizing

$P(X>0) = 0.59$
$E[X] = -0.12$

(f) [2 pts] Are you convinced that the strategy outlined above is a winning strategy? Explain your answer.

It's not a winning strategy. In spite of the fact that the probability of positive winnings is greater that 1/2, the player loses an average of \$.12 with every complete round

Name: ______________________

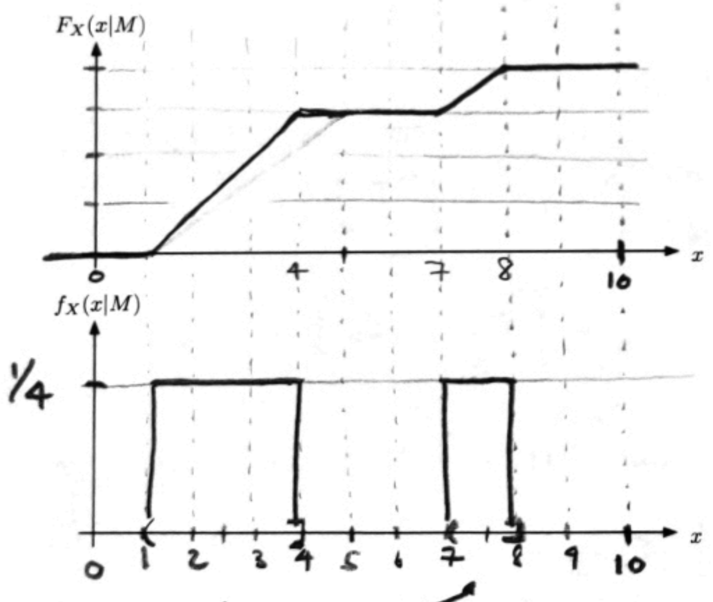
Problem 2. *Conditional CDF, PDF, Expectation, and Variance Associated with a Uniform Random Variable.* [24 pts. total]

Let X be a uniform random variable on the interval $(0, 10)$. Define an event $M = \{1 < X \leq 4\} \cup \{7 < X \leq 8\}$.

(a) [2 pts] Find $P(M)$.

$$P(M) = \frac{(4-1)+(8-7)}{10} = \frac{4}{10} = \frac{2}{5}$$

(b) [10 pts] Compute and sketch the conditional CDF $F_X(x|M)$ and the conditional PDF $f_X(x|M)$.



Obtain this by integrating the piecewise unif. density below.

conditional pdf is supported on these two intervals; it will still be unif.

Problem 2. (cont'd.) Name: ______________

(c) [6 pts] Find the conditional mean E[X|M].

$$E[X|M] = \int x f_X(x|M)\,dx = \frac{1}{4}\int_1^4 x\,dx + \frac{1}{4}\int_7^8 x\,dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2}\Big|_1^4 + \frac{x^2}{2}\Big|_7^8\right] = \frac{1}{4}\left[\frac{16}{2} - \frac{1}{2} + \frac{64}{2} - \frac{49}{2}\right]$$

$$= \frac{1}{8}\left[15 + 15\right] = \frac{30}{8} = \frac{15}{4} = 3.75$$

(d) [6 pts] Find the conditional variance Var[X|M].

$$E[X^2|M] = \frac{1}{4}\left[\frac{x^3}{3}\Big|_1^4 + \frac{x^3}{3}\Big|_7^8\right] = \frac{1}{4}\frac{1}{3}\left[64 - 1 + 512 - 343\right]$$

$$= \frac{1}{12}\left[63 + 169\right] = \frac{232}{12}$$

$$Var[X|M] = E[X^2|M] - \left(E[X|M]\right)^2$$

$$= \frac{232}{12} - \left(\frac{15}{4}\right)^2 = \frac{232}{12} - \frac{225}{16}$$

$$= \frac{928 - 675}{48} = \frac{253}{48} \cong 5.27$$

$$\begin{array}{r} {}^{3}64 \\ 8 \\ \hline 512 \end{array} \quad \begin{array}{r} {}^{6}49 \\ 7 \\ \hline 343 \end{array} \quad \begin{array}{r} 512 \\ -\,343 \\ \hline 169 \end{array} \quad \begin{array}{r} 169 \\ 63 \\ \hline 232 \end{array}$$

Name: ______________________

Problem 3. *Hypothesis Testing for a Signal in Gaussian Noise.* [20 pts. total]

At the output of a receiver there is observed a random variable Y ...

$$Y = S + N$$

where $S = \mu > 0$ if the signal is present (hypothesis H_1) or $S = 0$ if the signal is absent (hypothesis H_0) and N is Gaussian with mean zero and variance σ^2. The detector compares the observation to a threshold t and if it is larger it declares that H_1 is true. On the other hand, if the observation is less than the threshold t then the detector declares that H_0 is true. Sometimes, we use a notation such as

$$\text{If} \cdots \begin{cases} Y \geq t & \text{then declare } H_1 \text{ true} \\ Y < t & \text{then declare } H_0 \text{ true} \end{cases} .$$

Another way to write the same thing would be to use $\hat{H}_1$ and $\hat{H}_0$ to denote the events that the detector declares it thinks H_1 or H_0 is true, respectively. In other words ...

$$\{Y \geq t\} = \hat{H}_1 \quad \text{and} \quad \{Y < t\} = \hat{H}_0.$$

With this notation, the following important probabilities are defined

- The detection probability, which is the probability that the detector declares that H_1 is true given that it actually is true: $P_D = P(\hat{H}_1 | H_1)$.
- The false-alarm probability, which is the probability that the detector declares that H_1 is true given that it is not true: $P_{FA} = P(\hat{H}_1 | H_0)$.

(a) [10 pts] Find the value of the threshold t such that the false alarm probability is one percent, i.e., $P_{FA} = 0.01$. Use the table of normalized Gaussian cdf values given. The value of t will be expressed in terms of the noise standard deviation σ.

$$P_{FA} = P(\hat{H}_1 | H_0) = P(Y \geq t | H_0) \overset{\text{set}}{=} 0.01$$

Under H_0

$$Y = N \sim N(0, \sigma^2) \implies \frac{Y}{\sigma} \sim N(0,1)$$

Now for the false-alarm probability

$$0.01 = P(Y \geq t | H_0) = P\left(\frac{Y}{\sigma} \geq \frac{t}{\sigma} \middle| H_0\right) = 1 - \Phi(t/\sigma)$$

in terms of the unit normal cdf ...

Problem 3. (cont'd.) Name: ____________________

$$\Rightarrow \Phi\left(\frac{t}{\sigma}\right) = 1 - 0.01 = 0.99$$

$$\Rightarrow \Phi^{-1}\left(\Phi(t/\sigma)\right) = \Phi^{-1}(0.99)$$

$$\Rightarrow t = \sigma\,\Phi^{-1}(0.99) = 2.33\sigma$$

(b) [10 pts] For the value of t found in (a), find the value of the signal amplitude $\mu > 0$ such that the detection probability is 90 percent, i.e., $P_D = 0.9$. The required value of μ will also be expressed in terms of the noise standard deviation σ.

$$P_D = P(\hat{H}_1 | H_1) = P(Y \geq t | H_1)$$

Under H_1

$$Y = \mu + N \sim N(\mu, \sigma^2) \Rightarrow \frac{Y-\mu}{\sigma} \sim N(0,1)$$

$$P_D = P(Y \geq t | H_1) = P\left(\frac{Y-\mu}{\sigma} \geq \frac{t-\mu}{\sigma} \Big| H_1\right) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$

$$\stackrel{\text{Set}}{=} 0.9$$

$$\Rightarrow \Phi\left(\frac{t-\mu}{\sigma}\right) = 0.1$$

$$\Phi^{-1}\left(\Phi\left(\frac{t-\mu}{\sigma}\right)\right) = \Phi^{-1}(0.1) = -1.28$$

$$\frac{t-\mu}{\sigma} = -1.28 \Rightarrow \mu = +1.28\sigma + t$$

$$= 1.28\sigma + 2.33\sigma$$

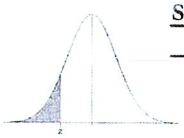
$$= 3.61\sigma$$

$$\begin{array}{r} 1.28 \\ 2.33 \\ \hline 3.61 \end{array}$$

Problem 3. (Gaussian CDF Table Part 1) Name: ____________________

Standard Normal Distribution Tables

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.



Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

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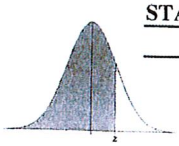
$\Phi^{-1}(0.10027) = -1.28$

Problem 3. (Gaussian CDF Table Part 2) Name: ______________



STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

$\Phi^{-1}(0.99010) = 2.33$

Name: ______________________

Problem 4. *Functions of an Exponential Random Variable.* [24 pts. total]

Let T be an exponential random variable parameter $\lambda > 0$, i.e., its pdf is ...

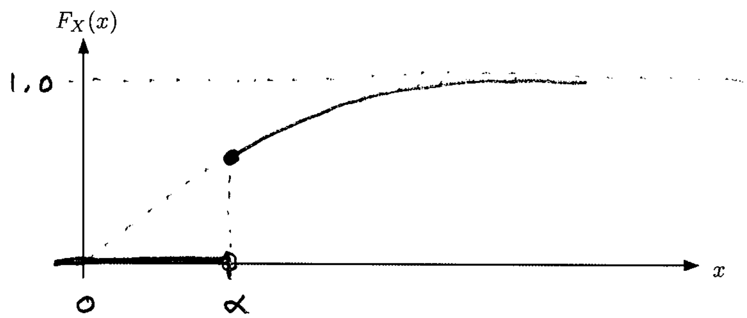
$$F_T(t) = \int_0^t \lambda e^{-\lambda \alpha} d\alpha = -\frac{\lambda}{\lambda} e^{-\lambda\alpha}\Big|_{\alpha=0}^{t}$$

$$= 1 - e^{-\lambda t} \text{ for } t \geq 0.$$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

For a number $\alpha > 0$ define new random variables $X = \max(T, \alpha)$ and $Y = \min(T, \alpha)$.

(a) [6 pts] Find an expression and sketch the plot of the cdf $F_X(x)$ of the random variable X.

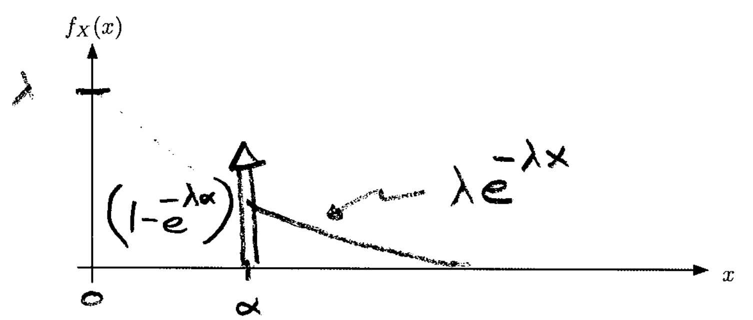


$$F_X(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x < \alpha \\ 1 - e^{-\lambda x} & \text{if } x \geq \alpha \end{cases}$$

There is a jump in the cdf of X because

$$P(X = \alpha) = P(T \leq \alpha) = 1 - e^{-\lambda\alpha}$$

(b) [6 pts] Find an expression and sketch the plot of the pdf $f_X(x)$ of the random variable X.

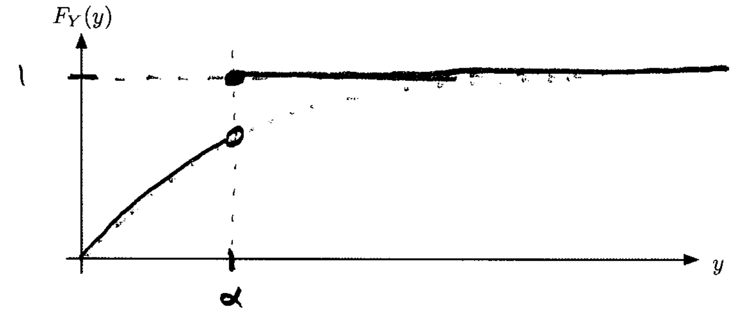


Because of the jump in the cdf, there will be a delta function in the pdf ...

$$f_X(x) = \frac{d}{dx} F_X(x) = (1 - e^{-\lambda\alpha})\delta(x - \alpha) + \lambda e^{-\lambda x} 1_{[\alpha, \infty)}(x)$$

Problem 4. (cont'd.) Name: ______________________

(c) [6 pts] Find an expression and sketch the plot of the cdf $F_Y(y)$ of the random variable Y.



$Y = \min(T, \alpha)$

If $y < \alpha$ then

$P(Y \le y) = P(T \le y) = F_T(y)$

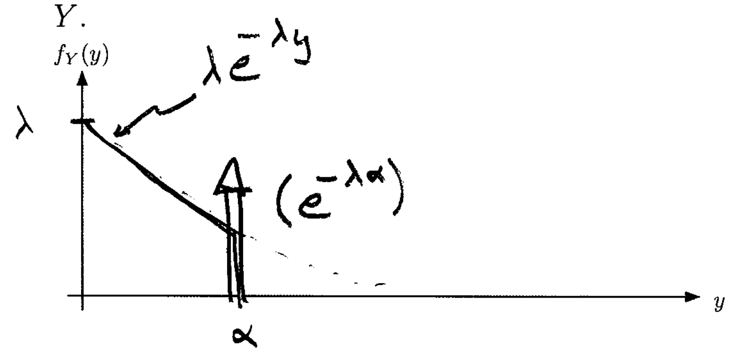
$= 1 - e^{-\lambda y}$

If $y > \alpha$ then

$P(Y \le y) = P(\Omega) = 1$

$\Rightarrow$ a jump at $y = \alpha$ of size $e^{-\lambda\alpha}$

(d) [6 pts] Find an expression and sketch the plot of the pdf $f_Y(y)$ of the random variable Y.



$$f_Y(y) = \frac{d}{dy} F_Y(y) = \lambda e^{-\lambda y} \, 1_{[0,\alpha)}(y) + e^{-\lambda\alpha} \delta(y-\alpha)$$