Session 21

Examples DTFS

OW 3.28 (c, d)

(c) x[n] periodic of per 4

$$x[n] = 1 - \sin\left(\frac{\pi n}{4}\right) \qquad 0 \le n \le 3$$

$$\frac{\pi}{4}(n+N) = \frac{\pi}{4}n + 2\pi K$$

(d) Same sig. per. of per 12

$$X_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-j2\pi kn/4} \qquad k = 0,1,2,3$$

$$= \frac{1}{4}\left[1 + \underbrace{\left(1 - \sin\frac{\pi}{4}\right)}_{\frac{\sqrt{2}-1}{\sqrt{2}}} e^{-j\pi k/2} + 0\, e^{-j\cdots} + \underbrace{\left(1 - \sin\frac{3\pi}{4}\right)}_{\frac{\sqrt{2}-1}{\sqrt{2}}} e^{-j3\pi k/2}\right]$$

$$X_0 = \frac{3-\sqrt{\ }}{4} \qquad X_1 = \frac{1}{4} \quad ,,,,$$

(d) $x[n] = 1 - \sin\frac{\pi n}{4} \qquad 0 \le n \le 11$ and periodically extend.

Play with Matlab.

OW 3.36 (b)

In general $x[n] = z^n \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h[n]\end{array}} \longrightarrow y[n] = H(z)z^n$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k] z^{-k}}_{H(z)}$$

(In general the values of z must be constrained st. sum converges ..)

Imagine $x[n]$ is N-periodic with $x[n] = \sum_{k=0}^{N-1} X_k e^{+j 2\pi k n/N}$ (*)

(*) Is a linear comb. of sigs of form $e^{j\omega_0 n}$ $\quad \omega_0 = \frac{2\pi k}{N}$

Therefore

$$y[n] = \sum_{k=0}^{N-1} \underbrace{X_k H\left(e^{j2\pi k/N}\right)}_{Y_k} e^{+j2\pi kn/N}$$

Proves: ① N-periodic input $\Longrightarrow$ N-periodic output

② $Y_k = H\left(e^{j2\pi k/N}\right) X_k$

To do the HW problem, need $H(z)$...

Two ways... Way #1 find $h[n]$ from the diff. eqn.

$$y[n] - \frac{1}{4} y[n-1] = x[n] \Longrightarrow h[n] = \left(\frac{1}{4}\right)^n u[n]$$

$\Longrightarrow$ get $H(e^{j\omega})$ from $h[n]$.

Way 2 Let $x[n] = e^{j\omega n} \Rightarrow y[n] = H(e^{j\omega})e^{j\omega n}$

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

$$H(e^{j\omega})e^{j\omega n} - \frac{1}{4}H(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

$$H(e^{j\omega}) - \frac{1}{4}H(e^{j\omega})e^{-j\omega} = 1$$

$$H(e^{j\omega})\left[1 - \frac{1}{4}e^{-j\omega}\right] = 1$$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$ is the DTFT of $h[n]$.

In case of N periodic input ...

$$H\left(e^{j2\pi k/N}\right) = \frac{1}{1-\frac{1}{4}e^{-j2\pi k/N}} \Longrightarrow \text{Need DTFS } X_k$$

$k = 0, 1, \ldots N-1$

$$x[n] = \cos\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{\pi}{2}n\right)$$

$$= \underbrace{\frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2}}_{\text{8-periodic}} + 2\left(\frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}\right)$$

8-periodic

$$e^{j2\pi kn/8}$$

$$= \underbrace{\frac{1}{2}e^{-j2\pi n/8}}_{k=-1} + \underbrace{\frac{1}{2}e^{+j2\pi n/8}}_{k=1} + \underbrace{e^{-j4\pi n/8}}_{k=-2} + \underbrace{e^{+j\overset{2\pi\cdot 2}{4\pi} n/8}}_{k=2}$$

$$X_k = \begin{cases} \frac{1}{2} & k=-1,\ k=7 \\ \frac{1}{2} & k=1 \\ 1 & k=2 \\ 1 & k=-2,\ k=6 \end{cases}$$

DTFS

$$x[n] = \sum_{k=0}^{N-1} X_k e^{+jk\left(\frac{2\pi}{N}\right)n}$$

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

3.48 $x[n] = x[n+N]$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

ⓐ $y[n] = x[n-n_0]$

$$= \sum_{k=0}^{N-1} a_k \underbrace{e^{jk\left(\frac{2\pi}{N}\right)(n-n_0)}}_{e^{jk\left(\frac{2\pi}{N}\right)n}\, e^{-jk\left(\frac{2\pi}{N}\right)n_0}}$$

$$= \sum_{k=0}^{N-1} \left(a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}\right) e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\Longrightarrow Y_k = a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}.$$

(b) $y[n] = x[n] - x[n-1]$

$$= \sum_{k=0}^{N-1} a_k e^{+jk\left(\frac{2\pi}{N}\right)n} - \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)(n-1)}$$

$$= \sum_{k=0}^{N-1} \underbrace{a_k\left(1 - e^{-j2\pi k/N}\right)}_{Y_k} e^{jk\left(\frac{2\pi}{N}\right)n}$$

(d) $w[n] = x[n] + x\left[n + \frac{N}{2}\right]$

$(-1)^n = e^{j\pi n}$

Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{+j\omega n}\, d\omega$$

Spectrum is periodic in ω of period 2π

"DC" are $\omega \cong 0, \pm 2\pi, \pm 4\pi, \ldots$

highest freqs are $\omega \simeq \pm\pi, \pm 3\pi, \pm 5\pi$

What is a fast DT signal? $x[n] = (-1)^n$

slow DT $x[n] = 1 \quad \forall n$

Relationship Between DTFT and DTFS

① $\tilde{x}[n]$ which is N-periodic.

$\updownarrow$

$\tilde{X}_k$ be the DTFS coeffs.

and now define $x[n] = \begin{cases} \tilde{x}[n] & n = 0, 1, \dots N-1 \\ 0 & \text{else} \end{cases}$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\omega n}$$

Remember

$$\tilde{X}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-jk\left(\frac{2\pi}{N}\right)n} = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$

OW 5.21 (b) $x[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1]$ Find $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}e^{j\omega}\right)^{-n}$$

$m = -n$

$$= \sum_{m=1}^{\infty} \left(\frac{1}{2}e^{j\omega}\right)^{m} = \frac{1}{1-\frac{1}{2}e^{j\omega}} - 1$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

$|a| < 1$

$$= \frac{\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}}$$