ECE 30100 – Signals and Systems

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DT Convolution Sum (OW pp. 75-90)

Fact: Any DT LTI system is completely characterized by its impulse response, i.e., $\cdots$

$$\delta[n] \rightarrow \boxed{\text{LTI}} \rightarrow h[n]$$

$\cdots$ for any input $x[n]$ the LTI system's output $y[n]$ is the DT convolution of the input and the impulse response

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = x * h[n]$$

where

$$x * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Notation

$x * h$

$x * h[\cdot]$

$x * h[n]$

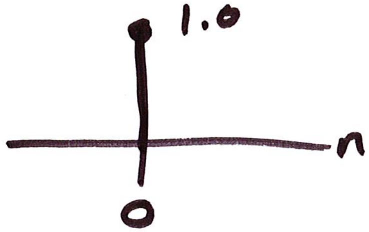
Convolution commutes: $x * h = h * x$

$$x*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{c.o.v. } m = n-k \Leftrightarrow k = n-m$$

$$= \sum_{m=+\infty}^{-\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} \overbrace{x[n-m]h[m]}^{h[m]x[n-m]}$$

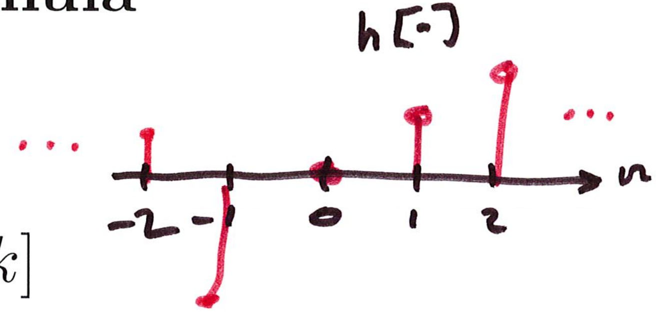
$$= h * x[n]$$

Proof of DT Convolution Sum Formula



$$\delta[\cdot] \rightarrow \boxed{\text{LTI}} \rightarrow h[\cdot]$$

$$\delta[\cdot - k] \rightarrow \boxed{\text{LTI}} \rightarrow h[\cdot - k]$$



An arbitrary input signal $x[\cdot]$ can be written uniquely as a weighted sum of shifted delta functions ...

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

→ Claim

$$= \begin{cases} 0 & n \neq k \\ x[n] & n = k \end{cases}$$

$$x[\cdot] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[\cdot - k]$$

$n = 17$

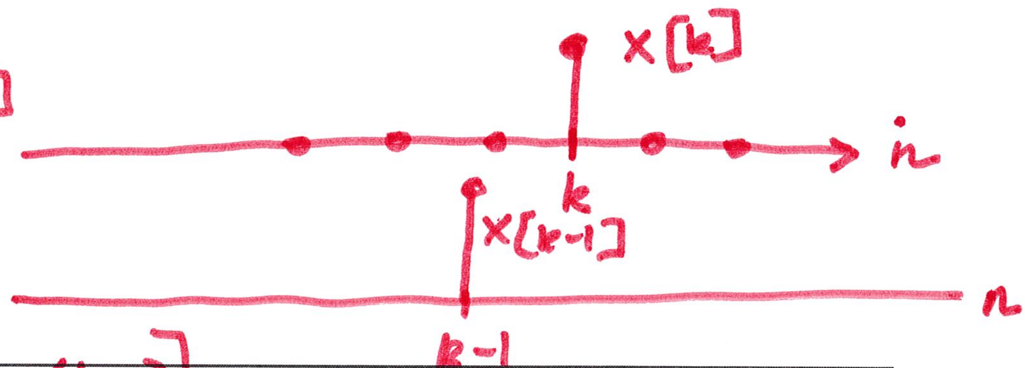
$$\sum_{k} x[k]\,\delta[17-k]$$

$$\| $$

$$x[17]$$

$x[k]\,\delta[n-k] \rightarrow$

$x[k-1]\,\delta[n-(k-1)] \rightarrow$



Arb. sign.

$$x[\cdot] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[\cdot - k]$$

Apply $x[\cdot]$ to $\longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h[\cdot]\end{array}} \longrightarrow x * h[\cdot]$

$$y[\cdot] = \text{LTI}\left\{x[\cdot]\right\} = \text{LTI}\left\{\sum_k x[k]\,\delta[\cdot - k]\right\}$$

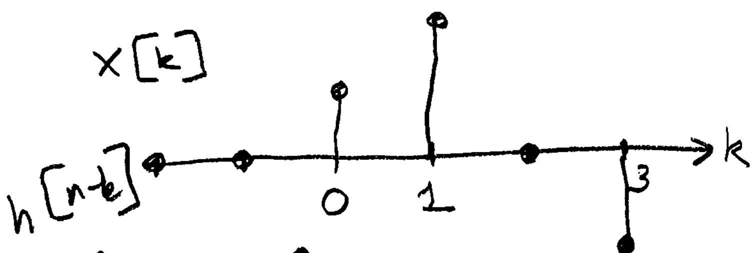
$$= \sum_k x[k]\,\text{LTI}\left\{\delta[\cdot - k]\right\} = \sum_k x[k]\,h[\cdot - k]$$

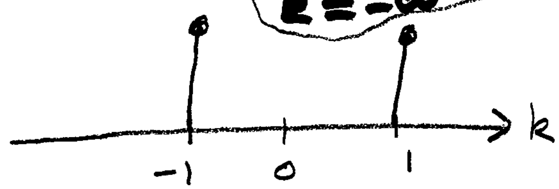
$$\Longrightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \qquad n \in \mathbb{Z}$$

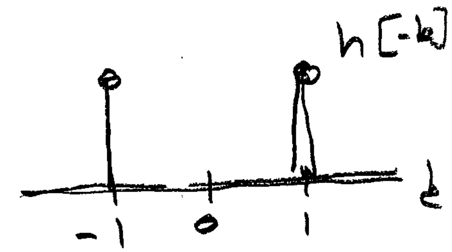
DT Convolution Example (OW Prob. 2.1(a))

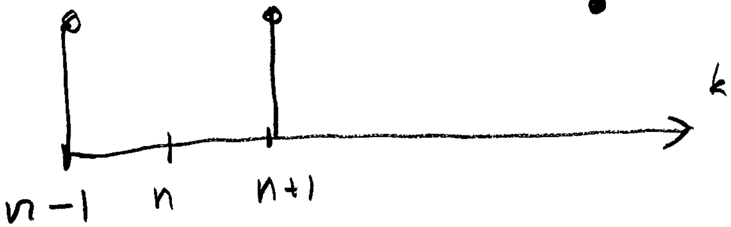
$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$$
$$h[n] = 2\delta[n+1] + 2\delta[n-1]$$

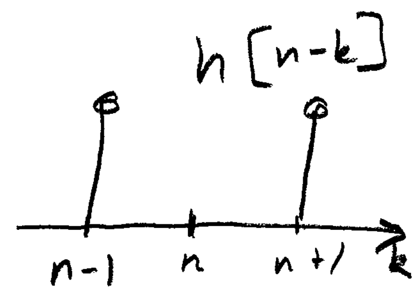
Find $y = x * h$ ... $\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$







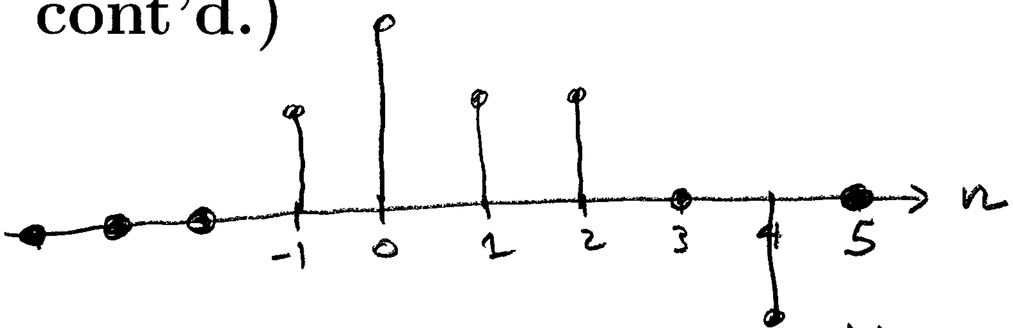




Case: $n+1 < 0 \Leftrightarrow n < -1 \Rightarrow y[n] = 0$

$n-1 > 3 \Leftrightarrow n > 4 \Rightarrow y[n] = 0$

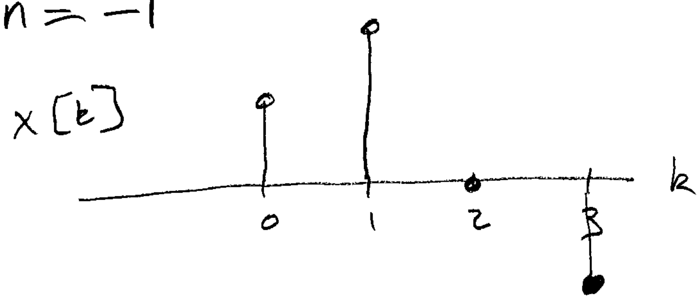
DT Convolution Example (OW Prob. 2.1(a) – cont'd.)

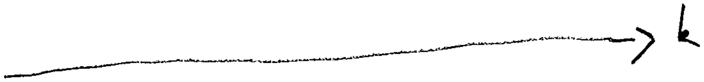


$y[n] = h * x[n]$

Case $-1 \le n \le 4$ have possible overlap.

$n = -1$





$h[n] = 2\delta[n+1] + 2\delta[n-1]$

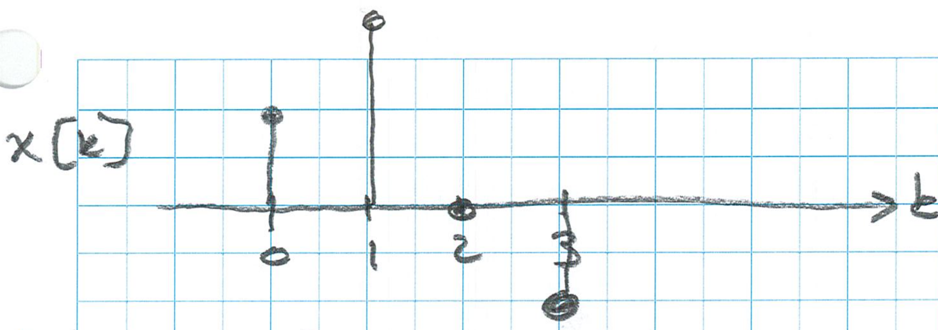
Any time you convolve a signal with a δ or shifted δ you get the signal back possibly shifted.

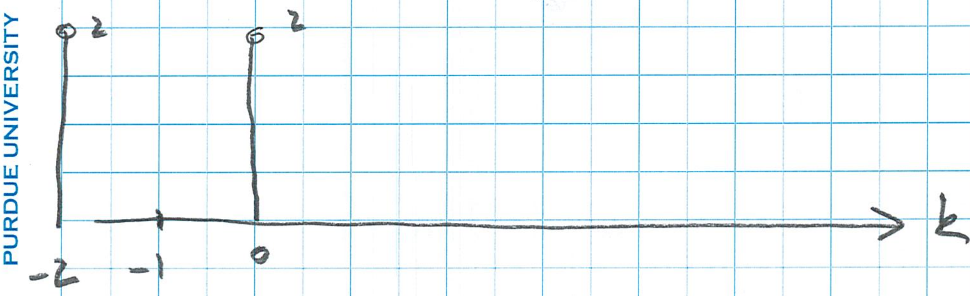
$$y[\cdot] = h * x[\cdot] = \{2\delta[\cdot+1] + 2\delta[\cdot-1]\} * x$$

$$= 2x[\cdot+1] + 2x[\cdot-1]$$

or with time var. n back in ...

$y[n] = 2x[n+1] + 2x[n-1]$



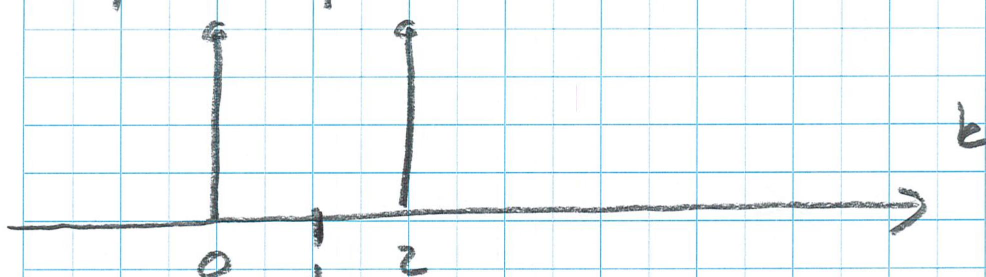


$h[-1-k]$

$x*h[-1] = 2$

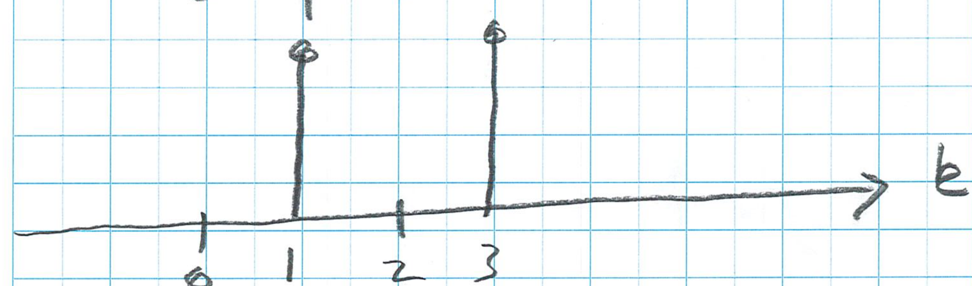
$h[0-k]$

$x*h[0] = 4$



$h[1-k]$

$x*h[1] = 2$



$2 \cdot 2 + (-1)2$

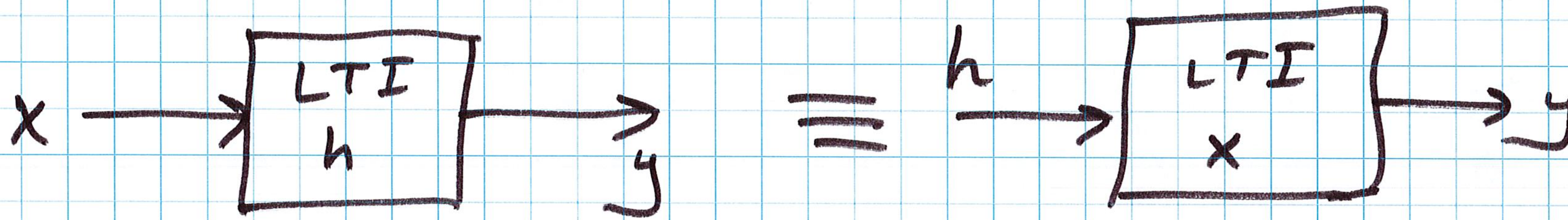
$= 2$

$x*h[2] = 2$

$\Rightarrow \quad x*h[3] = 0$

$x*h[4] = -2$

$$h * x = x * h$$

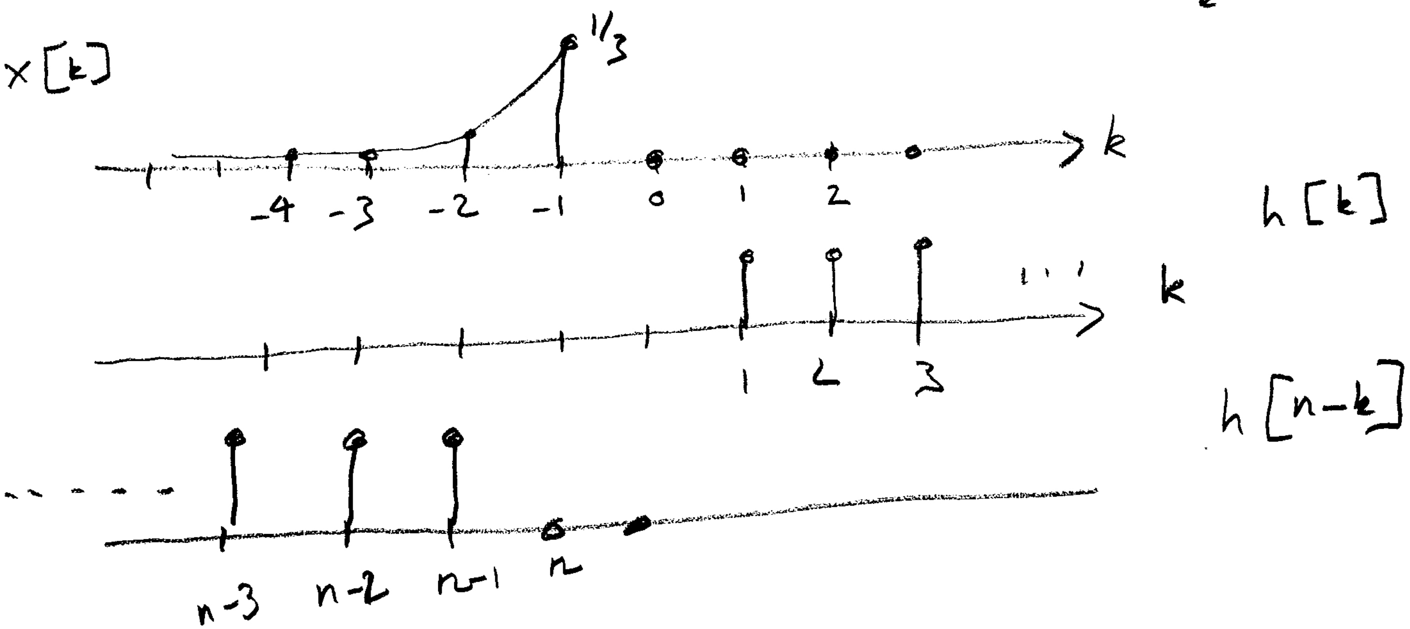


DT Convolution Example (OW Prob. 2.6)

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1]$$
$$h[n] = u[n-1]$$

Find $y = x * h$ ...

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



DT Convolution Example (OW Prob. 2.6 – cont'd.)

$n < 0$

$$y[n] = \sum_{k=-\infty}^{n-1} \left(\frac{1}{3}\right)^{-k} = \frac{1}{2} 3^n \quad n<0$$

$n \geq 0$

$$y[n] = \frac{1}{2}$$

$$\sum_{\ell=0}^{\infty} a^{\ell} = \frac{1}{1-a}$$

$0 < a < 1$