Session 20

$x[n] = x[n+N] \quad \forall n \in \mathbb{Z} \iff x[\cdot]$ is periodic with period N

We observed

$$x[n] = e^{j\omega_0 n}$$

is periodic $\iff \omega_0/2\pi =$ a rational #.

Further if fix an N and solve for all DT complex sinusoids with per N, then there are only N distinct N-periodic sinusoids

$$\left.\begin{array}{l} \phi_\ell[n] = e^{j2\pi \ell n/N} \qquad n \in \mathbb{Z} \\ \ell = 0, 1, 2 \ldots N-1 \end{array}\right\} \text{these } N \text{ sigs have fundamental freqs which are multiples of } 2\pi/N$$

$\Longrightarrow$ Next is DTFS.

$$x[\cdot] = \sum_{\ell} a_\ell \phi_\ell[\cdot]$$ a lin. comb. of the complex sinusoids.

Since $\phi_\ell[\cdot]$ are periodic in ℓ you can always write the l.c. in terms of N contiguous values of ℓ

e.g. $\ell = 0, 1, 2 \dots N-1$

or

$\ell = 2, 3, 4 \cdots N-1, N, N+1$

OW uses the

$$\sum_{\ell = \langle N \rangle}$$

notation to indicate this.

$$\int_T$$

$$\sum_{\ell=\langle N\rangle} \qquad \sum_{\ell=0}^{N-1}$$

<u>**FACT DTFS**</u> If $x[\cdot]$ is any N-periodic sequence $\exists$ a unique set of DTFS coeffs $a_0 \; a_1 \; \dots \; a_{N-1}$ st

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk(2\pi/N)n}$$

analysis eqN

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{+jk(2\pi/N)n}$$

synthesis eqN

CT version of this ...

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$x(t) = \sum_{k=\infty}^{\infty} X_k\, e^{+jk\omega_0 t}$$

"Proof"

$$x[n] = \sum_{k=0}^{N-1} a_k e^{+jk(2\pi/N)n} = \sum_{k=0}^{N-1} \left(\frac{1}{N} \sum_{m=0}^{N-1} x[m] e^{-jk(2\pi/N)m} \right) e^{jk(2\pi/N)n}$$

The rest follows from reversing the order of sum and ...

$$\sum_{n=0}^{N-1} e^{jk(2\pi/N)n} = \begin{cases} N & \text{if } k = 0, \pm N, \pm 2N \\ 0 & \text{otherwise} \end{cases}$$

For "otherwise" use

$$\sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a} \qquad a \neq 1$$

DTFS Examples

$x[n] = \sin \omega_0 n \quad n \in \mathbb{Z} \quad \text{where } \frac{2\pi}{\omega_0} = \text{rat.}$

(a) $\omega_0 = \frac{2\pi}{N} \implies$ periodic with fund period N

$$\sin \omega_0 n = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{j2}$$

$$= \frac{1}{j2} e^{j\frac{2\pi}{N}n} - \frac{1}{j2} e^{-j\frac{2\pi}{N}n}$$

$$= \sum_{k=0}^{N-1} a_k e^{+jk\left(\frac{2\pi}{N}\right)n}$$

$$\implies \underline{\underline{a_1 = \frac{1}{j2}}}, \quad a_{-1} = -\frac{1}{j2} = \underline{\underline{a_{N-1}}}$$

(b) $\omega_0 = \frac{2\pi M}{N}$ $\quad \gcd(M,N)=1$

$$\sin \omega_0 n = \sin \frac{2\pi M}{N} n = \frac{1}{j2} e^{jM\left(\frac{2\pi}{N}\right)n} - \frac{1}{j2} e^{-jM\left(\frac{2\pi}{N}\right)n}$$

$$a_M = \frac{1}{j2} \qquad a_{-M} = -\frac{1}{j2} = a_{N-M}$$

and all other DTFS coeffs are zero.

(c) $\omega_0 = \frac{2\pi M}{N}$ where M, N have factors in common

Above still works. But $x[n]$, while periodic-N, is also periodic with some smaller period. So could represent in terms of a smaller DTFS.

For example $N = \overset{=2K}{\text{even}}$, $M = 2$

$$\sin \omega_0 n = \frac{1}{j2} e^{j\cancel{2}\left(\frac{2\pi}{\cancel{2}K}\right)n} - \frac{1}{j2} e^{-j\cancel{2}\left(\frac{2\pi}{\cancel{2}K}\right)n}$$

after cancelling factor 2 $\Longrightarrow$ K point DTFS $\Longrightarrow$ $a_1 = \frac{1}{j2}$, $a_{-1} = -\frac{1}{j2} = a_{K-1}$

Periodic Convolution Property

$x[n]$ $y[n]$ both N-periodic

Define periodic (aka circular) convolution

$$z[n] = x \circledast y[n]$$

$$= \sum_{k=0}^{N-1} x[k]\, y[n-k]$$

↳ sum is over any period.

Can see $z[n]$ is also N periodic.

$$x[n] \longleftrightarrow X_k$$

$$y[n] \longleftrightarrow Y_k$$

$$z[n] \longleftrightarrow Z_k$$

$$Z_k = \frac{1}{N} \sum_{n=0}^{N-1} z[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$Z_k = \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{l=0}^{N-1} x[l]\,y[n-l]\right)e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l]\,N\,\underbrace{\frac{1}{N}\sum_{n=0}^{N-1} y[n-l]\,e^{-jk\left(\frac{2\pi}{N}\right)n}}_{n'=n-l}$$

$$\underbrace{\frac{1}{N}\sum_{n'=-l}^{N-1-l} y[n']\,e^{-jk\left(\frac{2\pi}{N}\right)n'}}_{Y_k}\,e^{-jk\left(\frac{2\pi}{N}\right)l}$$

$$= N Y_k \underbrace{\frac{1}{N}\sum_{l=0}^{N-1} x[l]\,e^{-jk(2\pi/N)l}}_{X_k}$$

$$z[n] = x \circledast y\,[n] \longleftrightarrow Z_k = N X_k Y_k$$

The Discrete-Time Fourier Transform

$x[n]$ no longer periodic

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\updownarrow$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{+j\omega n} \, d\omega$$