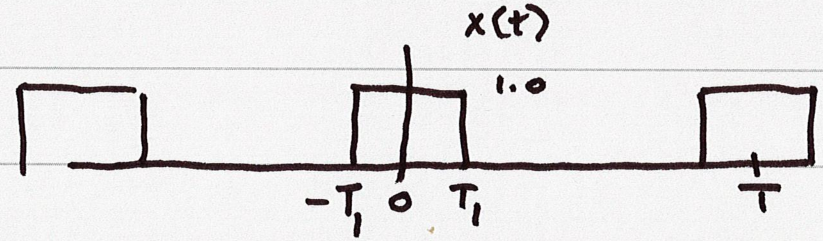
CT Fourier Transform (OW Ch 4) Session 13



$\longleftrightarrow$ $\omega_0 = 2\pi/T$ $\qquad a_k = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$

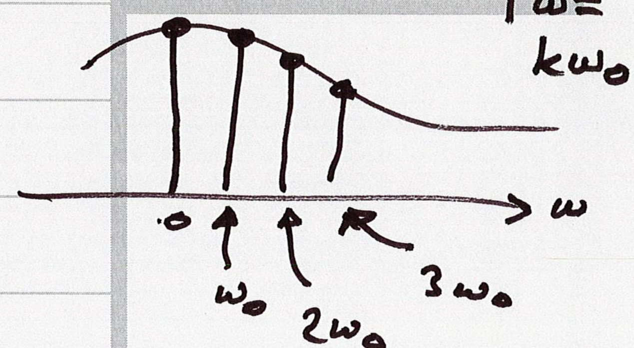
F.T. concerns aperiodic, but can think of the limit of the signal $x(t)$ as $T \to \infty$.

<u>Note</u> Envelope of $T a_k$ samples is the function

$$\frac{2\sin\omega T_1}{\omega}$$

$$T a_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}$$

$\omega_0 = \dfrac{2\pi}{T}$ as $T \to \infty$ these samples get close together



This observation can be used to make a Reimann sum approx to the standard integral which defines the F.T.

Analysis Equation: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Synthesis Equation: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t} d\omega$

ω is radians/sec.

$2\pi f = \omega$

↳ Usually specify freqs in Hz [1 cycle/sec]

C.O.V.

$\omega = 2\pi f$

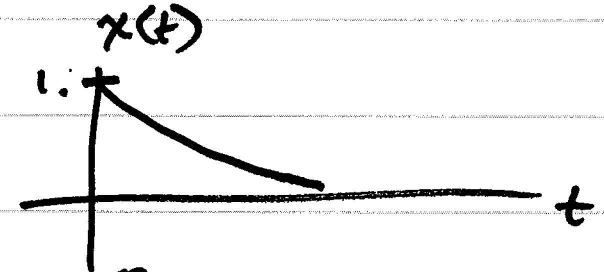
$d\omega = 2\pi df$

So what about $X(j\omega)$?

When you integrate or diff. wrt ω think of $X(j\omega)$ as being $\tilde{X}(\omega)$

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(t) = e^{-at}u(t) \quad \underline{a > 0}$$



$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t}\, dt = \int_0^\infty e^{-(a+j\omega)t}\, dt$$

$$= \cancel{\frac{1}{a(a+j\omega)}} \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t} \Big|_{t=0}^{\infty}$$

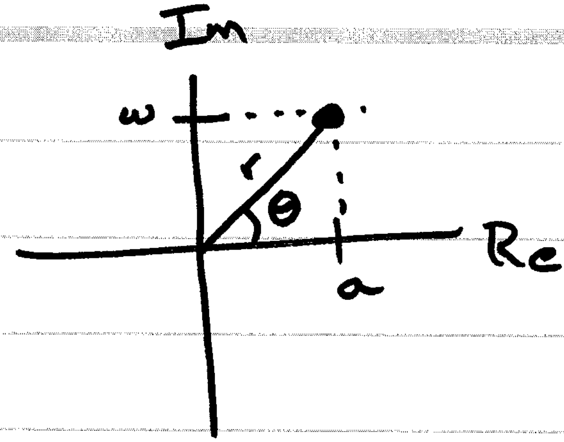
$$= \frac{-1}{a+j\omega}\left[0 - 1\right]$$

$$= \frac{1}{a+j\omega} \longrightarrow \text{rect. form } \frac{1}{a+j\omega} \frac{a-j\omega}{a-j\omega}$$

polar form $|X(j\omega)| e^{j\angle X(j\omega)}$

$$= \frac{a - j\omega}{a^2 + \omega^2} \quad Re = \frac{a}{a^2+\omega^2} \quad Im = \frac{-\omega}{a^2+\omega^2}$$

Examine den. first $\quad a + j\omega$



$a+j\omega$: $|a+j\omega| = \sqrt{a^2+\omega^2} = r$

$r\sin\theta = \omega$

$r\cos\theta = a$

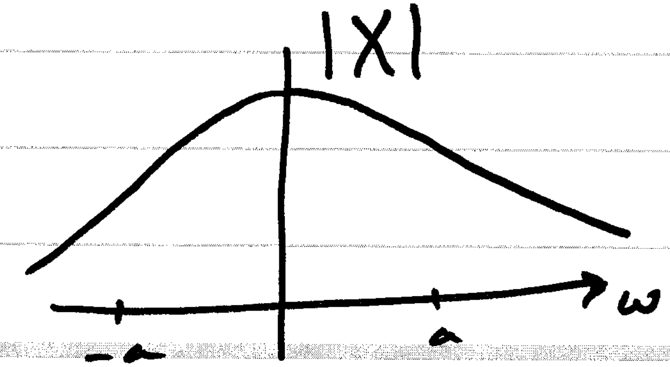
Take ratio

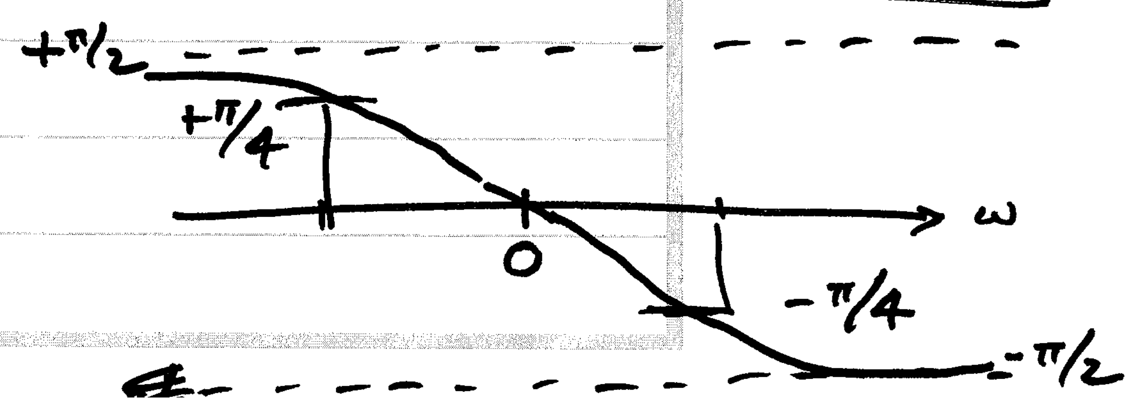
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\omega}{a}$$

$$\theta = \tan^{-1}(\omega/a)$$

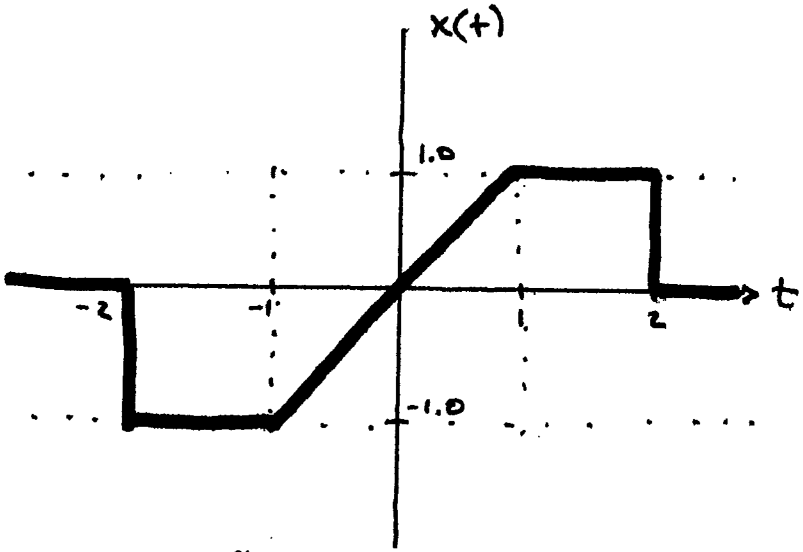
$$a+j\omega = \sqrt{a^2+\omega^2}\, e^{j\tan^{-1}(\omega/a)}$$

$$\frac{1}{a+j\omega} = \boxed{X(j\omega) = \frac{1}{\sqrt{a^2+\omega^2}}\, e^{-j\tan^{-1}(\omega/a)}}$$





Example: Problem 4.21 (g)



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= -\int_{-2}^{-1} e^{-j\omega t}\, dt + \int_{-1}^{1} t\, e^{-j\omega t}\, dt + \int_{1}^{2} e^{-j\omega t}\, dt$$

Look at the parts separately

$$\int_{1}^{2} e^{-j\omega t}\, dt = \frac{1}{-j\omega} e^{-j\omega t}\Big|_{1}^{2}$$

$$= \frac{1}{-j\omega}\left[e^{-j2\omega} - e^{-j\omega}\right]$$

$$-\int_{-2}^{-1} e^{-j\omega t}\, dt = \frac{-1}{-j\omega}\left[e^{j\omega} - e^{j2\omega}\right]$$

$$\int_{-1}^{1} t\, e^{-j\omega t}\, dt$$

parts

$\int u\, dv = uv - \int v\, du$

$u = t \quad du = dt$

$e^{-j\omega t} dt = dv$

$\frac{1}{-j\omega} e^{-j\omega t} = v.$

$\updownarrow$

$||$

$$\frac{t}{-j\omega} e^{-j\omega t}\Big|_{t=-1}^{1} - \int_{-1}^{1} \frac{1}{-j\omega} e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega} + e^{j\omega}\right] - \left(\frac{1}{-j\omega}\right)^2 e^{-j\omega t}\Big|_{-1}^{1}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega} + e^{j\omega}\right] + \left(\frac{1}{\omega^2}\right)\left[e^{-j\omega} - e^{j\omega}\right]$$

Putting the pieces together in order to simplify:

$$X(j\omega) = \frac{1}{j\omega}\left[e^{j\omega} - e^{j2\omega}\right]$$

$$- \frac{1}{j\omega}\left[e^{-j2\omega} - e^{-j\omega}\right]$$

$$- \frac{1}{j\omega}\left[e^{-j\omega} + e^{j\omega}\right] + \left(\frac{1}{\omega^2}\right)\left[e^{-j\omega} - e^{j\omega}\right]$$

$$\cancel{= \frac{1}{j\omega}\left[2e^{j\omega}\right] \quad \frac{1}{j\omega}\left[\right.}$$

$$= e^{j\omega}\left[\cancel{\frac{1}{j\omega}} - \cancel{\frac{1}{j\omega}} - \frac{1}{\omega^2}\right]$$

$$+ e^{-j\omega}\left[\cancel{\frac{1}{j\omega}} - \cancel{\frac{1}{j\omega}} + \frac{1}{\omega^2}\right]$$

$$+ e^{j2\omega}\left[-\frac{1}{j\omega}\right] + e^{-j2\omega}\left[-\frac{1}{j\omega}\right]$$

$$= -\frac{1}{\omega^2}\left[\frac{e^{j\omega} - e^{-j\omega}}{j2}\right](j2)$$

$$- \frac{1}{j\omega}\left[\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right]2$$

$$X(j\omega) = -\frac{j2}{\omega^2}\sin(\omega)$$

$$- \frac{2}{j\omega}\cos(2\omega)$$

$$= \frac{j2}{\omega}\left[\cos(2\omega) - \frac{\sin(\omega)}{\omega}\right].$$

$$X(j\omega) = \cos\left(4\omega + \pi/3\right) \qquad \omega \in \mathbb{R}$$

find $x(t)$

$$= \frac{e^{j(4\omega+\pi/3)} + e^{-j(4\omega+\pi/3)}}{2}$$

$$= \frac{1}{2}e^{j\pi/3}\,e^{j4\omega} + \frac{1}{2}e^{-j\pi/3}\,e^{-j4\omega}$$

Then I try to use the synth. formula directly ...

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,e^{j\omega t}\,d\omega$$

I get things like

$$\int_{-\infty}^{\infty} e^{j4\omega} e^{j\omega t}\,d\omega = \frac{1}{j(4+t)}\,e^{j(4+t)\omega}\Bigg|_{\omega=-\infty \;(-\Omega)}^{\infty \;(\Omega)} = \,?$$

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$$

$$x(t) = \delta(t - t_0)$$

$$\tilde{X}(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt = e^{-j\omega t_0}$$

$$\therefore\; e^{-j\omega t_0} \longleftrightarrow \delta(t - t_0)$$

$$e^{-j\omega 4} \longleftrightarrow \delta(t - 4)$$

$$e^{+j\omega 4} \longleftrightarrow \delta(t + 4)$$

$$X(j\omega) = \frac{1}{2}e^{j\pi/3} e^{j4\omega} + \frac{1}{2}e^{-j\pi/3} e^{-j4\omega}$$

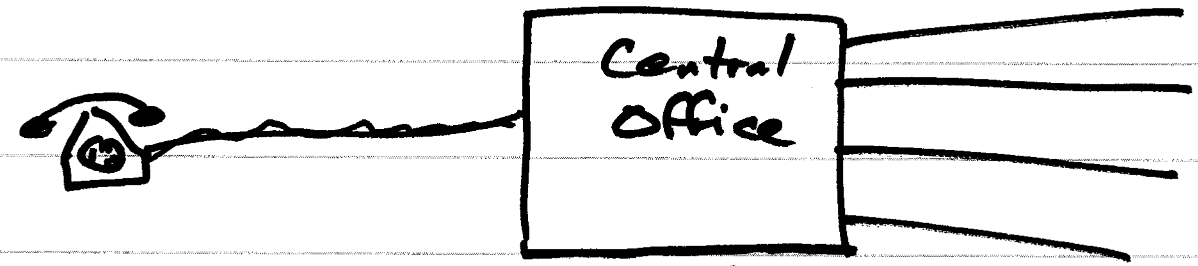
$$\updownarrow$$

$$x(t) = \left(\frac{1}{2}e^{j\pi/3}\right)\delta(t+4) + \left(\frac{1}{2}e^{-j\pi/3}\right)\delta(t-4)$$

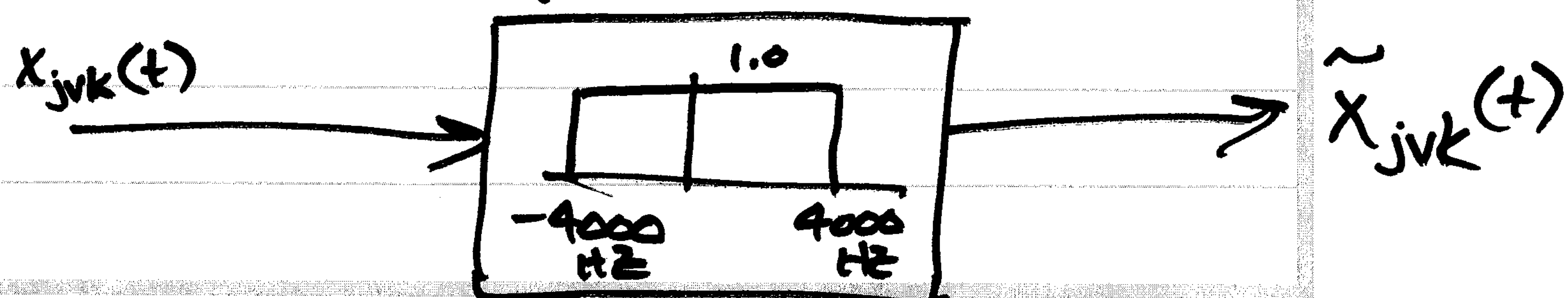
$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow y(t) = h * x(t)$$

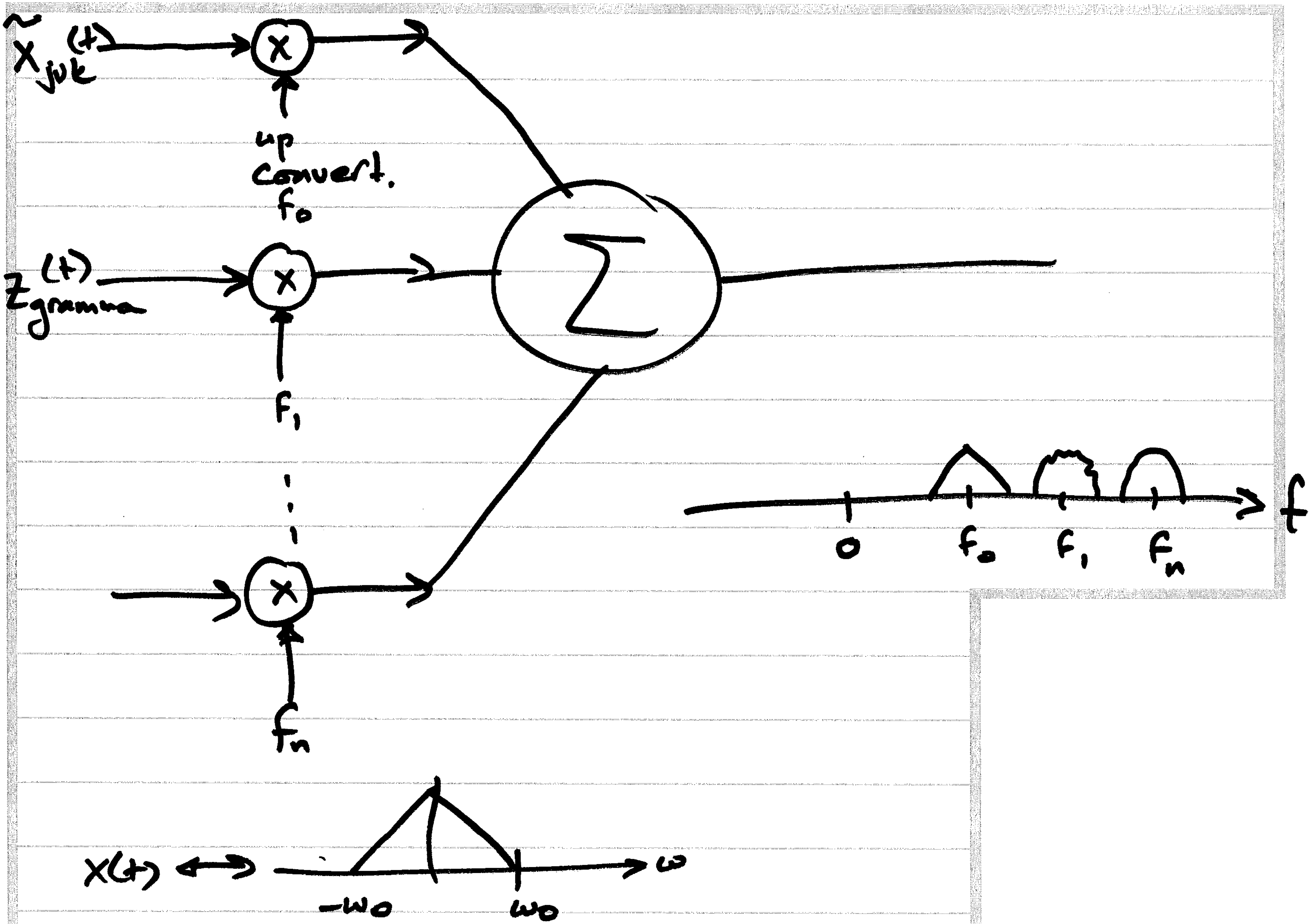
$$\Updownarrow$$

$$X(j\omega) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ H(j\omega)\end{matrix}} \longrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$



Bell Labs decided that 4 KHz was enough BW to understand speech.

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