Continuous Time Fourier Transform

$$x(t) \longleftrightarrow X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

analysis equation / forward CTFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

synthesis equation / inverse CTFT

Last time: Saw that $X(j\omega)$ is complex-valued in general...
$|X(j\omega)|$ & $\angle X(j\omega)$ [or real & imaginary parts]

However: IF $x(t)$ is real valued $\Longrightarrow X(j\omega) = X^*(-j\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

$$x^*(t) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega\right)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t}\, d\omega$$

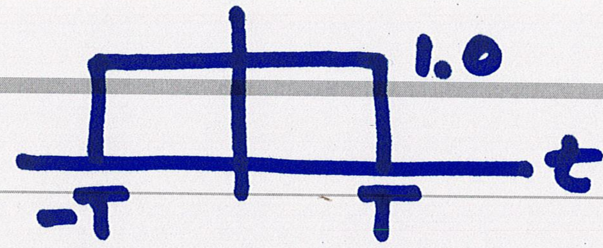
C.O.V. $\omega' = -\omega$, $d\omega' = -d\omega$

$$x^*(t) = \frac{1}{2\pi}\int_{+\infty}^{-\infty} X^*(-j\omega')e^{+j\omega' t}(-d\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X^*(-j\omega')}\, e^{+j\omega' t}\, d\omega'$$

$$\Longrightarrow \quad x^*(t) \longleftrightarrow X^*(-j\omega) = X^*(j(-\omega))$$

But if, in addition, $x(t)$ is real-valued, then $x(t) = x^*(t)$

$$\Longrightarrow \quad X(j\omega) = X^*(-j\omega)$$

CTFT{  }

$$x(t) = \begin{cases} 1.0 & |t| < T \\ 0 & \text{else} \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = \int_{-T}^{T} 1 e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega} e^{-j\omega t} \Big|_{t=-T}^{T} = \frac{1}{-j\omega}\left[e^{-j\omega T} - e^{+j\omega T}\right]$$

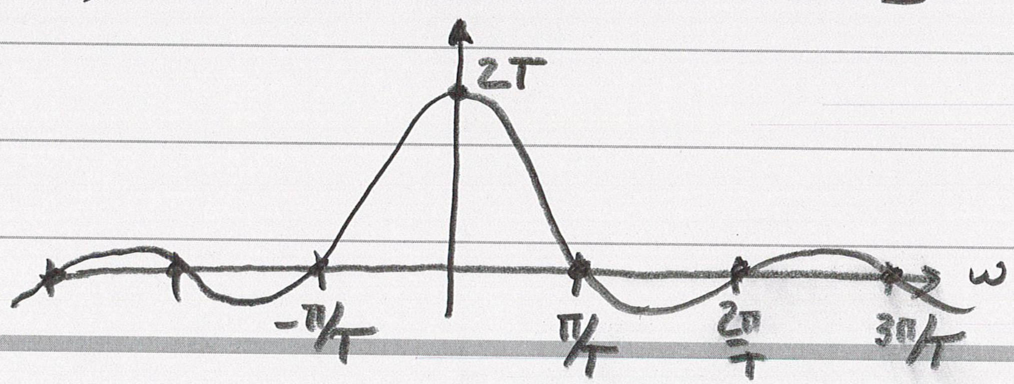
$$= \frac{2 \cdot 1}{2 j\omega}\left[e^{+j\omega T} - e^{-j\omega T}\right] = \frac{T}{T}\frac{2}{\omega} \sin \omega T$$

$$= 2T \frac{\sin(\omega T)}{\omega T} \longrightarrow$$

Note the text's def. of sinc function:

$$\operatorname{Sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

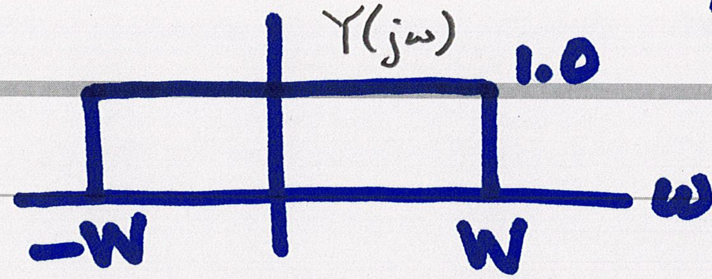
$\sin(\omega T) = 0$ if $\omega T = k\pi \quad k \in \mathbb{Z}$



$\omega = \frac{k\pi}{T} \quad k \neq 0$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = ?}$$

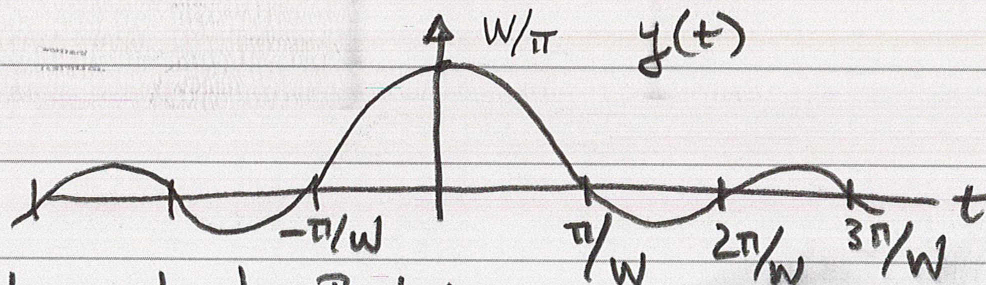
ICTFT{



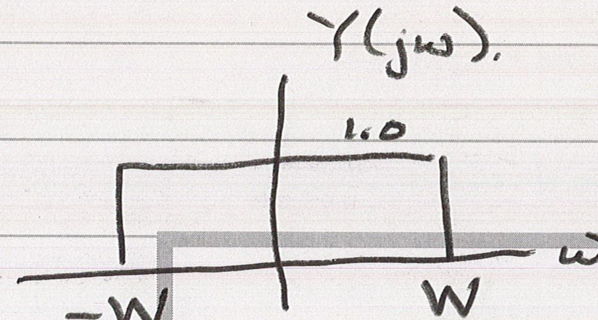
}

$$y(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{1}{2\pi}\frac{1}{jt}\, e^{j\omega t}\Big|_{\omega=-W}^{W}$$

$$= \frac{W}{\pi}\,\frac{\sin Wt}{Wt}$$



$\longleftrightarrow$



$Wt = \pi k \quad k \in \mathbb{Z},\ k \neq 0$

What if Tried $\text{CTFT}\left\{ \overbrace{\frac{W}{\pi} \frac{\sin Wt}{Wt}}^{y(t)} \right\}$?

$$Y(j\omega) = \int_{-\infty}^{\infty} \frac{W}{\pi} \frac{\sin Wt}{Wt} e^{-j\omega t}\, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{jWt} - e^{-jWt}}{t\,(j2)} e^{-j\omega t}\, dt$$

$$= \underbrace{\frac{1}{j2\pi} \int_{-\infty}^{\infty} \frac{e^{j(W-\omega)t}}{t}\, dt}_{?} - \underbrace{\frac{1}{j2\pi} \int_{-\infty}^{\infty} \frac{e^{-j(W+\omega)t}}{t}\, dt}_{?}$$

$\text{CTFT}\left\{\frac{1}{\sqrt{2\pi}\,\tau} e^{-t^2/2\tau^2}\right\} \quad t\in\mathbb{R}$

$x(t) =$

$$X(j\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\tau} e^{-t^2/2\tau^2} e^{-j\omega t}\, dt \qquad s = j\omega$$

$$\Rightarrow X(s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\tau} e^{-t^2/2\tau^2} e^{-st}\, dt$$

$$= \frac{1}{\sqrt{2\pi}\,\tau} \int_{-\infty}^{\infty} e^{-\frac{1}{2\tau^2}\left[t^2 + 2\tau^2 s t\right]} dt$$

$(t+\beta)^2 = t^2 + 2\beta t + \beta^2$

add + sub. $\beta^2 = (\tau^2 s)^2$ $\qquad \beta = \tau^2 s$

$$= \frac{1}{\sqrt{2\pi}\,\tau} \int_{-\infty}^{\infty} e^{-\frac{1}{2\tau^2}(t+\tau^2 s)^2} e^{-\frac{1}{2\tau^2}(-\tau^4 s^2)}\, dt$$

$$e^{-\frac{1}{2\tau^2}(-\tau^4 s^2)} = e^{\tau^2 s^2/2} \qquad s^2 = -\omega^2$$

$$= e^{-\tau^2\omega^2/2}$$

[More complete version of p6]

$$X(j\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\tau} e^{-t^2/2\tau^2} e^{-j\omega t}\, dt$$

Let $s = j\omega$ for short

$$= \frac{1}{\sqrt{2\pi}\,\tau} \int_{-\infty}^{\infty} e^{-\frac{1}{2\tau^2}\left[t^2 + 2\tau^2 s t\right]}\, dt$$

complete square by writting as

$$(t+\beta)^2 = t^2 + 2\beta t + \beta^2$$

$$\beta = \tau^2 s$$

$$e^{-\frac{1}{2\tau^2}\left[\underbrace{t^2 + 2\tau^2 s t + \tau^4 s^2 - \tau^4 s^2}_{(t+\tau^2 s)^2 - \tau^4 s^2}\right]}$$

$$\therefore X(s) = \frac{1}{\sqrt{2\pi}\,\tau} \int_{-\infty}^{\infty} e^{-\frac{1}{2\tau^2}(t+\tau^2 s)^2} e^{-\frac{1}{2\tau^2}(-\tau^4 s^2)}\, dt$$

[More complete version of p. 6]

$$X(s) = e^{-\frac{1}{2\tau^2}(-\tau^4 s^2)} \underbrace{\frac{1}{\sqrt{2\pi}\,\tau}\int_{-\infty}^{\infty} e^{-(t+\tau^2 s)^2/2\tau^2}\,dt}_{=1}$$

$$= e^{\tau^2 s^2/2} \qquad s = j\omega \Rightarrow s^2 = -\omega^2$$

$$X(j\omega) = e^{-\tau^2\omega^2/2}$$

$$\therefore \quad \frac{1}{\sqrt{2\pi}\,\tau}\, e^{-t^2/2\tau^2} \longleftrightarrow e^{-\tau^2\omega^2/2}$$

Gaussian Pulse in time $\longleftrightarrow$ Gaussian Pulse in freq.