ECE 30100 – Signals and Systems

J. V. Krogmeier

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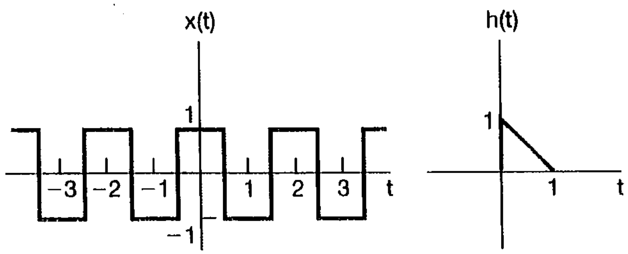
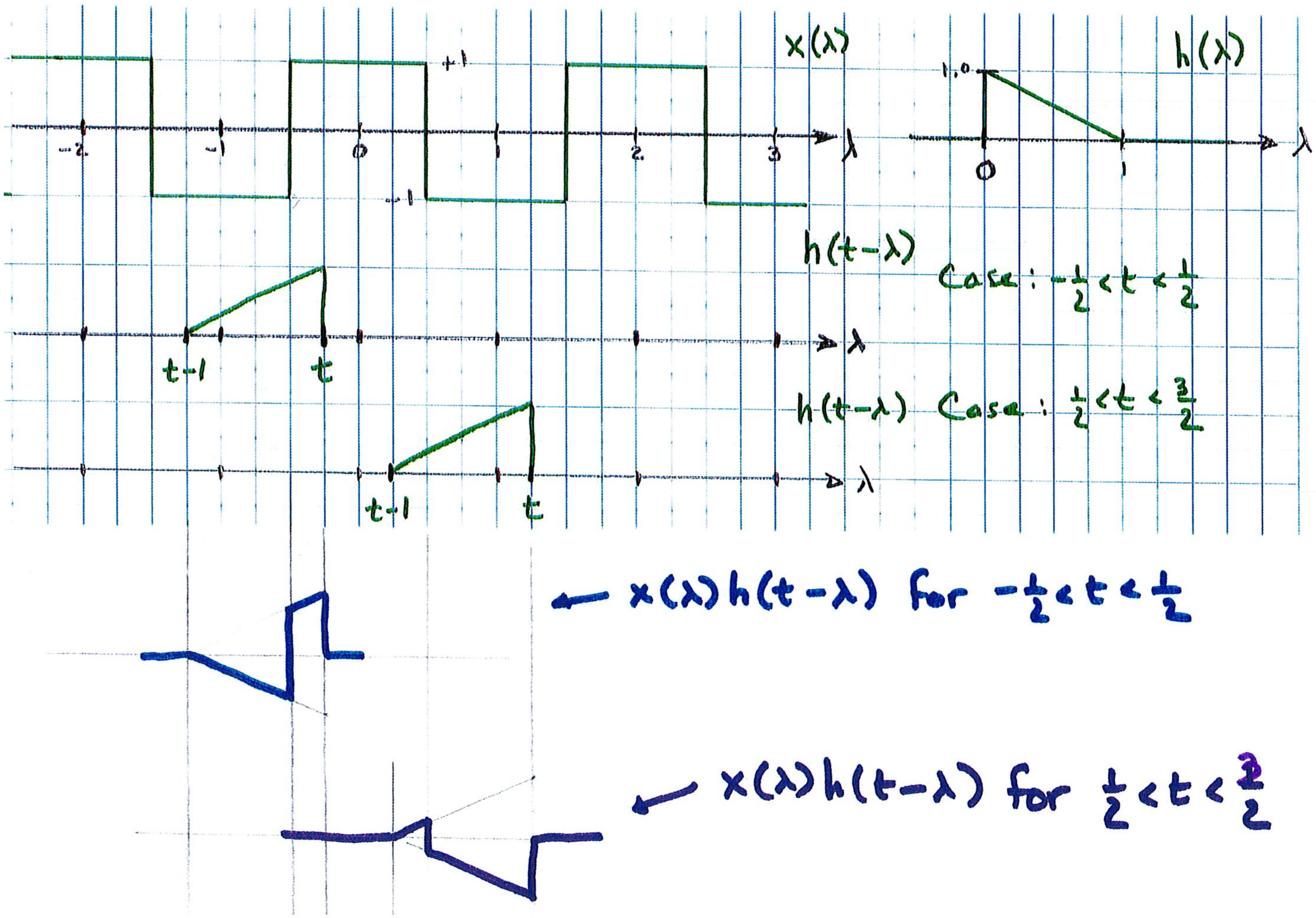
CT Convolution Example (OW Prob. 2.22)



Find $y = x * h$ ...

CT Convolution Example (OW Prob. 2.22 – cont'd.)



CT Convolution Example (OW Prob. 2.22 – cont'd.)

Compute via Integration $\quad h(\lambda) = 1 - \lambda \text{ for } 0 \le \lambda \le 1$

$$\therefore h(t-\lambda) = 1-(t-\lambda) = 1-t+\lambda \text{ for } 0 \le t-\lambda \le 1$$

$$\Longleftrightarrow t-1 \le \lambda \le t$$

$\boxed{\text{Case: } -\frac{1}{2} < t < \frac{1}{2}}$

$$x*h(t) = \int_{t-1}^{-1/2} -(1-t+\lambda)\,d\lambda + \int_{-1/2}^{t} (1-t+\lambda)\,d\lambda$$

$$= -\left[(1-t)\lambda + \frac{\lambda^2}{2}\right]_{\lambda=t-1}^{-1/2} + \left[(1-t)\lambda + \frac{\lambda^2}{2}\right]_{\lambda=-1/2}^{t}$$

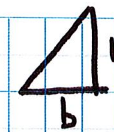
$\boxed{\text{Case: } \frac{1}{2} < t < \frac{3}{2}}$

$$x*h(t) = \int_{t-1}^{1/2} (1-t+\lambda)\,d\lambda + \int_{1/2}^{t} -(1-t+\lambda)\,d\lambda$$

$$= \left[(1-t)\lambda + \frac{\lambda^2}{2}\right]_{\lambda=t-1}^{1/2} - \left[(1-t)\lambda + \frac{\lambda^2}{2}\right]_{\lambda=1/2}^{t}$$

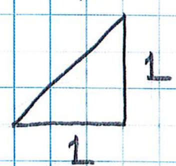
CT Convolution Example (OW Prob. 2.22 – cont'd.)

Compute Using Triangle Area Formula:

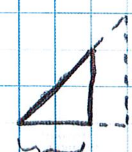


$\Rightarrow A = \frac{1}{2} bh$

The triangles that come up here are:



$\Rightarrow$ Area $= \frac{1}{2}$



$-\frac{1}{2}-(t-1)$

$= \frac{1}{2} - t$

$\Rightarrow$ Area $= \frac{1}{2}\left(\frac{1}{2}-t\right)^2$

$\triangleq T_1(t)$ defined for $-\frac{1}{2} < t < \frac{1}{2}$

$\frac{1}{2}-(t-1)$

$= \frac{3}{2} - t$

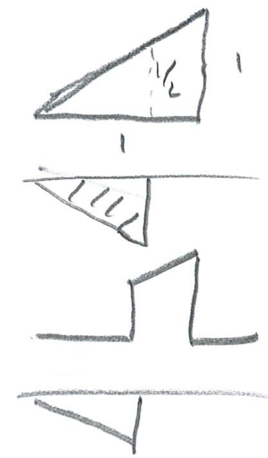
$\Rightarrow$ Area $= \frac{1}{2}\left(\frac{3}{2}-t\right)^2$

$\triangleq T_2(t)$ defined for $\frac{1}{2} < t < \frac{3}{2}$

Case: $-\frac{1}{2} < t < \frac{1}{2}$

Area$\{$  $\}$ $= \frac{1}{2} - 2T_1(t)$

$= \frac{1}{2} - \left(\frac{1}{2}-t\right)^2$



Case: $\frac{1}{2} < t < \frac{3}{2}$

Area$\{$  $\}$ $= -\frac{1}{2} + 2T_2(t)$

$= -\frac{1}{2} + \left(\frac{3}{2}-t\right)^2$

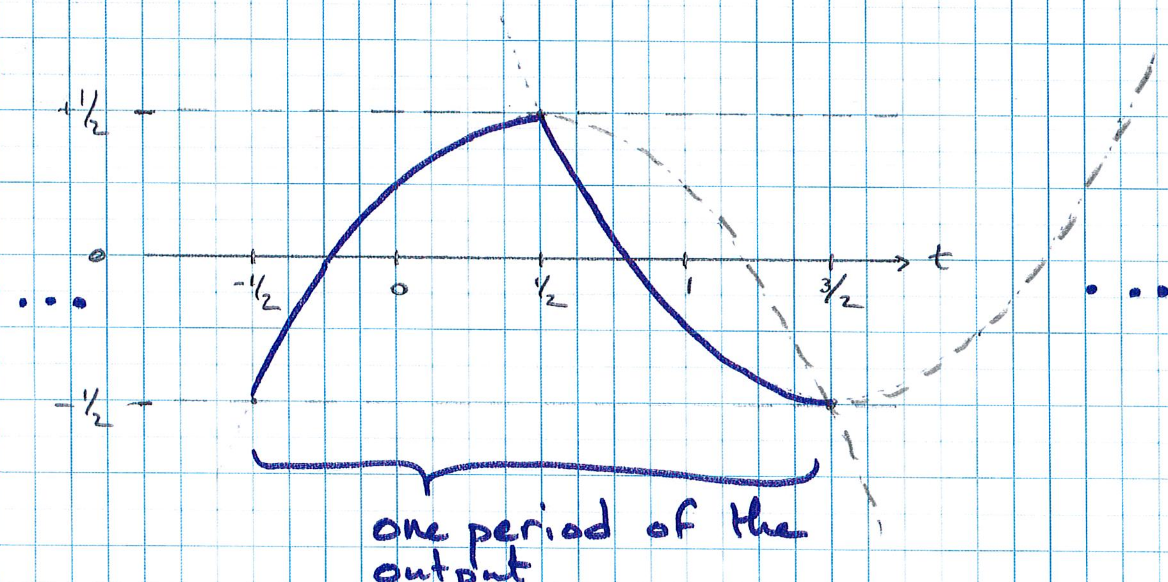
CT Convolution Example (OW Prob. 2.22 – cont'd.)

If want to Plot Output

Then actually easier to leave answer in form:

$$x * h(t) = \begin{cases} \frac{1}{2} - \left(\frac{1}{2} - t\right)^2 & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\ -\frac{1}{2} + \left(\frac{3}{2} - t\right)^2 & \text{for } \frac{1}{2} < t < \frac{3}{2} \end{cases}$$

and periodic for remaining t values.



Properties of LTI Systems (a.k.a., Properties of Convolution)

$$x * h = \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} x[k]\, h[n-k] & \text{DT} \\ \displaystyle\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau & \text{CT} \end{cases}$$

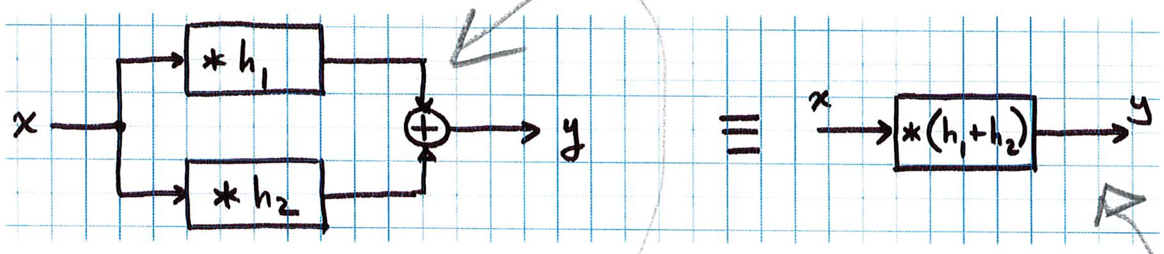
- Commutativity
- Distributivity
- Associativity
- Memoryless
- Invertibility
- Causality
- Stability

Commutative Property of LTI Systems

$$x \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ * h\end{matrix}} \longrightarrow y = x * h \qquad\qquad x * h = h * x$$

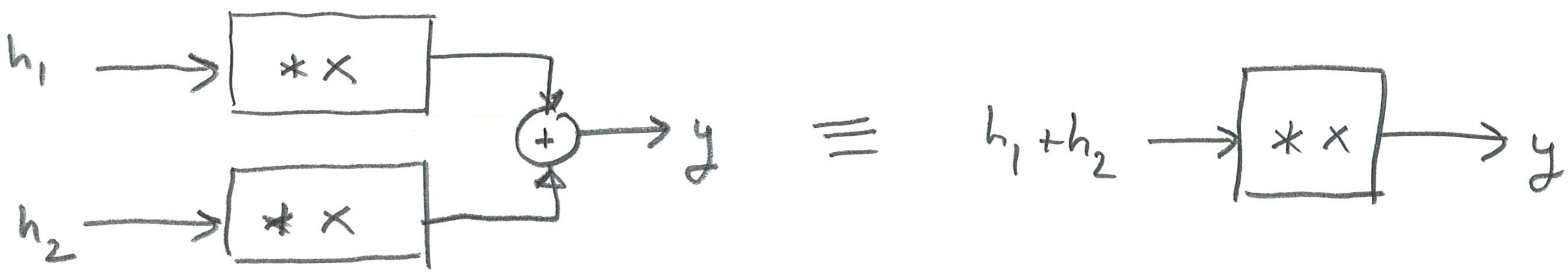
$$h \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ * x\end{matrix}} \longrightarrow y = h * x$$

Distributive Property of LTI Systems

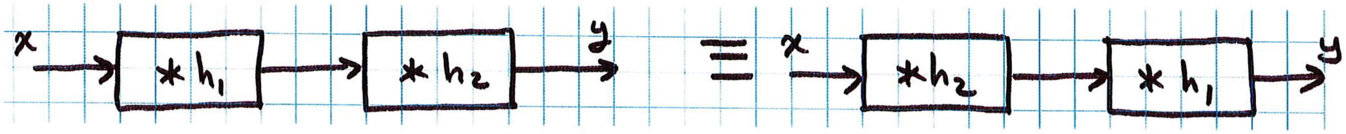


Convolution "Distributes" over Addition

$$x*(h_1+h_2) = x*h_1 + x*h_2$$

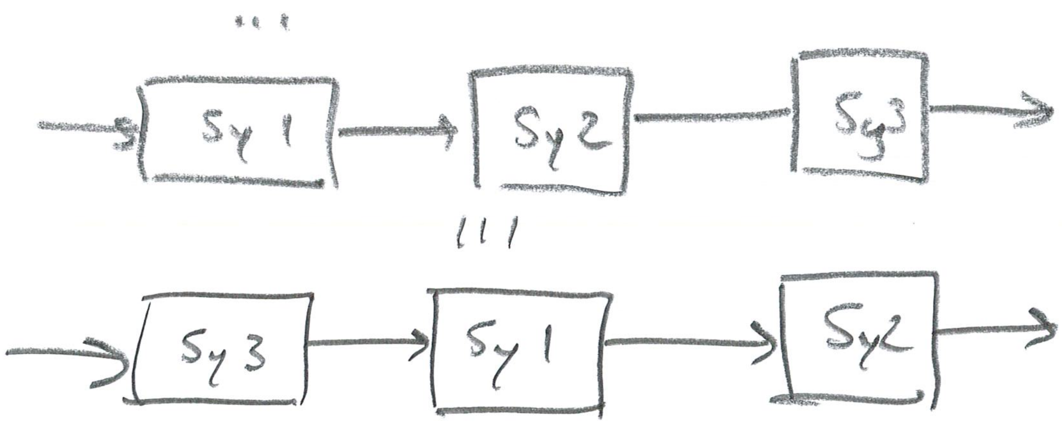


Associative Property of LTI Systems



$$x * (h_1 * h_2) = (x * h_1) * h_2 = (x * h_2) * h_1$$

Ideal case that in a syst. with a cascade of subsystems we can reorder the blocks



Memoryless Property of LTI Systems

$$y[n] = \sum_k x[k]\,h[n-k]$$

$$y[n] = \text{Funct}\big(x[n]\big)$$

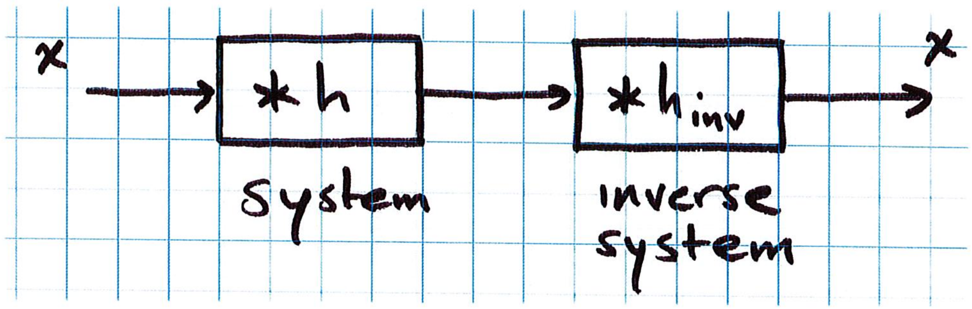
$$= \sum_{k \neq n} x[k]\,h[n-k] + x[n]\,h[0]$$

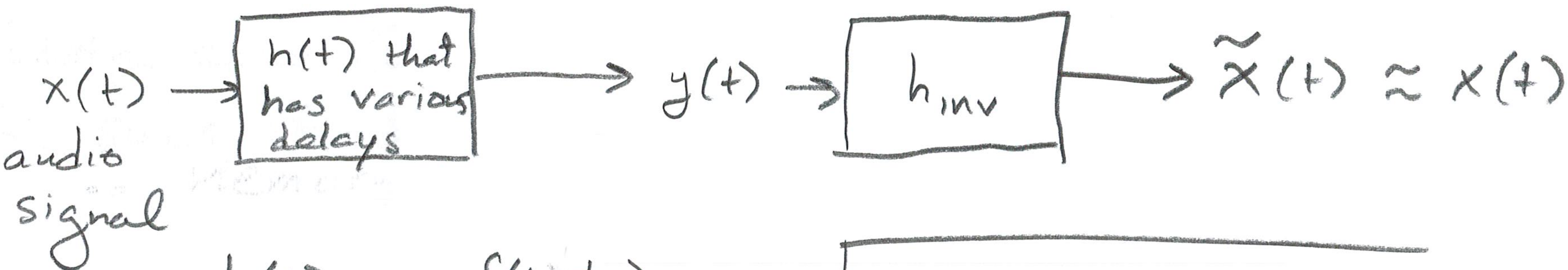
$\therefore$ memoryless $\iff h[n] = K\delta[n]$

C.T.

memoryless $\iff h(t) = K\delta(t)$

Invertibility Property of LTI Systems





$$h(t) = \alpha_0 \delta(t - t_0) + \alpha_1 \delta(t - t_1) + \alpha_2 \delta(t - t_2) \vdots$$

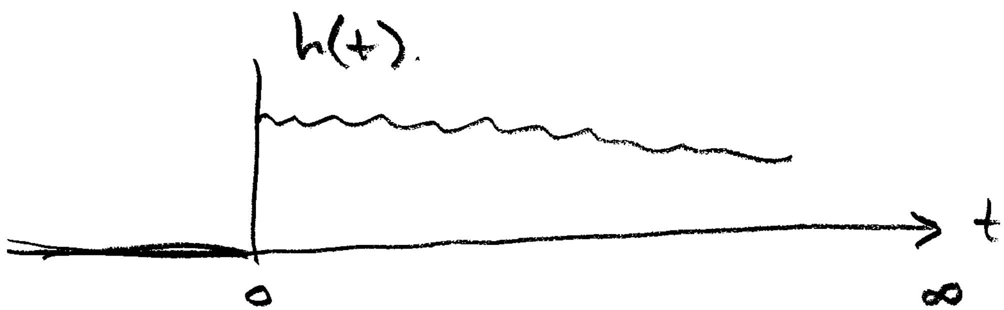
$$h * h_{inv} = \delta$$

Causality Property of LTI Systems

LTI system with h is causal $\Longleftrightarrow$ Impulse resp. is a causal signal

$$h(t) = 0 \quad t < 0$$

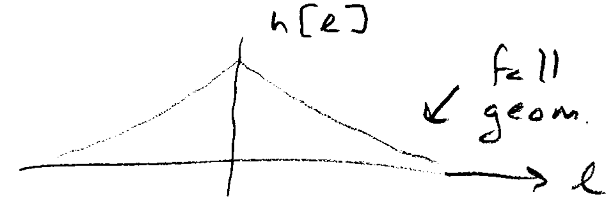
$$h[k] = 0 \quad k < 0$$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \overset{\text{caus. input}}{=} \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\underset{\text{Caus } h}{=} \int_{\cancel{0}\ -\infty}^{t} x(\tau)\, \underline{h(t-\tau)}\, d\tau$$

C.T. $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

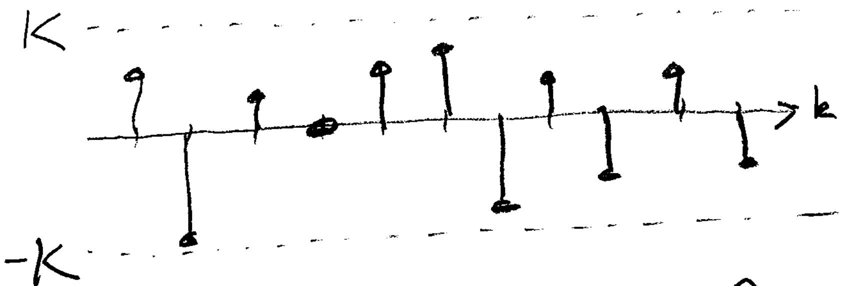


Stability Property of LTI Systems

Bounded input – Bounded Output (BIBO)

D.T. $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{l=-\infty}^{\infty} x[n-l]\, h[l]$

Bounded input $\Longleftrightarrow$ $\exists\, K > 0$ st $|x[k]| < K \quad \forall k \in \mathbb{Z}$



Try to find a bound for $y[n]$:

Triangle inequality

$$|y[n]| = \left| \sum_{l=-\infty}^{\infty} x[n-l]\, h[l] \right| \leq \sum_{l=-\infty}^{\infty} |x[n-l]| \cdot |h[l]|.$$

$$\leq \sum_{l=-\infty}^{\infty} K\, |h[l]| = K \sum_{l=-\infty}^{\infty} |h[l]|$$

$\therefore \sum_{l=-\infty}^{\infty} |h[l]| < \infty \Longrightarrow$ BIBO stable.

Invertibility Example – DT

$h * h_{inv} = \delta$ --- Given h, how to solve for h_{inv}?

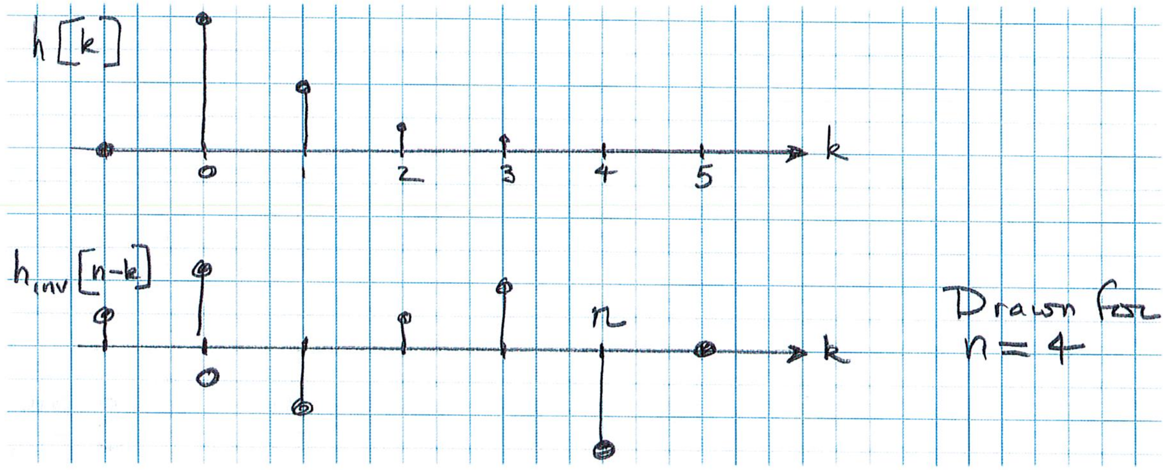
$h[n] = a^n u[n]$

$$\delta[n] = \sum_k h[k]\, h_{inv}[n-k] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

$\boxed{\text{Can we guess a solution?}}$

Invertibility Example – DT (cont'd.)

$$\sum_{k=0}^{n} a^k h_{inv}[n-k] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$



$n=0$: $\displaystyle\sum_{k=0}^{0} a^k h_{inv}[0-k] = 1 \cdot h_{inv}[0] = 1 \Rightarrow h_{inv}[0] = 1$

$n=1$: $\displaystyle\sum_{k=0}^{1} a^k h_{inv}[1-k] = 1 \cdot h_{inv}[1] + a \cdot \underbrace{h_{inv}[0]}_{1} = 0$

$$h_{inv}[1] = -a$$

$n=2$: Do it, then see $h_{inv}[2] = 0$

Invertibility Example – CT

$h * h_{inv} = \delta$ --- solve for h_{inv} in CT

$h(t) = e^{-at}u(t)$

Solve for $h_{inv}(t)$ s.t. $\delta(t) = \int_{-\infty}^{\infty} h(\lambda)\, h_{inv}(t-\lambda)\, d\lambda$

Can show: $h_{inv}(t) = \delta'(t) + a\,\delta(t)$

$$\int_{-\infty}^{\infty} h(\lambda)\left[\delta'(t-\lambda) + a\,\delta(t-\lambda)\right] d\lambda$$

$$h'(t) = \frac{d}{dt}\left\{e^{-at}u(t)\right\} = -ae^{-at}u(t) + e^{-at}\overset{\delta(t)}{u'(t)} = -ae^{-at}u(t) + \delta(t)$$

$x(t) \longrightarrow \boxed{\frac{d}{dt}(\cdot)} \longrightarrow$ is LTI with impulse resp. $\delta'(t)$